

STRONGLY E -INVERSIVE SEMIRINGS

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Abstract

E -inversive semigroups have been the topic of research for many years. Properties of E -inversive semigroups were studied by Edward [1], Mitsch [9] and many others. In [2], Ghosh defined E -inversive semiring and studied its properties. According to him, an additively commutative semiring is called E -inversive semiring if and only if its additive reduct is an E -inversive semigroup. In this paper, we define strongly E -inversive semiring and study its properties.

Keywords: E -inversive semigroup, E -inversive semiring, strongly E -inversive semiring, skew-ring.

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1. INTRODUCTION

A semigroup S is said to be E -inversive if for every $a \in S$ there is an element $x \in S$ such that $ax \in E(S)$, where $E(S)$ is the set of all idempotents of the semigroup S . The notion of E -inversive semigroup was first introduced by Thierrin [13]. Later on, properties of E -inversive semigroups were studied by Mitsch [9], Lallement and Petrich [6], Hall and Munn [4], Margolis and Pin [8]. Subdirect products of E -inversive semigroups were first studied by Mitsch [9]. In 1999, the author defined E -inversive semiring [2]. According to him, an additively commutative semiring is said to be E -inversive if its additive reduct is an E -inversive semigroup. Some properties of E -inversive semirings were studied by Ghosh in that same paper

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[2]. In this paper, we define strongly E -inversive semiring and study some of its interesting properties.

The preliminaries and prerequisites, we need for this paper, are discussed in Section 2. In Section 3, we define strongly E -inversive semiring and study its basic properties. Finally, Section 4 is devoted to the study of E -unitary covers of strongly E -inversive semirings.

2. PRELIMINARIES

A semiring $(S, +, \cdot)$ is a type $(2, 2)$ -algebra such that both the additive reduct $(S, +)$ and the multiplicative reduct (S, \cdot) are semigroups and multiplication distributes over addition from either side, that is, $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ for all $a, b, c \in S$. We do not assume that the additive reduct $(S, +)$ is commutative. Following [3], we denote a semiring $(S, +, \cdot)$ as a skew-ring if its additive reduct $(S, +)$ is a group, not necessarily an abelian group. A semiring $(S, +, \cdot)$ is said to be a hemiring if the additive reduct $(S, +)$ is a monoid.

Let $(S, +, \cdot)$ be a semiring. An element $a \in S$ is called:

- *additively regular* if there exists an element $x \in S$ such that $a + x + a = a$;
- *additively completely regular* if there exists an element $z \in S$ such that $a + z + a = a$ and $z + a = a + z$;
- *completely regular* [12] if there exists $x \in S$ such that $a = a + x + a$, $a + x = x + a$ and $a(a + x) = a + x$.

If $a \in S$ is additively regular, we denote the set of all inverses of a in the semigroup $(S, +)$ by $V^+(a)$. A semiring S is said to be completely regular if every element a of S is completely regular. A semiring S is called idempotent semiring if both the reducts $(S, +)$ and (S, \cdot) are bands. A subsemiring I of a semiring S is called an ideal of S if $IS \subseteq I$ and $SI \subseteq I$. An ideal I of a semiring S is called a k -ideal of S if $a \in I$ and either $a + x \in I$ or $x + a \in I$ for some $x \in S$ implies $x \in I$. Throughout this paper, we always let $E^+(S)$ be the set of all additive idempotents of the semiring S . Let S and T be two semirings. A mapping $\varphi : S \rightarrow T$ is called a *semiring homomorphism* if $(a+b)\varphi = a\varphi + b\varphi$ and $(ab)\varphi = (a\varphi)(b\varphi)$ hold for all $a, b \in S$. If $\varphi : S \rightarrow T$ is a semiring homomorphism, then the kernel of φ , denoted by $\ker \varphi$, is defined by $\ker \varphi = \{s \in S \mid s\varphi \in E^+(T)\}$. A subsemiring T of the direct product $S_1 \times S_2$ of two semirings S_1 and S_2 is called subdirect product of S_1 and S_2 if the two projection mappings $\pi_1 : T \rightarrow S_1$ and $\pi_2 : T \rightarrow S_2$ respectively given by $\pi_1(s_1, s_2) = s_1$ and $\pi_2(s_1, s_2) = s_2$ (where $s_1 \in S_1, s_2 \in S_2$) are surjective.

In a semiring S , we denote Green's relations on the semigroup $(S, +)$ by $\mathcal{L}^+, \mathcal{R}^+, \mathcal{J}^+, \mathcal{D}^+$ and \mathcal{H}^+ . In fact, the relations $\mathcal{L}^+, \mathcal{R}^+, \mathcal{J}^+, \mathcal{D}^+$ and \mathcal{H}^+ are all

congruences on the multiplicative reduct (S, \cdot) . Thus, if any one of these happens to be a congruence on the additive reduct $(S, +)$, it will be a semiring congruence on the semiring $(S, +, \cdot)$. A completely regular semiring S is said to be completely simple [12] if $\mathcal{J}^+ = S \times S$.

Before closing this section, we recall an important result which will be helpful in our further discussion.

Theorem 1 [12]. *The following conditions on a semiring are equivalent:*

- (i) S is completely regular;
- (ii) every \mathcal{H}^+ -class is a skew-ring;
- (iii) S is union (disjoint) of skew-rings;
- (iv) S is a b -lattice of completely simple semirings.

3. BASIC PROPERTIES OF STRONGLY E -INVERSIVE SEMIRINGS

In this section, we define strongly E -inversive semiring and study some of its properties.

Definition. A semiring S is said to be a strongly E -inversive semiring if for each $a \in S$, there exists an element $x \in S$ such that (i) $a + x \in E^+(S)$, (ii) $ax = xa$, (iii) $a(a + x) = a + x$ and (iv) $a(x + a) = x + a$.

There are plenty of examples of strongly E -inversive semirings; for instance, every ring is a strongly E -inversive semiring, every skew-ring is a strongly E -inversive semiring, every completely regular semiring is a strongly E -inversive semiring (as every completely regular semiring is union of skew-rings), every idempotent semiring is a strongly E -inversive semiring. However, the converse of the above statement is not true in general as the following example shows.

Example 2. Define on the set \mathbb{N} , the following operations \oplus and \odot :

$$\begin{aligned} a \oplus b &= \min\{a + b, M\} \\ a \odot b &= \min\{ab, M\}, \end{aligned}$$

where $M \in \mathbb{N}$ is a fixed natural number. It is easy to verify that M is an additive idempotent element as well as a multiplicative idempotent. Now, for any $a \in \mathbb{N}$ choose an element $x \in \mathbb{N}$ such that $a + x > M$. Then $a \oplus x = x \oplus a = M$, $a \odot x = x \odot a$, $a \odot (a \oplus x) = a \odot M = M = a \oplus x$, $a \odot (x \oplus a) = a \odot M = M = x \oplus a$. Hence $(\mathbb{N}, \oplus, \odot)$ is a strongly E -inversive semiring. Since any element $b(> M) \in \mathbb{N}$ is not additively regular, it follows that this semiring is not a quasi completely regular semiring [7] and hence not a completely regular semiring. Moreover, it is interesting to point out that this semiring is not even an idempotent semiring.

Example 3. Any semiring S with (S, \cdot) a band is strongly E -inverse. In fact for any $a \in S$, $a + a \in E^+(S)$, $a \cdot a = a \cdot a$, $a(a + a) = a + a$.

Theorem 4. A semiring S is a strongly E -inverse semiring if and only if for all $a \in S$ there exists $y \in S$ such that $a + y, y + a \in E^+(S)$; $ay = ya$; $(a + y)a = a + y$; $(y + a)a = y + a$.

Proof. First suppose that the semiring S is a strongly E -inverse semiring. Then for any $a \in S$, there exists an element $x \in S$ such that $a + x \in E^+(S)$, $ax = xa$, $a(a + x) = a + x$ and $a(x + a) = x + a$. Now, if we choose $y = x + a + x$, then it can be easily verified that $y + a + y = y$ and hence $a + y, y + a \in E^+(S)$. Since $ax = xa$, we must have $ay = ya$. Again, $(a + y)a = a^2 + ya = a^2 + ay = a(a + y) = a(a + x + a + x) = a(a + x) + a(a + x) = a + x + a + x = a + y$. Hence $(a + y)a = a + y$. Similarly, we can show that $(y + a)a = y + a$.

Converse part follows easily. ■

Theorem 5. A semiring S is a strongly E -inverse semiring if and only if for all $a \in S$ there exists $y \in S$ such that $y + a + y = y$, $ay = ya$, $a(a + y) = a + y$, $a(y + a) = y + a$.

Proof. First suppose that the semiring S is a strongly E -inverse semiring. Then for any $a \in S$, there exists an element $x \in S$ such that $a + x \in E^+(S)$, $ax = xa$, $a(a + x) = a + x$ and $a(x + a) = x + a$. Now, if we choose $y = x + a + x$, then it can be easily verified that $y + a + y = y$, $ay = ya$, $a(a + y) = a + y$ and $a(y + a) = y + a$.

Converse part is obvious. ■

Remark 6. The definition of strongly E -inverse semiring is not one-sided. Moreover, a semiring S is strongly E -inverse if and only if $W(a) = \{x \in S : x + a + x = x, ax = xa, a(a + x) = a + x, a(x + a) = x + a\} \neq \emptyset$ for every $a \in S$. Now following [9], we get that if $(S, +, \cdot)$ is a strongly E -inverse semiring then $(S, +)$ is an E -inverse semigroup as $I(a) = \{x \in S : a + x, x + a \in E^+(S)\} \neq \emptyset$ for every $a \in S$ and it is obvious that $W(a) \subseteq I(a)$ for all $a \in S$.

Though the additive reduct of a strongly E -inverse semiring is always an E -inverse semigroup, but the converse may not be true as the following example shows.

Example 7. Consider the semiring $(\mathbb{N}, +, \cdot)$, where addition ‘+’ of two elements is maximum of two elements in \mathbb{N} and multiplication ‘.’ is the usual multiplication of natural numbers. Then $(\mathbb{N}, +)$ is an E -inverse semigroup and $(\mathbb{N}, +, \cdot)$ is an E -inverse semiring but not a strongly E -inverse semiring.

Proposition 8. Every left (right, both sided) k -ideal of a strongly E -inverse semiring is strongly E -inverse.

Proof. Let I be a left k -ideal of a strongly E -inverse semiring S and $a \in I$. Since S is strongly E -inverse, it follows that $W(a) \neq \emptyset$. Let $x \in W(a)$. As I is a left ideal of S and $a \in I$, we must have $a^2, xa \in I$. This implies $a^2 + xa \in I$, i.e., $a + x \in I$. Again, I is a left k -ideal of S and $a, a + x \in I$ imply that $x \in I$. Consequently, I is a strongly E -inverse semiring. ■

The following well known result can be found, for instance, in [10].

Lemma 9. *The following conditions on a semigroup (S, \cdot) are equivalent.*

- (1) S is a rectangular band;
- (2) S is regular and satisfies the identity $ab = axb$;
- (3) S is a completely simple band.

Proposition 10. *Let S be a strongly E -inverse semiring such that the multiplicative reduct (S, \cdot) is a completely simple semigroup and $e \in W(e)$, for all $e \in E^+(S)$. Then (S, \cdot) is a rectangular band. Conversely, if S is a semiring such that (S, \cdot) is a rectangular band, then S is a strongly E -inverse semiring such that $e \in W(e)$, for all $e \in E^+(S)$.*

Proof. Since $E^+(S)$ is an ideal of (S, \cdot) and (S, \cdot) is simple, it follows that $S = E^+(S)$. Let $a \in S$. Then $a \in E^+(S)$ and hence by the given condition, it follows that $a \in W(a)$. This implies $a(a + a) = a + a$, $a^2 = a$. Thus (S, \cdot) is a band and hence by Lemma 9, it follows that (S, \cdot) is a rectangular band.

Conversely, if S is a semiring whose multiplicative reduct (S, \cdot) is a rectangular band, then by Example 3, it at once follows that S is a strongly E -inverse semiring such that $e \in W(e)$, for all $e \in E^+(S)$. ■

Proposition 11. *If for every element a in a semiring S there is exactly one $x \in S$ such that $a + x \in E^+(S)$, then S is a skew-ring.*

Proof. From [9, Proposition 2], it follows that $(S, +)$ is a group and hence S is a skew-ring. ■

Proposition 12. *Let S be a strongly E -inverse semiring without zero. Then the following conditions are equivalent:*

- (i) S is weakly additive cancellative (i.e., $a + x = b + x$ and $x + a = x + b$ imply that $a = b$) and for each $a \in S$ there exists some $x \in W(a)$ satisfying $x(a + x) = a + x, x(x + a) = x + a$;
- (ii) S is a completely simple semiring.

Proof. (i) \implies (ii) Let $a \in S$. Then there exists an element $x \in W(a)$ such that $x + a + x = x, ax = xa, a(a + x) = a + x = x(a + x), a(x + a) = x + a = x(x + a)$. Now, clearly $(a + x + a) + x = a + x$ and $x + (a + x + a) = x + a$. Since S is

weakly additive cancellative, we must have $a + x + a = a$ and thus S is additively regular. Now $(a + x)(x + a) = a(x + a) + x(x + a) = x + a + x + a = x + a$. Again, $(a + x)(x + a) = (a + x)x + (a + x)a = x(a + x) + a(a + x) = a + x + a + x = a + x$. Hence $a + x = x + a$. Therefore, S is a completely regular semiring. By [5, Theorem 3.3.3], it follows that $(S, +)$ is completely simple, so $\mathcal{J}^+ = S \times S$. Hence S is a completely simple semiring.

(ii) \implies (i) Since S is a completely simple semiring, so $(S, +)$ is a completely simple semigroup. Then by [5, Theorem 3.3.3], we must have S is weakly additive cancellative. Again, since S is completely simple, it follows that S is a completely regular semiring and hence by Theorem 1, it follows that S is union of skew-rings. Let $a \in S$. Then H_a , the \mathcal{H}^+ -class containing the element a is a skew-ring. Let $y \in H_a$ be the inverse of a in the group $(H_a, +)$. Then clearly $y \in W(a)$ such that $y(a + y) = a + y, y(y + a) = y + a$. ■

Proposition 13. *Let S be a strongly E -inverse semiring. Then the following are equivalent:*

- (i) S is left additive cancellative;
- (ii) S is a right skew-ring.

Proof. Since S is strongly E -inverse semiring, so its additive reduct is an E -inverse semigroup. Again, a semiring is called right skew-ring if its additive reduct is a right group. Hence the result holds from [9, Proposition 4]. ■

Definition. A nonempty subset I of a semiring S is said to be a bi-ideal of S if $a \in I$ and $x \in S$ imply that $a + x, x + a, ax, xa \in I$. A semiring S is said to be an ideal extension of a semiring T if T is a bi-ideal of S .

Theorem 14. *Let S be an ideal extension of a semiring T . If S is strongly E -inverse, then T is also strongly E -inverse.*

Proof. Let $t \in T$. Then $t \in S$. Since S is strongly E -inverse, there exists an element $x \in S$ such that $x + t + x = x, xt = tx, t(t + x) = t + x$ and $t(x + t) = x + t$. Since T is a bi-ideal of S and $t \in T, x \in S$, it follows that $x = x + t + x \in T$. Hence T is strongly E -inverse. ■

The converse of Theorem 14 can be proved by taking additional condition as follows.

Theorem 15. *Let S be an ideal extension of a strongly E -inverse semiring T such that $se_t = e_ts = e_t$ for all $s \in S \setminus T, e_t \in E^+(T)$ and S is weakly additive cancellative. Then S is also strongly E -inverse.*

Proof. Let $s \in S$. If $s \in T$ then obviously s is a strongly E -inverse element. Let $s \notin T$, so $s \in S \setminus T$. Then $s+t \in T$ for all $t \in T$ as T is a bi-ideal of S . Since T is strongly E -inverse, there exists an element $x \in T$ such that $x+(s+t)+x = x$. Then $t+x+s+t+x = t+x$ which implies that $s+t+x, t+x+s \in E^+(T)$. Hence $s(s+t+x), (s+t+x)s, s(t+x+s), (t+x+s)s \in E^+(T)$. By given condition, we have $s(s+t+x) = (s+t+x)s = s+t+x$ and $s(t+x+s) = (t+x+s)s = t+x+s$ which imply that $s^2 + s(t+x) = s^2 + (t+x)s$ and $s(t+x) + s^2 = (t+x)s + s^2$. Hence $s(t+x) = (t+x)s$ as S is weakly additive cancellative. Now if we choose $t+x = x_s$, then $x_s + s + x_s = x_s, sx_s = x_ss, s(s+x_s) = s+x_s, s(x_s+s) = x_s+s$. Hence S is strongly E -inverse. ■

Corollary 16. *Let S be an ideal extension of a strongly E -inverse semiring T such that $|E^+(T)| = 1$ and S is weakly additive cancellative. Then S is strongly E -inverse semiring such that for all $s \in S$ there exists $x \in S$ satisfying (i) $s+x = x+s \in E^+(S)$, (ii) $sx = xs$ and (iii) $s(s+x) = s+x = x(s+x)$.*

4. E -UNITARY COVERS OF STRONGLY E -INVERSE SEMIRINGS

It is easy to verify that the direct product of strongly E -inverse semirings is again a strongly E -inverse semiring. But the subdirect product of even two strongly E -inverse semirings may not be strongly E -inverse. In this section, we characterize all those subdirect products of two strongly E -inverse semirings which are again strongly E -inverse. Also, we study E -unitary covers of strongly E -inverse semirings.

Proposition 17. *Homomorphic image of a strongly E -inverse semiring is strongly E -inverse.*

Proof. Let $\varphi : S \rightarrow T$ be a semiring epimorphism between two semirings S and T such that S is strongly E -inverse. Let $b \in T$ be arbitrary. Then there exists an element $a \in S$ such that $b = a\varphi$. Now, for the element $a \in S$, there exists an element $x \in S$ such that $x \in W(a)$. Let $y = x\varphi$. Then one can easily verify that $y \in W(b)$. Since $b \in T$ is arbitrary, it follows that T is strongly E -inverse. ■

Corollary 18. *If S is a strongly E -inverse semiring such that it is a subdirect product of two semirings A and B , then both A and B are strongly E -inverse.*

Remark 19. Subdirect product of two strongly E -inverse semirings may not be strongly E -inverse. This follows from the following example.

Example 20. Let $A = B = (\mathbb{Z}, +, \cdot)$ and S be the subsemiring of $A \times B$ generated by $\{(1, 1), (-1, -3)\}$. Then S is a subdirect product of strongly E -inverse semirings A and B , but S is not strongly E -inverse, since $E^+(S) = \{(0, 0)\}$ and

for the element $(1, 1) \in S$, there is no element $(x, y) \in S$ such that $(1, 1) + (x, y) = (0, 0) \in E^+(S)$.

In [9], Mitsch established necessary and sufficient condition for subdirect products of two E -inversive semigroups to be again E -inversive. Now, we establish a necessary and sufficient condition for subdirect products of two strongly E -inversive semirings to be again strongly E -inversive. For this purpose, let us first define the following definition.

Definition (surjective subhomomorphism). Suppose S, T be two strongly E -inversive semirings and $\psi : S \rightarrow \mathcal{P}(T)$ (the power set of T) is a mapping. Then ψ is called surjective subhomomorphism of S onto T if the following conditions are satisfied:

- (1) $s\psi \neq \emptyset$ for all $s \in S$,
- (2) $s_1\psi + s_2\psi \subseteq (s_1 + s_2)\psi$ and $(s_1\psi)(s_2\psi) \subseteq (s_1s_2)\psi$ for all $s_1, s_2 \in S$,
- (3) $\bigcup_{s \in S} s\psi = T$,
- (4) for every $t \in s\psi$, there exist $x \in W(s)$ and $y \in W(t)$ such that $y \in x\psi$.

Theorem 21. Let S, T be two strongly E -inversive semirings and ψ be a surjective subhomomorphism of S onto T . Then $\pi(S, T, \psi) = \{(s, t) \in S \times T : t \in s\psi\}$ is a strongly E -inversive semiring which is a subdirect product of S and T . Conversely, every strongly E -inversive semiring which is a subdirect product of two strongly E -inversive semirings can be obtained in this way.

Proof. Let $A = \pi(S, T, \psi)$. Then by condition (2) of the definition of surjective subhomomorphism, it follows that A is a semiring. Again, following [9, Theorem 7], it follows that $(A, +)$ is an E -inversive semigroup which is a subdirect product of S and T . To show A is strongly E -inversive, let $(s, t) \in A$. Then $t \in s\psi$ and hence by condition (4), there exist $x \in W(s)$ and $y \in W(t)$ such that $y \in x\psi$. This implies $(x, y) \in A$. Moreover, it is easy to verify that $(x, y) \in W((s, t))$. Hence A is a strongly E -inversive semiring.

Conversely, let B be a strongly E -inversive semiring which is a subdirect product of two strongly E -inversive semirings S and T . Then $(B, +)$ is an E -inversive semigroup. Let $\alpha : S \rightarrow \mathcal{P}(T)$ be defined by $s\alpha = \{t \in T : (s, t) \in B\}$ for every $s \in S$. Since B is a subsemiring of $S \times T$, we can easily verify that the condition (2) holds. Let $t \in s\alpha$, where $s \in S$ and $t \in T$. Then $(s, t) \in B$, hence there exists $(x, y) \in B$ such that $(x, y) \in W((s, t))$ as B is a strongly E -inversive semiring. Therefore, $(x, y) + (s, t) + (x, y) = (x, y)$, $(s, t)(x, y) = (x, y)(s, t)$, $(s, t)((s, t) + (x, y)) = (s, t) + (x, y)$, $(s, t)((x, y) + (s, t)) = (x, y) + (s, t)$. Then we get, $x + s + x = x$, $sx = xs$, $s(s + x) = s + x$, $s(x + s) = x + s$. Hence $x \in W(s)$ and similarly we get $y \in W(t)$. As $(x, y) \in B$, so $y \in x\alpha$. Therefore, the condition (4) holds. Finally, since $(B, +)$ is an E -inversive semigroup, by [9, Theorem 7], it follows that the conditions (1), (3) hold and $B = \pi(S, T, \alpha)$. ■

Definition. A subset T of a strongly E -inverse semiring S is called an ideal of S if all $a, b \in T$ and all $s \in S$ imply that $a + b, sa, as \in T$. An ideal T of S is said to be full if $E^+(S) \subseteq T$. An ideal T of S is called a normal ideal if $s + T + s' \subseteq T$ and $s' + T + s \subseteq T$ for all $s \in S$ and $s' \in I(s)$.

Lemma 22. *Let S be a strongly E -inverse semiring. Then the least skew-ring congruence σ on S is given by : for $a, b \in S$,*

$$a \sigma b \text{ if and only if } x + a = b + y \text{ for some } x, y \in N,$$

where N is the intersection of all full normal ideals of S .

Proof. Following the proof of [9, Proposition 9], it is easy to verify that for every full normal ideal T , the relation σ_T on S given by : for $a, b \in S$, $a \sigma_T b$ if and only if $x + a = b + y$ for some $x, y \in T$ is a group congruence on $(S, +)$. Let $a \sigma_T b$ and $c \in S$. This implies $xc + ac = bc + yc$ with $xc, yc \in T$ and thus $ac \sigma_T bc$. Similarly, we can show that $ca \sigma_T cb$. Therefore, σ_T is a semiring congruence on S such that $(S/\sigma_T, +)$ is a group. Consequently, S/σ_T is a skew-ring and hence σ_T is a skew-ring congruence on S .

Conversely, let δ be a skew-ring congruence on S and let $T = \{t \in S : t_\delta = 0_\delta\}$, where 0_δ is the zero element of the skew-ring S/δ . Obviously, T is an ideal of S such that $E^+(S) \subseteq T$. Now for each $s \in S$ and $s' \in I(s)$, s'_δ is the additive inverse of $s_\delta \in S/\delta$. Again, for any $t \in T$, $(s + t + s')_\delta = (s + s')_\delta = 0_\delta$. Hence $s + t + s' \in T$. Similarly, it can be verified that $s' + t + s \in T$. Therefore, T is a full normal ideal of S . It is easy to show that $\sigma_T \leq \delta$. Let $a \delta b$, then $(b' + a)_\delta (b' + b)_\delta$, where $b' \in I(b)$ and hence $b' + a \in T$. Now $(b + b') + a = b + (b' + a)$ implies that $a \sigma_T b$, so $\sigma_T = \delta$. Now it is clear that $\sigma = \sigma_N$, where N is the intersection of all full normal ideals of S . ■

Definition. A semiring S with additive idempotents is said to be E -unitary if its additive semigroup reduct $(S, +)$ is an E -unitary semigroup, i.e., $e + s, e \in E^+(S), s \in S$ imply $s \in E^+(S)$. One can easily verify that this condition is equivalent to : $s + e, e \in E^+(S), s \in S$ imply $s \in E^+(S)$. A semiring T is called E -unitary cover of a semiring S if T is E -unitary and if there is an epimorphism $\varphi : T \rightarrow S$ such that φ is additive idempotent separating, i.e., for any two elements $e, f \in E^+(T)$, $e\varphi = f\varphi$ implies $e = f$. In addition, if there is a least skew-ring congruence δ on T and a skew-ring R such that $T/\delta \cong R$, then T is called E -unitary cover of S through R .

Proposition 23. *If σ is the least skew-ring congruence on a strongly E -inverse, E -unitary semiring S , then $\ker \sigma = N = E^+(S)$, where N is the intersection of all full normal ideals of S .*

Proof. Using Lemma 22 and following the proof of [9, Corollary 10], in a similar way it can be easily verified that $E^+(S)$ is a full normal ideal of S and $E^+(S) = N = \ker \sigma$. ■

Definition. Suppose S be a strongly E -inversive semiring and R be a skew-ring; a subhomomorphism ψ of S into R is called nullary if $0 \in s\psi$ implies that $s \in E^+(S)$, where $s \in S$ and 0 is the zero element of the skew-ring R .

Lemma 24. Let S be a strongly E -inversive semiring, R be a skew-ring and ψ be a nullary surjective subhomomorphism of S onto R . Then $\pi_2 \circ \pi_2^{-1}$ is the least skew-ring congruence on the strongly E -inversive semiring $T = \pi(S, R, \psi)$, where $\pi_2 : T \rightarrow R$ is the projection mapping defined by $\pi_2(s, r) = r$ for all $(s, r) \in T$. Hence $T/\sigma = T/\pi_2 \circ \pi_2^{-1} \cong R$, where σ is the least skew-ring congruence on T .

Proof. From Theorem 21, it follows that T is strongly E -inversive semiring which is a subdirect product of S and R and hence R is a homomorphic image of T under the projection mapping defined by $\pi_2(s, r) = r$ for all $(s, r) \in T$. Now, the congruence induced by π_2 on T , i.e., $\pi_2 \circ \pi_2^{-1}$ on T is a skew-ring congruence as R is a skew-ring. Then $\sigma \subseteq \pi_2 \circ \pi_2^{-1}$, where σ is the least skew-ring congruence on the strongly E -inversive semiring T .

Suppose $(s_1, r_1) \pi_2 \circ \pi_2^{-1} (s_2, r_2)$, where $(s_1, r_1), (s_2, r_2) \in T$. This implies $(s_1, r_1)\pi_2 = (s_2, r_2)\pi_2$, i.e., $r_1 = r_2$ and thus $(s_1, r_2) \in T$. Since T is strongly E -inversive, then there exists $(u, v) \in T$ such that $(u, v) + (s_1, r_2) \in E^+(T) = \{(e, 0) : e \in E^+(S)\}$, where 0 is the zero element of the skew-ring R . Therefore, $v = -r_2$ and so $(u, -r_2) \in T$ and $(u + s_1, 0) \in E^+(T)$. Again, $(s_2, r_2) + (u, -r_2) = (s_2 + u, 0) \in T$ implies $0 \in (s_2 + u)\psi$. Since ψ is nullary, therefore $s_2 + u \in E^+(S)$. Let N be the intersection of all full normal ideals of T . Then $(s_2 + u, 0), (u + s_1, 0) \in E^+(T) \subseteq N$ and $(s_2 + u, 0) + (s_1, r_1) = (s_2 + u + s_1, r_1) = (s_2, r_1) + (u + s_1, 0) = (s_2, r_2) + (u + s_1, 0)$ imply that $(s_1, r_1)\sigma (s_2, r_2)$. Hence $\pi_2 \circ \pi_2^{-1} \subseteq \sigma$ and so $\pi_2 \circ \pi_2^{-1} = \sigma$. Consequently, $T/\sigma = T/\pi_2 \circ \pi_2^{-1} \cong R$. ■

Theorem 25. Let S be a strongly E -inversive semiring, R be a skew-ring and ψ be a nullary surjective subhomomorphism of S onto R . Then $\pi(S, R, \psi)$ is a strongly E -inversive, E -unitary cover of S through R .

Proof. By Theorem 21, it follows that $\pi(S, R, \psi)$ is a strongly E -inversive semiring. Similar to the proof of [9, Theorem 8], we can prove that $\pi(S, R, \psi)$ is an E -unitary cover of S . Finally, by Lemma 24, it follows that $\pi(S, R, \psi)/\sigma \cong R$, where σ is the least skew-ring congruence on $\pi(S, R, \psi)$ and hence $\pi(S, R, \psi)$ is a strongly E -inversive, E -unitary cover of S through R . ■

A construction of all unitary, surjective subhomomorphisms of an inverse semigroup onto a group was described by Petrich and Reilly in [11]. Later on,

this result was generalized by Mitsch in [9] for E -inversive semigroups. In the following theorem, we make an extension of this idea to strongly E -inversive semirings.

Theorem 26. *Let S be a strongly E -inversive semiring and R be a skew-ring. Then ψ is a nullary surjective subhomomorphism of S onto R if and only if $\psi = \alpha^{-1} \circ \beta$ for some strongly E -inversive semiring T and some surjective homomorphisms $\alpha : T \rightarrow S$ and $\beta : T \rightarrow R$ such that $\ker \beta \subseteq \ker \alpha$.*

Proof. First we assume that T is a strongly E -inversive semiring and $\psi = \alpha^{-1} \circ \beta$, where $\alpha : T \rightarrow S$ and $\beta : T \rightarrow R$ are surjective homomorphisms such that $\ker \beta \subseteq \ker \alpha$. As α, β both are surjective, it follows that $s\psi \neq \emptyset$ and ψ is surjective, i.e., $\bigcup_{s \in S} s\psi = R$. Suppose $r_1 \in s_1\psi$ and $r_2 \in s_2\psi$, where $s_1, s_2 \in S$. Then there exist $t_1, t_2 \in T$ such that $t_1\alpha = s_1$, $t_1\beta = r_1$, $t_2\alpha = s_2$, $t_2\beta = r_2$. Now, $r_1 + r_2 = (t_1 + t_2)\beta$ and $(t_1 + t_2)\alpha = s_1 + s_2$ imply that $(r_1 + r_2) \in (s_1 + s_2)(\alpha^{-1} \circ \beta) = (s_1 + s_2)\psi$ and similarly $(r_1 r_2) \in (s_1 s_2)\psi$. Therefore, $s_1\psi + s_2\psi \subseteq (s_1 + s_2)\psi$ and $(s_1\psi)(s_2\psi) \subseteq (s_1 s_2)\psi$ for all $s_1, s_2 \in S$. Again, as T is a strongly E -inversive semiring, so $W(t_1) \neq \emptyset$. Let $t'_1 \in W(t_1)$. Let $t'_1\alpha = s_3$ and $t'_1\beta = r_3$. Then clearly $s_3 \in W(s_1)$, $r_3 \in W(r_1)$ and obviously $r_3 \in s_3(\alpha^{-1} \circ \beta) = s_3\psi$ and thus ψ is a surjective subhomomorphism. Now, suppose $0 \in s\psi$. Then there exists some $t \in T$ such that $t\alpha = s$ and $t\beta = 0$. This implies $t \in \ker \beta \subseteq \ker \alpha$ and hence $s = t\alpha \in E^+(S)$. Consequently, ψ is nullary surjective subhomomorphism of S onto R .

Conversely, we assume that ψ is a nullary surjective subhomomorphism from S onto R . By Theorem 25, it follows that $\pi(S, R, \psi)$ is a strongly E -inversive, E -unitary cover of S through R and by Theorem 21, we have $\pi(S, R, \psi)$ is a subdirect product of S and R . Let $T = \pi(S, R, \psi)$ and let $\pi_1 : T \rightarrow S$ and $\pi_2 : T \rightarrow R$ be the projection mappings. By Lemma 24, $\pi_2 \circ \pi_1^{-1}$ is the least skew-ring congruence on T and so by Proposition 23, we have $\ker \pi_2 = E^+(T)$. Clearly, $E^+(T) \subseteq \ker \pi_1$ and so by the sufficiency of this theorem $\gamma = \pi_1^{-1} \circ \pi_2$ is a nullary surjective subhomomorphism from S onto R and it can be easily verified that $\pi(S, R, \gamma) = \{(t\pi_1, t\pi_2) : t \in T\} = T$. Hence $s\gamma = \{r \in R : (s, r) \in \pi(S, R, \gamma)\} = \{r \in R : (s, r) \in T\} = s\psi$ for all $s \in S$. Therefore, $\psi = \gamma = \pi_1^{-1} \circ \pi_2$. ■

We are now in a position to prove the converse of the Theorem 25.

Theorem 27. *Let S be a strongly E -inversive semiring, R be a skew-ring and T be a strongly E -inversive, E -unitary cover of S through R . Then there is a nullary surjective subhomomorphism ψ of S onto R such that $\pi(S, R, \psi)$ is a homomorphic image of T and $\pi(S, R, \psi)$ is a strongly E -inversive, E -unitary cover of S through R . Moreover, if T is a subdirect product of S and R , then $T \cong \pi(S, R, \psi)$.*

Proof. Using Theorem 21, Proposition 23, Theorem 25 and Theorem 26, the proof follows similar to the proof of [9, Theorem 13]. ■

Following [9], a subdirect product H of a semigroup K and a group G is called full if $(e, 1) \in H$ for every $e \in E(K)$, where 1 is the identity element of the group G . A subdirect product T of a semiring S and a skew-ring R is called full if $(e, 0) \in T$ for every $e \in E^+(S)$, where 0 is the zero element of the skew-ring R .

In [9], Mitsch gave the construction of all E -inversive semigroups which are full subdirect products of a semilattice and a group. Similar to semigroup, using the concept of surjective subhomomorphism, we here describe the construction of all strongly E -inversive semirings which are full subdirect products of an idempotent semiring and a skew-ring.

Theorem 28. *Let I be an idempotent semiring, R be a skew-ring and $\mathcal{H}(R)$ be the collection of all subhemirings of R . Suppose $\theta : I \longrightarrow \mathcal{H}(R) \subseteq \mathcal{P}(R)$ is a mapping such that θ is a surjective subhomomorphism of I onto R . Then $S = \{(\alpha, r) \in I \times R : r \in \alpha\theta\}$ is a strongly E -inversive semiring which is a full subdirect product of I and R . Conversely, every such semiring can be constructed in this manner.*

Proof. Suppose $S = \{(\alpha, r) \in I \times R : r \in \alpha\theta\}$, where θ is a surjective subhomomorphism of I onto R . Then by Theorem 21, it follows that S is a strongly E -inversive semiring which is a subdirect product of I and R . Also, S is full as $\alpha\theta$ is a subhemiring of R for all $\alpha \in I$.

Conversely, suppose that S is a strongly E -inversive semiring which is a full subdirect product of an idempotent semiring I and a skew-ring R . We define $\theta : I \longrightarrow \mathcal{H}(R) \subseteq \mathcal{P}(R)$ by $\alpha\theta = \{r \in R : (\alpha, r) \in S\}$ for all $\alpha \in I$. Since $(\alpha, 0) \in S$ for all $\alpha \in I$, it follows that $0 \in \alpha\theta$ and thus $\alpha\theta \neq \emptyset$. Moreover, $\bigcup_{\alpha \in I} \alpha\theta = R$, since S is a subdirect product of I and R . It is easy to verify that $\alpha\theta$ is a subhemiring of R for all $\alpha \in I$. Suppose $r_1 \in \alpha\theta$ and $r_2 \in \beta\theta$. Then $(\alpha, r_1), (\beta, r_2) \in S$. Since S is a semiring, we must have $(\alpha + \beta, r_1 + r_2) = (\alpha, r_1) + (\beta, r_2) \in S$ and $(\alpha\beta, r_1 r_2) = (\alpha, r_1)(\beta, r_2) \in S$ and thus $\alpha\theta + \beta\theta \subseteq (\alpha + \beta)\theta$ and $(\alpha\theta)(\beta\theta) \subseteq (\alpha\beta)\theta$. Let $r \in \alpha\theta$. Then $(\alpha, r) \in S$. As S is strongly E -inversive, so there exists $(\beta, p) \in S$ such that $(\beta, p) \in W((\alpha, r))$. It is easy to verify that $\beta \in W(\alpha)$ and $p \in W(r)$. Obviously, $p \in \beta\theta$ as $(\beta, p) \in S$. Therefore, θ is a surjective subhomomorphism of I onto R and finally, one can easily verify that $S \cong \{(\alpha, r) \in I \times R : r \in \alpha\theta\}$. ■

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