Discussiones Mathematicae
General Algebra and Applications 43 (2023) 363–373
https://doi.org/10.7151/dmgaa.1426

DISTRIBUTIVE CATEGORIES OF COALGEBRAS

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Abstract

We prove that the category of coalgebras for an endo-functor F is distributive or extensive, provided that F preserves pullbacks along monomorphisms and the underlying category is distributive or extensive.

Keywords: coalgebra, coproduct, distributive category, pullback.

2020 Mathematics Subject Classification: 68Q85, 18A32, 18B50, 18A30.

1. Introduction

A category $\mathcal{C}$ with finite products and coproducts is said to be distributive, if for all objects A, B and C in $\mathcal{C}$, the canonical morphism

$$\delta : A \times B + A \times C \longrightarrow A \times (B + C)$$

is an isomorphism. The category Set of sets and mappings, Top of topological spaces and continuous mappings and Hty of topological spaces and homotopy classes of mappings, are distributive. The ordered set $\mathcal{P}(X)$ of subsets of a nonempty set X, viewed as a category is distributive. More generally, every distributive lattice viewed as a preorder is a distributive category.

There are two notions of distributive category in the literature: distributivity and extensiveness (see [3] and [4]). The difference consists in how many types of limits the category in question is supposed to have. A category $\mathcal{C}$ with finite coproducts is called extensive, if for each pair A, B of objects in $\mathcal{C}$, the canonical functor

$$\mathcal{C}/_A \times \mathcal{C}/_B \longrightarrow \mathcal{C}/_{A+B}$$

is an equivalence. This implies the existence of certain pullbacks (see Proposition 4.2). Any extensive category with products is distributive. However, there are extensive categories that are not distributive. For instance, the free category with coproducts on the category comprising only two parallel arrows is extensive but not distributive (see 4.1 [3]).

An extensive category with a terminal object is not necessarily distributive. As illustration, the category of manifolds of dimension less than five is extensive with singletons as terminal objects. But this category is not distributive because it does not have products: the product of two manifolds of dimension 4 being a manifold of dimension 8.

The category of coalgebras for a *Set*-endofunctor F is distributive if and only if, it has finite products and F preserves preimages; that is pullbacks along injective mappings (see Theorem 1 [7]). An observation is that the category *Set* is distributive and coproducts in *Set* are universal. Furthermore, the preservation of preimages by F is equivalent to the property that, in the category of F-coalgebras, each morphism into a coproduct induces a split of its domain.

Given an endofunctor F on a category $\mathcal{C}$ which preserves pullbacks along monomorphisms. We prove that the category of F-coalgebras is extensive whenever $\mathcal{C}$ is so. If more, F is a covarietor, the category $\mathcal{C}_F$ of F-coalgebras is distributive provided that $\mathcal{C}$ is distributive with universal coproducts.

2. Pullbacks and preserving functors

We review pullbacks and their presevation. The reader is referred to [10] for more on this subject.

Let $f : A \to C$ and $g : B \to C$ be two morphisms with the same codomain. A *pullback* (or *fiber product*) for the pair (f, g), also called a pullback of g along f, is a commutative diagram

$$\begin{tikzcd} P \arrow[r, "p_1"] \arrow[d, "p_2"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

with the following property: if $u : D \to A$ and $v : D \to B$ are morphisms with $f \circ u = g \circ v$, then there is exactly one morphism $w : D \to P$ with $u = p_1 \circ w$ and $v = p_2 \circ w$. Particularly, the *intersection* of f and g is the pullback of g along f when f and g are monomorphisms. A *weak pullback* of f and g is a cone (P, p_1, p_2) so that for every other cone (Q, q_1, q_2) with $f \circ q_1 = g \circ q_2$, there is at least one morphism $w : Q \to P$ such that $p_1 \circ w = q_1$ and $p_2 \circ w = q_2$.

Let F be a functor and a pair (f, g) of morphisms. We say that F preserves pullbacks, if it transforms every pullback into a pullback; i.e., for every pullback

(P, p_1, p_2) of f and g we get (FP, Fp_1, Fp_2) is a pullback of Ff and Fg. However, if at least one of f and g is a monomorphism, we say that F preserves pullbacks along monomorphisms. The functor F is said to preserve weak pullbacks, if it transforms every weak pullback of f and g into a weak pullback of Ff and Fg. Every functor which preserves weak pullbacks also preserves pullbacks along monomorphisms (see [9], p. 6).

Every functor F which preserves pullbacks along monomorphisms, also preserves monomorphisms. Indeed, the following diagram is a pullback given that m is a monomorphism.

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "1_A"'] & A \arrow[d, "m"] \\ A \arrow[r, "m"'] & B \end{tikzcd}$$

Given two morphisms $u, v : D \to A$ such that $F(m) \circ u = F(m) \circ v$. Since the functor F preserves pullbacks along monomorphisms, there is exatly one arrow $w : D \to FA$ such that $F(1_A) \circ w = u$ and $F(1_A) \circ w = v$. By the fact that $F(1_A) = 1_{FA}$, the equality $u = v$ holds. Thus $F(m)$ is a monomorphism.

A functor which transforms every regular monomorphism into a regular monomorphism is said to preserve regular monomorphisms. But every equalizer is an intersection in a finitely complete category (see 7.8.7 [12]). Thus a functor which preserves pullbacks along monomophisms also preserves regular monomorphisms provided that equalizers are intersections.

3. DISTRIBUTIVE CATEGORIES

We will consider in this part the distributivity of the category of coalgebras. First of all, we recall some basic facts about distribitive categories.

Definition 3.1. A category $\mathcal{C}$ with finite products and coproducts is said to be *distributive*, if for all A, B and C in $\mathcal{C}$, the canonical morphism

$$\delta : A \times B + A \times C \longrightarrow A \times (B + C)$$

is an isomorphism.

This definition implies a condition concerning the initial object; that is, the product projection $p : A \times 0 \to 0$ is inversible in any distributive category (see Proposition 3.2 [3]). Accordingly, coproduct injections are monomorphisms in any distributive category (see Proposition 3.3 [3]).

Definition 3.2. Consider an endofunctor F on a category $\mathcal{C}$. An F-*coalgebra* is given by an object A in $\mathcal{C}$ together with a morphism $a : A \to FA$ in $\mathcal{C}$.

A *homomorphism* $f : (A, a) \to (B, b)$ of F-coalgebras is a commutative diagram:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\ FA \arrow[r, "F(f)"'] & FB \end{tikzcd}$$

We denote by $\mathcal{C}_F$ the category of F-coalgebras and their homomorphisms.

In a category with finite products, a *binary relation* from A to B is a regular subobject of $A \times B$. This is represented by a regular monomorphism $m : R \rightarrowtail A \times B$ or equivalently, by a pair of arrows

$$\begin{tikzcd} & A \\ R \arrow[ur, "r_1"] \arrow[dr, "r_2"'] & \\ & B \end{tikzcd}$$

with the property that the induced arrow $\langle r_1, r_2 \rangle : R \to A \times B$ is a regular monomorphism.

Definition 3.3. By a *bisimulation* between F-coalgebras (A, a) and (B, b) is meant a binary relation $(A \xleftarrow{r_1} R \xrightarrow{r_2} B)$ such that there is a morphism $r : R \to FR$ making both r_1 and r_2 homomorphisms of F-coalgebras.

An endofunctor $F : \mathcal{C} \to \mathcal{C}$ is called a *covarietor* if the forgetful functor $U : \mathcal{C}_F \to \mathcal{C}$ which maps an F-coalgebra (A, a) to its carrier A has a right adjoint.

Lemma 3.4. *Let $F : \mathcal{C} \to \mathcal{C}$ be a covarietor and let H be the right adjoint of the forgetful functor $U : \mathcal{C}_F \to \mathcal{C}$. If (A, a) is an F-coalgebra, then the cofree coalgebra structure on HA is given by*

$$\rho : HA \xrightarrow{\varepsilon_A} A \xrightarrow{a} FA \xrightarrow{F(\eta_{(A,a)})} F(HA)$$

where $\eta : Id_{\mathcal{C}_F} \Rightarrow H \circ U$ and $\varepsilon : U \circ H \Rightarrow Id_{\mathcal{C}}$ are respectively the unit and the counit of the adjunction $U \dashv H$.

Proof. Let (A, a) be an F-coalgebra. Then $\eta_{(A,a)}$ has a left inverse given by the triangle equality $\varepsilon_A \circ \eta_{(A,a)} = 1_A$. Also, $\eta_{(A,a)}$ is a homomorphism as it corresponds by adjunction to the identity morphism $1_A : A \to A$. More precisely, $\eta_{(A,a)}$ is a homorphism for the cofree coalgebra structure on HA. Besides $\eta_{(A,a)}$ is a homomorphism for the coalgebra structure $\rho : HA \xrightarrow{\varepsilon_A} A \xrightarrow{a} FA \xrightarrow{F(\eta_{(A,a)})}$

$F(HA)$ on HA. This follows from the commutative diagram below:

$$\begin{tikzcd}
A \arrow[r, "\eta_{(A,a)}"] \arrow[dr, equal] \arrow[ddd, "a"'] & HA \arrow[d, "\varepsilon_A"'] \arrow[ddd, bend left, "\rho"] \\
& A \arrow[d, "a"'] \\
& FA \arrow[d, "F(\eta_{(A,a)})"'] \\
FA \arrow[r, "F(\eta_{(A,a)})"'] & F(HA)
\end{tikzcd}$$

Likewise, ε_A is a homomorphism for the coalgebra structure ρ on HA as the following diagram commutes.

$$\begin{tikzcd}
HA \arrow[r, "\varepsilon_A"] \arrow[d, "\varepsilon_A"] \arrow[ddd, bend right, "\rho"'] & A \arrow[ddd, "a"] \\
A \arrow[d, "a"] & \\
FA \arrow[d, "F(\eta_{(A,a)})"] \arrow[ddr, equal] & \\
F(HA) \arrow[r, "F(\varepsilon_A)"'] & FA
\end{tikzcd}$$

Then $\eta_{(A,a)} \circ \varepsilon_A$ is a homomorphism as it is a composite of homomorphisms. Denote by $\psi : HA \to F(HA)$ the cofree coalgebra structure on HA. Since $\eta_{(A,a)}$ is a homomorphism for both coalgebra structures ρ and ψ on HA, the equalities $\rho \circ \eta_{(A,a)} = F(\eta_{(A,a)}) \circ a = \psi \circ \eta_{(A,a)}$ hold. It follows that $(\rho \circ \eta_{(A,a)}) \circ \varepsilon_A = (\psi \circ \eta_{(A,a)}) \circ \varepsilon_A$; that is, $\rho \circ (\eta_{(A,a)} \circ \varepsilon_A) = \psi \circ (\eta_{(A,a)} \circ \varepsilon_A)$. But $\eta_{(A,a)} \circ \varepsilon_A = 1_{HA}$; this is because $\eta_{(A,a)} \circ \varepsilon_A$ is a homomorphism which corresponds by adjunction to ε_A. The equality $\rho = \psi$ follows. This means that ρ is nothing else but the cofree coalgebra structure on HA. ∎

Now we are interested in the existence of limits in coalgebras.

Lemma 3.5. *Let $F : \mathcal{C} \to \mathcal{C}$ be an endofunctor on a category $\mathcal{C}$. Then the category $\mathcal{C}_F$ is finitely complete, provided that*

(i) *$\mathcal{C}$ is finitely complete with epi-(regular mono) factorizations.*

(ii) *F is a covarietor and preserves pullbacks along monomorphisms.*

Proof. Suppose we are given F-coalgebras (A, a) and (B, b). The product of A and B in $\mathcal{C}$ exists as $\mathcal{C}$ is finitely complete. We want to prove that the product of (A, a) and (B, b) exists in $\mathcal{C}_F$. Let H denote the right adjoint of the forgetful functor $U : \mathcal{C}_F \to \mathcal{C}$. There is an arrow $\varepsilon_{A\times B} : H(A \times B) \to A \times B$, where $\varepsilon : U \circ H \Rightarrow Id_{\mathcal{C}}$ is the counit of the adjunction $U \dashv H$. Let $p_1 : A \times B \to A$ be the first projection of the product of A and B. The following digram commutes as ε is a natural transformation.

$$\begin{tikzcd}
H(A \times B) \arrow[r, "\varepsilon_{A\times B}"] \arrow[d, "H(p_1)"'] & A \times B \arrow[d, "p_1"] \\
HA \arrow[r, "\varepsilon_A"'] & A
\end{tikzcd}$$

In addition, ε_A is a homomorphism due to Lemma 3.4. Besides the functor H transforms every $\mathcal{C}$-morphism into a homomorphism. So $H(p_1)$ is a homomorphism for the cofree coalgebra structure on HA. It follows that the composite $\varepsilon_A \circ H(p_1)$ is a homomorphism. Consequently, $p_1 \circ \varepsilon_{A\times B}$ is a homomorphism due to $p_1 \circ \varepsilon_{A\times B} = \varepsilon_A \circ H(p_1)$. The same argument may be used to prove that $p_2 \circ \varepsilon_{A\times B}$ is a homomorphism. Since F preserves pullbacks along monomorphisms, it preserves regular monomorphisms as $\mathcal{C}$ is finitely complete. Factorize $\varepsilon_{A\times B}$ into an epimorphism e followed by a regular monomorphism $m : Q \rightarrowtail A \times B$.

$$\begin{tikzcd}
H(A \times B) \arrow[rr, "\varepsilon_{A\times B}"] \arrow[dr, "e"'] & & A \times B \\
& Q \arrow[ur, "m"'] &
\end{tikzcd}$$

However, every strong monomorphism is regular by the epi-(regular mono) factorizations. As in addition every regular monomorphism is strong, epi-(regular mono) factorizations and epi-(strong mono) factorizations coincide. Thus Q is a bisimulation between (A, a) and (B, b) because the forgetful functor $U : \mathcal{C}_F \to \mathcal{C}$ creates epi-(regular mono) factorizations (see Corollary 4.13 and the proof of Proposition 5.5 in [1]). There is therefore a coalgebra structure $\sigma : Q \to FQ$ turning the projections $q_1 = p_1 \circ m$ and $q_2 = p_2 \circ m$ into homomorphisms.

We are going to verify the universal property of the product in $\mathcal{C}_F$. Consider a span $((S, s) \xrightarrow{t_i} (A_i, a_i))_{i=1,2}$ in $\mathcal{C}_F$. By the universal property of products in $\mathcal{C}$, there is a unique arrow $h : S = U(S, s) \to A_1 \times A_2$ such that $p_1 \circ h = t_1$ and $p_2 \circ h = t_2$. The arrow h corresponds by adjunction to a unique arrow $\bar{h} : (S, s) \to H(A_1 \times A_2)$; that is, $\bar{h}$ is a homomorphism. Similarly, the $\mathcal{C}$-morphism $e : H(A_1 \times A_2) \to U(Q, \sigma) = Q$ corresponds to the homomorphism $\eta_{(Q,\sigma)} \circ e : H(A_1 \times A_2) \to (Q, \sigma) \to H(U(Q, \sigma))$, where $\eta : Id_{\mathcal{C}_F} \Rightarrow H \circ U$ is the unit of the adjunction $U \dashv H$. But $\eta_{(Q,\sigma)}$ is a homomorphism and a section due to the equality $\varepsilon_Q \circ \eta_{(Q,\sigma)} = 1_Q$. Hence e is a homomorphism as

$F(\eta_{(Q,\sigma)})$ is a section (see Lemma 2.4 [11]). Thereafter the composite arrow $e \circ \bar{h} : S = U(S, s) \to H(A_1 \times A_2) \to Q$ is a homomorphism. Besides the equality $\varepsilon_{A_1 \times A_2} \circ \bar{h} = h$ holds as the composite arrow $\varepsilon_{A_1 \times A_2} \circ \bar{h}$ corresponds by adjunction to $\bar{h}$. One deduces that the following diagram commutes.

$$\begin{tikzcd}
S \arrow[rr, equal] \arrow[d, "\bar{h}"'] & & S \arrow[ddl, "h"'] \arrow[dd, "t_i"] \\
H(A_1 \times A_2) \arrow[d, "e"'] \arrow[dr, "\varepsilon_{A_1 \times A_2}"] & & \\
Q \arrow[r, "m"'] \arrow[rr, bend right, "q_i"'] & A_1 \times A_2 \arrow[r, "p_i"'] & A_i
\end{tikzcd}$$

Hence $q_i \circ (e \circ \bar{h}) = t_i$; $i = 1, 2$. Also, $e \circ \bar{h}$ is the only arrow with this property. This proves that $((Q, \sigma) \xrightarrow{q_i} (A_i, a_i))_{i=1,2}$ is the product of (A_1, a_1) and (A_2, a_2). As a result, $\mathcal{C}_F$ has a product for any two objects. Likewise $\mathcal{C}_F$ has a terminal object which is $H1$ (1 being the terminal object of $\mathcal{C}$). So $\mathcal{C}_F$ has finite products. Furthermore, $\mathcal{C}_F$ has finite intersections as the forgetful functor $U : \mathcal{C}_F \to \mathcal{C}$ creates finite intersections. This is because F preserves pullbacks along monomorphisms. Consequently, $\mathcal{C}_F$ is finitely complete (see Theorem 12.4 [2] and Proposition 7.8.8 [12]). ■

Proposition 3.6. *Let $F : \mathcal{C} \to \mathcal{C}$ be an endofunctor on a distributive category $\mathcal{C}$. Then the category $\mathcal{C}_F$ is distributive provided that*

(i) *$\mathcal{C}$ is finitely complete with epi-(regular mono) factorizations and has universal (binary) coproducts;*

(ii) *F is a covarietor and preserves pullbacks along monomorphisms.*

Proof. Given F-coalgebras (A, a), (B_1, b_1) and (B_2, b_2). By Lemma 3.5, the product of (A, a) and (B_k, b_k) exists; $k = 1, 2$. Denote by $p_k^1 : (A, a) \times (B_k, b_k) \to (A, a)$ and $p_k^2 : (A, a) \times (B_k, b_k) \to (B_k, b_k)$ the projections of the product of (A, a) and (B_k, b_k). By the universal property of coproducts, there is a unique arrow $p_1 : \sum_{k=1,2}(A, a) \times (B_k, b_k) \longrightarrow (A, a)$ such that $p_1 \circ s_1 = p_1^1$ and $p_1 \circ s_2 = p_1^2$; s_1 and s_2 being the injections of the coproduct of $(A, a) \times (B_1, b_1)$ and $(A, a) \times (B_2, b_2)$. Similarly, there is a unique arrow $p_2 : \sum_{k=1,2}(A, a) \times (B_k, b_k) \longrightarrow (\sum_{k=1,2} B_k, b)$ such that $p_2 \circ s_1 = e_1 \circ p_2^1$ and $p_2 \circ s_2 = e_2 \circ p_2^2$, where $(\sum_{k=1,2} B_k, b)$ together with arrows e_1 and e_2 is the coproduct of (B_1, b_1) and (B_2, b_2).

Let us prove that $(A, a) \times (B_1, b_1) + (A, a) \times (B_2, b_2)$ is isomorphic to $(A, a) \times ((B_1, b_1) + (B_2, b_2))$. To this end, consider a cone $((Q, q), \varphi : (Q, q) \to (A, a), \psi :$

$(Q,q) \to (\sum_{k=1,2} B_k, b))$. Since coproducts in $\mathcal{C}$ are universal, there is Q_1 and Q_2 in $\mathcal{C}$ such that $Q = Q_1 + Q_2$ if pulling back along ψ.

$$\begin{tikzcd}
Q_1 \arrow[r, "\pi_1^1"] \arrow[d, "\pi_2^1"'] & Q \arrow[d, "\psi"] & Q_2 \arrow[l, "\pi_1^2"'] \arrow[d, "\pi_2^2"] \\
B_1 \arrow[r, "e_1"'] & B_1 + B_2 & B_2 \arrow[l, "e_2"]
\end{tikzcd}$$

Besides, coproduct injections are monomorphisms; this follows from the fact that the category $\mathcal{C}$ is distributive. Hence, Q_1 and Q_2 are respectively equipped with a coalgebra structures q_1 and q_2 because the endofunctor F preserves pullbacks along monomorphisms. Let $\mu : (Q_k, q_k) \longrightarrow (A,a) \times (B_k, b_k)$ be the unique factorization such that $p_1^k \circ \mu = \varphi \circ \pi_1^k$ and $p_2^k \circ \mu = \pi_2^k$; $k = 1,2$. There is a unique arrow $\theta : (Q,q) \to \sum_{k=1,2}(A,a) \times (B_k, b_k)$ such that $\theta \circ \pi_1^k = s_k \circ \mu$; $k = 1,2$.

$$\begin{tikzcd}
(A,a)\times(B_k,b_k) \arrow[rr, "p_1^k"] \arrow[drrr, bend left, "s_k"] \arrow[ddr, "p_2^k"'] & & (A,a) & \\
& (Q_k,q_k) \arrow[ul, dashed, "\mu"] \arrow[r, "\pi_1^k"] \arrow[d, "\pi_2^k"] & (Q,q) \arrow[u, "\varphi"'] \arrow[r, "\theta"'] \arrow[d, "\psi"'] & \sum_{k=1,2}(A,a)\times(B_k,b_k) \arrow[ul, "p_1"'] \arrow[dl, "p_2"] \\
& (B_k,b_k) \arrow[r, "e_k"'] & (\sum_{k=1,2} B_k, b) &
\end{tikzcd}$$

Therefore, we have $p_1 \circ \theta \circ \pi_1^k = p_1 \circ s_k \circ \mu = p_1^k \circ \mu = \varphi \circ \pi_1^k$; $k = 1,2$. Whence $p_1 \circ \theta = \varphi$ since the pair (π_1^1, π_1^2) is an epi-sink. The same argument allows to check that $p_2 \circ \theta = \psi$. Consequently, $(\sum_{k=1,2}(A,a) \times (B_k, b_k), p_1, p_2)$ is a cone on (A,a) and $(\sum_{k=1,2} B_k, b)$ with the universal property of products. This implies that $(A,a) \times (B_1, b_1) + (A,a) \times (B_2, b_2)$ is isomorphic to $(A,a) \times ((B_1, b_1) + (B_2, b_2))$. Thus the category $\mathcal{C}_F$ is distributive. ∎

Notice that the distributivity lifts from $\mathcal{C}$ to $\mathcal{C}_F$ if F preserves finite products. This follows from the familiar facts that the forgetful functor $U : \mathcal{C}_F \to \mathcal{C}$ reflects isomorphisms, creates coproducts (always) and creates (binary) products if F preserves those.

Example. Consider the functor $()^3_2 : \mathit{Set} \to \mathit{Set}$ defined on objects as follows: for a set,

$$A^3_2 = \{(a_1, a_2, a_3) \in A^3 / \mid \{a_1, a_2, a_3\} \mid \leq 2\}$$

and for each mapping $f : A \longrightarrow B$,

$$f_2^3(a_1, a_2, a_3) = (f(a_1), f(a_2), f(a_3))$$

It is a covarietor as a *Set*-endofunctor preserving mono sources (see Theorem 8.10 [8]). Also, it preserves pullbacks along monomorphisms (see Example 4.12 [6]). However, it does not preserve finite products (see Example 2.4 [6]). Consequently, the category $Set_{()_2^3}$ is distributive by virtue of Proposition 3.6.

Example. Given $M : Set \to Set$ the functor which assigns to each set A the free commutative monoid generated by A; that is, $M(A)$ is the set of finite multisets of elements of A. The M-coalgebras can be regarded as transistion systems in which we have several different ways of making the transition from one given state to another. Also, the functor M does not preserve finite products. Furthermore, the functor M is a covarietor which preserves pullbacks along monomorphisms; this is because it preserves weak pullbacks and generates a cofree comonad (see Remark 1.5 and Example 2.6 in [9]). Hence the category Set_M is distributive due to Proposition 3.6.

4. EXTENSIVE CATEGORIES

An extensive category is one which coproducts exist and are well behaved (see Slogan 2.3 [3]).

Definition 4.1. A category $\mathcal{C}$ with finite coproducts is called *extensive*, if for each pair A, B of objects in $\mathcal{C}$, the canonical functor

$$\mathcal{C}/_A \times \mathcal{C}/_B \longrightarrow \mathcal{C}/_{A+B}$$

$$(f : X \to A, g : Y \to B) \longmapsto f + g : X + Y \to A + B$$

is an equivalence.

Any category with finite coproducts and pullbacks along coproduct injections is extensive, if and only if the coproducts are universal and disjoint (see Proposition 2.14 [3]).

Proposition 4.2 [3]. *A category $\mathcal{C}$ with coproducts is extensive, if and only if it has pullbacks and every diagram*

$$\begin{tikzcd} X \arrow[r, "i_X"] \arrow[d, "f"'] & Z \arrow[d, "h"] & Y \arrow[l, "i_Y"'] \arrow[d, "g"] \\ A \arrow[r, "e_A"'] & A + B & B \arrow[l, "e_B"] \end{tikzcd}$$

comprises a pair of pullback squares in $\mathcal{C}$ just when the top row is a coproduct diagram in $\mathcal{C}$.

An extensive category with products is distributive, but the converse is not true. A counter-example is the ordered set $\mathcal{P}(X)$ of subsets of a nonempty set X. It is a distributive category but not extensive, because it does not have disjoint coproducts.

Proposition 4.3. *Let $F : \mathcal{C} \to \mathcal{C}$ be an endofunctor on an extensive category $\mathcal{C}$. Then the category $\mathcal{C}_F$ is extensive, provided that F preserves pullbacks along monomorphisms.*

Proof. Since the category $\mathcal{C}$ has finite coproducts, the coproducts of any two F-coalgebras (A, a) and (B, b) exists in the category $\mathcal{C}_F$ because the forgetful functor $U : \mathcal{C}_F \to \mathcal{C}$ creates colimits (see Proposition 1.1 [5]). Let the following diagram

$$\begin{tikzcd}
(X,x) \arrow[r, "i_X"] \arrow[d, "f"'] & (Z,z) \arrow[d, "h"] & (Y,y) \arrow[l, "i_Y"'] \arrow[d, "g"] \\
(A,a) \arrow[r, "e_A"'] & (A+B, a+b) & (B,b) \arrow[l, "e_B"]
\end{tikzcd}$$

comprise a pair of commutative squares with the top row a coproduct diagram in $\mathcal{C}_F$. Then $(X \xrightarrow{i_X} Z \xleftarrow{i_Y} Y)$ is a coproduct diagram in $\mathcal{C}$. By the fact that $\mathcal{C}$ is extensive, (X, i_X, f) and (Y, i_Y, g) are respectively the pullback of h and e_A, and the pullback of h and e_B. In addition, e_A and e_B are monomorphisms. Each square above is therefore a pullback diagram as the forgetful functor $U : \mathcal{C}_F \to \mathcal{C}$ creates pullbacks along monomorphisms; this is because F preserves pullbacks along monomorphisms. Hence $\mathcal{C}_F$ is extensive due to Proposition 4.2. ∎

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Received 14 August 2021
Revised 29 June 2022
Accepted 29 June 2022