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A STUDY ON IDEAL ELEMENTS IN ORDERED Γ-SEMIRINGS

M. Murali Krishna Rao

Department of Mathematics
Sankethika Engineering College
Visakhapatnam-530 041, Andhra Pradesh, India
e-mail: mmarapureddy@gmail.com

Noorbhasha Rafi

Department of Mathematics, Bapatla Engineering College
Bapatla-522 101, Andhra Pradesh, India
e-mail: rafimaths@gmail.com

and

Rajendra Kumar Kona

Department of Mathematics
GIS, GITAM (Deemed to be University)
Visakhapatnam - 530 045, Andhra Pradesh, India
e-mail: rkkona1972@gmail.com

Abstract

The aim of this paper is to study the structures of ordered Γ-semigroups not only with the ideal elements but also with the generalization of ideal elements. The ideal elements play an important and necessary role in studying the structure of ordered semigroups. We introduce the notion of (ideal, interior ideal, quasi ideal, bi-ideal, quasi interior ideal and weak interior ideal) elements of ordered Γ-semirings. We study the properties of ideal elements, relations between them and characterize the ordered Γ-semirings, regular ordered Γ-semirings and simple ordered Γ-semirings using ideal elements. We prove that if M be a simple ordered Γ-semiring, then every element of M is an ideal element of M.

Keywords: bi-ideal elements, interior ideal elements, Bi-ideal elements, quasi interior ideal elements, weak interior ideal elements, ordered Γ-semirings.

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1. Introduction

The algebraic structures play a prominent role in mathematics with wide range of applications. Generalization of ideals of algebraic structures and ordered algebraic structure plays a very remarkable role and also necessary for further advance studies and applications of various algebraic structures. Many mathematicians proved important results and characterization of algebraic structures by using the concept and the properties of generalization of ideals in algebraic structures.

The author [15]–[?] introduced and studied (weak interior, bi-quasi, quasi-interior and bi-quasi interior) ideals of Γ-semirings, semirings, Γ-semigroups, semigroups as a generalization of (bi, quasi and interior) ideal of algebraic structures and characterized regular algebraic structures as well as simple algebraic structures using these ideals. In 1995, M.Murali Krishna Rao [13, ?] introduced the notion of Γ-semiring as a generalization of Γ-ring, ternary semiring and semiring. As a generalization of ring, the notion of a Γ-ring was introduced by Nobusawa [12] in 1964. In 1981, Sen [23] introduced the notion of a Γ-semigroup as a generalization of semigroup. The notion of a ternary algebraic system was introduced by Lehmer [10] in 1932. In 1971, Lister [11] introduced ternary ring. The set of all negative integers Z is not a semiring with respect to usual addition and multiplication but Z forms a Γ-semiring where $\Gamma = Z$. The important reason for the development of Γ-semiring is a generalization of results of rings, Γ-rings, semirings, semigroups and ternary semirings. The notion of a semiring was introduced by Vandiver [28] in 1934.

We know that the notion of a one sided ideal of any algebraic structure is a generalization of an ideal. The quasi ideals are generalization of left ideal and right ideal whereas the bi-ideals are generalization of quasi ideals. In 1952, the concept of bi-ideals was introduced by Good and Hughes [2] for semigroups. The notion of bi-ideals in rings and semigroups were introduced by Lajos and Szasz [8]–[?]. Bi-ideal is a special case of $(m-n)$ ideal. In 1976, the concept of interior ideals was introduced by Lajos for semigroups. Steinfeld [25] first introduced the notion of quasi ideals for semigroups and then for rings. Iseki and Izuka [4, ?, 5] introduced the concept of quasi ideal for a semiring. Quasi-ideals, bi-ideals in Γ-semirings was studied by Jagtap and Pawar [6]. Henriksen [3] and Shabir *et al.* [24] studied quasi-ideals in semirings.

In this paper, we introduce the notion of (ideal, interior-ideal, quasi-ideal, bi-ideal, quasi-interior ideal and weak interior ideal) elements of ordered Γ-semirings. We study the properties of ideal elements and relations between them and characterize the ordered Γ-semirings, regular ordered Γ-semirings and simple ordered Γ-semirings using ideal elements.

2. Preliminaries

In this section, we will recall some of the fundamental concepts and definitions, which are necessary for this paper.

Definition 2.1 [1]**.** A set S together with two associative binary operations called addition and multiplication (denoted by $+$ and $\cdot$ respectively will be called a semiring provided

(i) addition is a commutative operation.

(ii) multiplication distributes over addition both from the left and from the right.

(iii) there exists $0 \in S$ such that $x + 0 = x$ and $x \cdot 0 = 0 \cdot x = 0$ for all $x \in S$.

Definition 2.2 [13]**.** Let $(M, +)$ and $(\Gamma, +)$ be commutative semigroups. Then we call M a Γ-semiring, if there exists a mapping $M \times \Gamma \times M \to M$ (the image of (x, α, y) will be denoted by $x\alpha y, x, y \in M, \alpha \in \Gamma$) such that it satisfying the following axioms for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$

(i) $x\alpha(y + z) = x\alpha y + x\alpha z$,

(ii) $(x + y)\alpha z = x\alpha z + y\alpha z$,

(iii) $x(\alpha + \beta)y = x\alpha y + x\beta y$,

(iv) $x\alpha(y\beta z) = (x\alpha y)\beta z$.

Every semiring R is a Γ-semiring with $\Gamma = R$ and ternary operation $x\gamma y$ defined as the usual semiring multiplication.

Definition 2.3. A Γ-semiring M is said to be commutative Γ-semiring if $x\alpha y = y\alpha x$, for all $x, y \in M$ and $\alpha \in \Gamma$.

Definition 2.4. A non-empty subset A of a Γ-semiring M is called

(i) a Γ-subsemiring of M if $(A, +)$ is a subsemigroup of $(M, +)$ and $A\Gamma A \subseteq A$.

(ii) a quasi ideal of M if A is a Γ-subsemiring of M and $A\Gamma M \cap M\Gamma A \subseteq A$.

(iii) a bi-ideal of M if A is a Γ-subsemiring of M and $A\Gamma M\Gamma A \subseteq A$.

(iv) an interior ideal of M if A is a Γ-subsemiring of M and $M\Gamma A\Gamma M \subseteq A$.

(v) a left (right) ideal of M if A is a Γ-subsemiring of M and $M\Gamma A \subseteq A(A\Gamma M \subseteq A)$.

(vi) an ideal if A is a Γ-subsemiring of $M, A\Gamma M \subseteq A$ and $M\Gamma A \subseteq A$.

(vii) a bi-interior ideal of M if A is a Γ-subsemiring of M and $M\Gamma A\Gamma M \cap A\Gamma M\Gamma A \subseteq A$.

(viii) a left bi-quasi ideal (right bi-quasi ideal) of M if A is a Γ-subsemiring of M and $M\Gamma A \cap A\Gamma M\Gamma A \subseteq A$ $(A\Gamma M \cap A\Gamma M\Gamma A \subseteq A)$.

(ix) a bi-quasi ideal of M if A is a Γ-subsemiring of M and A is a left bi-quasi ideal and a right bi-quasi ideal of M.

(x) a left quasi-interior ideal (right quasi-interior ideal) of M if A is a Γ-subsemiring of M and $M\Gamma A\Gamma M\Gamma A \subseteq A$ ($A\Gamma M\Gamma A\Gamma M \subseteq A$).

(xi) a quasi-interior ideal of M if A is a Γ-subsemiring of M and $A$is a left quasi-interior ideal and a right quasi-interior ideal of M.

(xii) a bi-quasi-interior ideal of M if A is a Γ-subsemiring of M and $A\Gamma M\Gamma A\Gamma M\Gamma A \subseteq A$.

(xiii) a left (right) tri- ideal of M if A is a Γ-subsemiring of M and $A\Gamma M\Gamma A\Gamma A \subseteq A$ ($A\Gamma A\Gamma M\Gamma A \subseteq A$).

(xiv) a tri- ideal of M if A is a Γ-subsemiring of M and $A\Gamma M\Gamma A\Gamma A \subseteq A$ and $A\Gamma A\Gamma M\Gamma A \subseteq A$.

(xv) a left (right) weak-interior ideal of M if A is a Γ-subsemiring of M and $M\Gamma A\Gamma A \subseteq A(A\Gamma A\Gamma M \subseteq A)$.A weak-interior ideal of M if A is a Γ-subsemiring of M and A is a left weak-interior ideal and a right weak-interior ideal)of M.

(xvi) a k-ideal of M if A is an ideal of M and $x \in M$, $x + y \in A, y \in A$ then $x \in A$.

(xvii) a $m-k$-ideal of M if A is an ideal of M and $x \in A$, $xy \in A, 1 \neq y \in M$then $y \in A$.

Definition 2.5. A Γ-semiring M is said to have zero element if there exists an element $0 \in M$ such that $0 + x = x = x + 0$ and $0\alpha x = x\alpha 0 = 0$, for all $x \in M, \alpha \in \Gamma$.

Definition 2.6. An element $a \in \Gamma$-semiring M is said to be idempotent if there exists $\alpha \in \Gamma$ such that $a = a\alpha a$.

Definition 2.7. Let M be a Γ-semiring. An element $1 \in M$ is said to be unity if for each $x \in M$ there exists $\alpha \in \Gamma$ such that $x\alpha 1 = 1\alpha x = x$.

Definition 2.8. A Γ-semiring M is called an ordered Γ-semiring if it admits a compatible relation $\leq$, i.e., $\leq$ is a partial ordering on M satisfying the following conditions. If $a \leq b$ and $c \leq d$, then

(i) $a + c \leq b + d$

(ii) $a\alpha c \leq b\alpha d$

(iii) $c\alpha a \leq d\alpha b$, for all $a, b, c, d \in M, \alpha \in \Gamma$.

Example 2.9. Let $M = [0, 1], \Gamma = N$, $+$ and ternary operation be defined as $x + y = \max\{x, y\}, x\gamma y = \min\{x, \gamma, y\}$ for all $x, y \in M$, $\gamma \in \Gamma$. Then M is an ordered Γ-semiring with respect to usual ordering.

Definition 2.10. Let M be an ordered Γ-semiring and A be a non-empty subset of M. A is called a Γ-subsemiring of an ordered Γ-semiring M if A is a sub-semigroup of $(M, +)$ and $A\Gamma A \subseteq A$.

Definition 2.11. Let M be an ordered Γ-semiring. A non-empty subset A of M is called a left (right) ideal of an ordered Γ-semiring M if A is closed under addition and $M\Gamma A \subseteq A$ $(A\Gamma M \subseteq A)$ and for any $a \in M$, $b \in A$, $a \leq b$ then $a \in A$. A is called an ideal of M if it is both left ideal and right ideal.

Definition 2.12. An ordered Γ-semiring M is called regular if for each $a \in M$ there exist $x \in M, \alpha, \beta \in \Gamma$ such that $a \leq a\alpha x\beta a$.

3. Ideal elements in ordered Γ-semirings

In this section, we introduce the notion of (ideal, interior ideal, bi-ideal, quasi interior ideal and weak interior ideal) elements of ordered Γ-semirings. We study the properties of ideal elements and relations between them and characterize the ordered Γ-semirings using ideal elements.

Definition 3.1. An element a of an ordered Γ-semiring M is called a Γ-subsemiring element if $a\alpha a \leq a$, for all $\alpha \in \Gamma$.

Definition 3.2. An element a of an ordered Γ-semiring M is called a left (right) ideal element of M, if $x\alpha a \leq a(a\alpha x \leq a)$, for all $x \in M$, $\alpha \in \Gamma$.

Definition 3.3. An element of an ordered Γ-semiring M is called an ideal element of M, if it is both a left ideal element and a right ideal element of M.

Definition 3.4. Let M be an ordered Γ-semiring. An element a of M is said to be bi-ideal element of M if $a\alpha a \leq a$, $a\alpha x\beta a \leq a$, for all $x \in M$, $\alpha, \beta \in \Gamma$.

Definition 3.5. An element a of an ordered Γ-semiring M is called a quasi ideal element of M, if $a\alpha a \leq a$, there exist elements $x, y \in M$, and $\alpha, \beta \in \Gamma$ such that $x\alpha a = a\alpha y \leq a$.

Definition 3.6. Let M be an ordered Γ-semiring. An element a of M is said to be quasi interior ideal element of M if $a\alpha a \leq a$, $a\alpha x\beta a\gamma y \leq a$ and $x\alpha a\beta y\gamma a \leq a$, for all $x, y \in M$ $\alpha, \beta, \gamma \in \Gamma$.

Definition 3.7. A left ideal element b is said to be minimal if for every left ideal element a of an ordered Γ-semiring M, $a \leq b \Rightarrow a = b$.

Definition 3.8. Let M be an ordered Γ-semiring. An element a of M is said to be interior ideal element if $a\alpha a \leq a$, $x\alpha a\beta y \leq a$, for all $x, y \in M$, $\alpha, \beta \in \Gamma$.

Definition 3.9. An element a of an ordered Γ-semiring M is called a left (right) weak interior ideal of M if $a\alpha a \leq a$, $x\alpha a\beta a \leq a(a\beta a\alpha x \leq a)$, for all $x \in M$, $\alpha, \beta \in \Gamma$.

Definition 3.10. An element of an ordered Γ-semiring M is called a weak interior ideal element if $a\alpha a \leq a$, $a\alpha a\beta x \leq a$ and $x\alpha a\beta a \leq a$, for all $x \in M$, $\alpha, \beta \in \Gamma$.

Example 3.11. Let M and Γ be additive abelian semigroups of all 2×2 matrices, ternary operation is defined as $M \times \Gamma \times M \to M$ by $(x, \alpha, y) \to x\alpha y$ using usual matrix multiplication for all $x, y \in M$ and $\alpha \in \Gamma$. We define $A \leq B \Leftrightarrow a_{ij} \geq b_{ij}$, for all i, j. Then M is an ordered Γ-semiring.

(a) Let $M = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in N \cup \{0\} \right\}$
$\Gamma = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \mid \alpha, \beta \in N \right\}$ and $I = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid a \in N \right\}$.
Then $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}$ is an ideal element of an ordered Γ-semiring M.

(b) Let $M = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in N \cup \{0\} \right\}$
$\Gamma = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mid \alpha, \beta, \gamma, \delta \in N \right\}$ and $L = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \mid a, b \in N \right\}$.
Then $\left\{ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \right\}$ is a left ideal element of an ordered Γ-semiring M.

(c) Let $M = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in N \cup \{0\} \right\}$
$\Gamma = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mid \alpha, \beta, \gamma, \delta \in N \right\}$ and $R = \left\{ \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} \mid c, d \in N \right\}$.
Then $\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\}$ is a right ideal element of an ordered Γ-semiring M.

(d) Let $M = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in N \cup \{0\} \right\}$
$\Gamma = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mid \alpha, \beta, \gamma, \delta \in N \right\}$ and $B = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid a \in N \right\}$.
Then $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}$ is an interior ideal element of an ordered Γ-semiring M.

(e) Let $M = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in N \cup \{0\} \right\}$
$\Gamma = \left\{ \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \mid a, c \in N \right\}$ and the set $Q = \left\{ \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \mid a, c \in N \right\}$.

Then $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}$ is a quasi-interior ideal element of an ordered Γ-semiring M.

(f) Let $M = \left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in N \cup \{0\}\right\}$
$\Gamma = \left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in N\right\}$ and the set $Q = \left\{\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \mid a, c \in N\right\}$.
Then $\left\{\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\right\}$ is a quasi ideal element of an ordered Γ-semiring M.

(g) Let $M = \left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in N\right\}$
$\Gamma = \left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in N\right\}$ and the set $W = \left\{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid a \in N\right\}$.
Then $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right\}$ is a weak-interior ideal element of an ordered Γ-semiring M.

Definition 3.12. Let M be an ordered Γ-semiring. A $x \in M$ is said to be a maximal element if $y \in M$ and $x \leq y$ then $x = y$.

Definition 3.13. Let M be an ordered Γ-semiring. M is said to be a simple ordered Γ-semiring if M has no proper ideals.

Theorem 3.14. *If a is an ideal element of an ordered Γ-semiring M, then a is an interior ideal element of M.*

Proof. Suppose a is an ideal element of the Γ-semiring M. Then $a\alpha x \leq a$ and $y\beta a \leq a$, for all $x, y \in M$ and $\alpha, \beta \in \Gamma$. That implies $x\alpha a\beta y \leq x\alpha a \leq a$ and hence a is an interior ideal element of M. ∎

Theorem 3.15. *If a is a left ideal element of an ordered Γ-semiring M, then a is a quasi ideal element of M.*

Proof. Suppose that a is a left ideal element of M. Then $x\alpha a \leq a$, for all $\alpha \in \Gamma$, $x \in M$. Let $y, z \in M$, and $\beta, \gamma \in \Gamma$ such that $z\gamma a = a\beta y$. Then $z\gamma a \leq a$. Therefore $z\gamma a = a\beta y \leq a$. ∎

Theorem 3.16. *If a is a quasi ideal element of an ordered Γ-semiring M, then a is a bi-ideal element of M.*

Proof. Suppose that a is a quasi ideal element of M. Then there exist elements $x, y \in M$, and $\alpha, \beta \in \Gamma$ such that $x\alpha a = a\beta y \leq a$. Then $a\beta y \leq a$, and $a\beta y\alpha a \leq a\alpha a \leq a$. That implies $a\beta y\alpha a \leq a$. Hence a is a bi-ideal element of M. ∎

Theorem 3.17. *If a is an ideal element of an ordered Γ-semiring M, then a is a bi-ideal element of M.*

Proof. Suppose that a is an ideal element of M. Then $a\alpha x \leq a$, and $x\alpha a \leq a$, for all $\alpha \in \Gamma$, $x \in M$. That implies $a\alpha x\alpha a \leq x\alpha a \leq a$. Hence a is a bi-ideal element of M. ∎

Theorem 3.18. *Let M be an ordered Γ-semiring M. If a is a maximal element of M then a is an ideal element of M.*

Proof. Suppose a is a maximal element of an ordered Γ-semiring M. Then $a\alpha x \leq a$ and $x\alpha a \leq a$, for all $\alpha \in \Gamma$, $x \in M$. Therefore a is an ideal element of M. ∎

Theorem 3.19. *If a is a minimal element of an ordered Γ-semiring M, then a is a not an ideal element of M.*

Proof. Suppose a is a minimal element of M. Then $a \leq x$, for all $x \in M$. Therefore $a \leq a\alpha x$, for all $x \in M$ and $\alpha \in \Gamma$. ∎

Theorem 3.20. *If a is an interior ideal element of an idempotent ordered Γ-semiring M, then a is an ideal element of M.*

Proof. Suppose a is an interior ideal element and a is an α-idempotent of the ordered Γ-semiring M. Then $x\beta a\alpha y \leq a$, for all $\alpha, \beta \in \Gamma$ and $x, y \in M$. That implies $x\beta a\alpha a \leq a$ and hence $x\beta a \leq a$, for all $x \in M$. Similarly we can prove that $a\beta x \leq a$. Hence a is an ideal element of M. ∎

Theorem 3.21. *Every interior ideal element of an ordered Γ-semiring M is a quasi interior ideal element.*

Proof. Suppose a is an interior ideal element of M. Then $x\alpha a\beta y \leq a$, for $x, y \in M$, $\alpha, \beta \in \Gamma$. That implies $a\alpha x\beta a\beta y \leq a$ and $x\alpha a\beta y\gamma a \leq a$. Hence every interior ideal element is a quasi interior ideal element. ∎

Theorem 3.22. *Every left ideal element of an ordered Γ-semiring M is a left quasi interior element of M.*

Proof. Suppose x is a left ideal element of M. Then $x\alpha a \leq a$, for $x \in M$, $\alpha \in \Gamma$. That implies $x\alpha y\beta y\gamma a \leq a$. Hence a is left quasi ideal element of M. ∎

Theorem 3.23. *If a is a quasi interior ideal element of a regular ordered Γ-semiring M then a is an ideal element of M.*

Proof. Suppose a is a quasi interior ideal element of M. Then $a\alpha x\beta a\gamma y \leq a$, $x\alpha a\beta y\gamma a \leq a$, for all $x, y \in M$, $\alpha, \beta\gamma \in \Gamma$ and $a \leq a\alpha b\beta a$, for some $b \in M$, $\alpha, \beta \in \Gamma$. Suppose $x \in M$, $\gamma \in \Gamma$. Then $a\gamma x \leq a\alpha b\beta a\gamma x \leq a$ and $x\gamma a \leq x\gamma a\alpha b\beta a \leq a$. Hence a is an ideal element of M. ∎

Theorem 3.24. *Every interior ideal element of an ordered* Γ*-semiring* M *is a quasi interior ideal element.*

Proof. Suppose a is an interior ideal element of M. Then $x\alpha a\beta y \leq a$, for all $x, y \in M$, $\alpha, \beta \in \Gamma$. That implies $a\alpha x\beta a\gamma y \leq a$ and $x\alpha a\beta a\gamma a \leq a$. Hence every interior ideal element is a quasi interior ideal element. ■

Theorem 3.25. *Every left ideal element of an ordered* Γ*-semiring* M *is a left quasi interior element of* M*.*

Proof. Suppose x is a left ideal element of M. Then $x\alpha a \leq a$, for all $x \in M$, $\alpha \in \Gamma$. That implies $x\alpha y\beta y\gamma a \leq a$. Hence a is left quasi ideal element of M. ■

Theorem 3.26. *If* a *is a quasi interior ideal element of a regular ordered* Γ*-semiring* M *then* a *is an ideal element of* M*.*

Proof. Suppose a is a quasi interior ideal element of M. Then $a\alpha x\beta a\gamma y \leq a$, $x\alpha a\beta y\gamma a \leq a$, for all $x, y \in M, \alpha, \beta, \gamma \in \Gamma$. and $a \leq a\alpha b\beta a$, for some $b \in M, \alpha, \beta \in \Gamma$. Suppose $x \in M$, $\gamma \in \Gamma$. Then $a\gamma x \leq a\alpha b\beta a\gamma x \leq a$ and $x\gamma a \leq x\gamma a\alpha b\beta a \leq a$. Hence a is an ideal element of M. ■

Theorem 3.27. *If* a *is an ideal element of an ordered* Γ*-semiring*M *then* a *is a weak interior ideal element of* M*.*

Proof. Suppose a is an ideal element of M. Then $a\alpha x \leq a$ and $x\alpha a \leq a$, for all $x \in M$, $\alpha \in \Gamma$. That implies $x\alpha a\beta a \leq a\beta a \leq a$ and $a\beta a\alpha x \leq a$, for all $\alpha, \beta \in \Gamma$. Therefore $a\beta a\alpha x \leq a\beta a \leq a$. Hence ideal element of M is a weak interior ideal element of M. ■

Corollary 3.28. *Let* a *be an interior ideal element of an ordered* Γ*-semiring* M*. Then* a *is a weak interior ideal element of* M*.*

Theorem 3.29. *Let* M *be an ordered* Γ*-semiring. If* a *is a weak interior and idempotent element of* M *then* a *is an interior ideal element of* M*.*

Proof. Let a be an interior ideal and idempotent element of M. Then there exists $\gamma \in \Gamma$ such that $a\gamma a = a$. Suppose $x, y \in M$ and α, $\beta \in \Gamma$. Now $x\alpha a\beta y = x\alpha a\gamma a\gamma a\gamma a\beta y \leq a\gamma a = a$. ■

Theorem 3.30. *If* a *and* b *are minimal left ideal elements of an ordered* Γ*-semiring* M*, then* $a\beta b$ *is a minimal left ideal element of ordered* Γ*-semiring* M*.*

Proof. Suppose a and b are minimal left ideals of the ordered Γ-semiring M. Then $x\alpha a \leq a$ and $x\alpha b \leq b$, $\alpha \in \Gamma$, $x \in M$. That implies $x\alpha a\beta b \leq a\beta b$, for all $x \in M$. Therefore $a\beta b$ is the left ideal element of M. Suppose c is any left ideal element of M and $c \leq a\beta b$. Then $c \leq a\beta b \leq b$. Therefore $a\beta b = c$. Hence $a\beta b$ is the minimal left ideal of an ordered Γ-semiring M. ■

Theorem 3.31. *Let M be an ordered regular Γ-semiring. If a is a bi-ideal element of M and a commutes with every element of M then a is an ideal element.*

Proof. Let M be an ordered Γ-semiring, a be a bi-ideal element of M and a commutes with every element of M. Then $a \leq a\alpha x\beta a$, for some $x \in M$, $\alpha, \beta \in \Gamma$. That implies $a \leq a\alpha x\beta a \leq a$. Therefore $a\alpha x\beta a = a$. Suppose $y \in M$, $\gamma \in \Gamma$. Then $a\gamma y = a\alpha x\beta a\gamma y = a\alpha x\beta y\gamma a \leq a$, for all $y \in M$, $\gamma \in \Gamma$. Hence a is an ideal element of M. ∎

Theorem 3.32. *Let M be a simple ordered Γ-semiring. Then every element of M is an ideal element of M.*

Proof. Let $I = \{a \mid a \in M,\ a\alpha x \leq a$ and $x\alpha a \leq a$, for all $x \in M$, $\alpha \in \Gamma\}$ and $a, b \in I$. Then $(a+b)\alpha x = a\alpha x + b\alpha x \leq a+b$ and $x\alpha(a+b) = x\alpha a + x\alpha b \leq a+b$. Therefore $a+b \in I$. For $\gamma \in \Gamma$, $a\gamma b\alpha x = a\gamma(b\alpha x) \leq a\gamma b$ and $x\alpha(a\gamma b) = (x\alpha a)\gamma b \leq a\gamma b$. Therefore $a\gamma b \in I$. Hence I is a Γ-subsemiring of M. Suppose $a \in M$, $b \in I$, $\alpha, \gamma \in \Gamma$. Then $(a\alpha b)\gamma x = a\alpha(b\gamma x) \leq a\alpha b$ and $x\gamma(a\alpha b) = (x\gamma a)\alpha b \leq b\alpha a = a\alpha b$. Suppose that $x \in M$, $a \in I$, $\alpha \in \Gamma$ and $x \leq a$. Then $y\alpha x \leq y\alpha a \leq a$ and $x\alpha y \leq a\alpha y \leq a$. Hence $x \in I$. Therefore I is an ideal of the ordered Γ-semiring. Hence $I = M$. ∎

Corollary 3.33. *Let M be a left (right) simple ordered Γ-semiring. Then every element of M is a left (right) ideal element of M.*

Theorem 3.34. *If a is an ideal element of an ordered Γ-semiring M, then a is an interior-ideal element.*

Proof. Suppose a is an ideal element of M. Then $a\alpha x \leq a$ and $x\beta a \leq a$, for all $x \in M$ and $\beta, \alpha \in \Gamma$. That implies $x\alpha a\beta y \leq x\alpha a \leq a$ and hence a is interior ideal element of M. ∎

Theorem 3.35. *If a is an interior ideal element and idempotent element of an ordered Γ-semiring M, then a is an ideal element of M.*

Proof. Suppose a is an interior ideal element and a is an α-idempotent. Then $x\beta a\beta y \leq a$, for all $\beta \in \Gamma$ and $x, y \in M$. That implies $x\beta a\alpha a \leq a$ and hence $x\beta a \leq a$, for all $x \in M$. Similarly we can prove that $a\beta x \leq a$. Hence a is an ideal element of M. ∎

Theorem 3.36. *Let M be a regular ordered Γ-semiring. Then a is an interior ideal element of M if and only if a is an ideal element of M.*

Proof. Assume that a be an interior ideal element of M. Then $x\alpha a\beta y \leq a$, for all $x, y \in M$, $\alpha, \beta \in \Gamma$. We have that $a \leq a\alpha x\beta a$, for some $x \in M$, $\alpha, \beta \in \Gamma$. Then $a\alpha x \leq a\alpha x\beta a\alpha x \leq a$, by the definition of interior ideal element of M. Similarly we can prove that $x\alpha a \leq a$. Hence a is an ideal element of M. Conversely, assume that a is an ideal element of M. Suppose $y, z \in M$ and $\alpha, \beta \in \Gamma$. Then $y\alpha a\beta z \leq a\beta z \leq a$. Hence a is an interior ideal element of the ordered Γ-semiring M. ∎

4. Conclusion

In this paper, we studied the structures of ordered Γ-semirings M with the generalization of ideal elements. The ideal elements play an important and necessary role in studying the structure of ordered semigroups. We introduced the notion of (ideal, interior-ideal, quasi-ideal, bi-ideal, bi- interior ideal, quasi-interior ideal and weak interior ideal) elements of ordered Γ-semirings. We studied the properties of ideal elements and relations between them. We characterized the ordered Γ-semirings, regular ordered Γ-semirings and simple ordered Γ-semirings using ideal elements.

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