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ON SOME SUBGROUP LATTICES OF DIHEDRAL, ALTERNATING AND SYMMETRIC GROUPS

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Abstract

In this paper, the collections of all pronormal subgroups of D_n and Hall subgroups for groups A_n, S_n and D_n are studied. It is proved that the collection of all pronormal subgroups of D_n is a sublattice of $L(D_n)$. It is also proved that the collection of all Hall subgroups of D_n, A_n and S_n do not form sublattices of respective $L(D_n)$, $L(A_n)$ and $L(S_n)$.

Keywords: group, pronormal subgroup, Hall subgroup, lattice of subgroups, strong lattice.

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1. Introduction and notation

Throughout this article, G denotes a finite group. It is known that the set of all subgroups of a given finite group G forms a lattice denoted by $L(G)$ with $H \wedge K = H \cap K$ and $H \vee K = < H, K >$ for subgroups H, K of G. The interrelations between the theory of lattices and the theory of groups have been studied by many researchers, see Pálfy [10], Schmidt [14], Suzuki [18]. For the

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group theoretic concepts and notations, we refer to Birkhoff [1], Luthar and Passi [8], Schmidt [14].

There are a few types of subgroups such as, pronormal subgroups, Hall subgroups etc, whose collections may form lattices and these lattices can be used to study the properties of groups. Accordingly, the study of collection of pronormal subgroups of D_n and Hall subgroups of S_n, A_n and D_n has been carried out.

The following notations are used throughout this article.

- $LH(G)$ – Collection of all Hall subgroups of G.
- $LN(G)$ – Collection of all normal subgroups of G, which is a sublattice of $L(G)$.
- $LPrN(G)$ – Collection of all pronormal subgroups of G.
- $|G|$ – Order of G.
- $|L(G)|$ – Number of subgroups of G= Cardinality of $L(G)$.
- $o(a)$ – Order of an element a of G.
- e – Neutral (Identity) element in G.
- $[m, r]$ – lcm of m and r.
- (m, r) – gcd of m and r.
- $< H, K >$ – Subgroup generated by the subgroups H and K of G.
- $n_p(G)$ – Number of Sylow p-subgroups of G.

In what follows, n is always a positive natural number.

We use the notation D_n for the dihedral group of order $2n$ generated by two elements, say a and b, such that $a^n = e = b^2$; we write $D_n =< a, b >$. S_n denotes the symmetric group on n symbols and A_n denotes the alternating group on n symbols which is a normal subgroup of S_n.

The lattice depicted in Figure 1.1 is the Hasse diagram of $L(S_4)$; see also [17].

Note that the lattice $LPrN(S_4)$ depicted in Figure 1.2 is not a sublattice of the lattice $L(S_4)$, since $M_{18} \wedge M_{19} = K_2$ in $L(S_4)$ and $M_{18} \wedge M_{19} = K_1$ in $LPrN(S_4)$. As such, $LPrN(S_n)$ is not necessarily a sublattice of $L(S_n)$, in general.

We show that the situation is different for D_n.

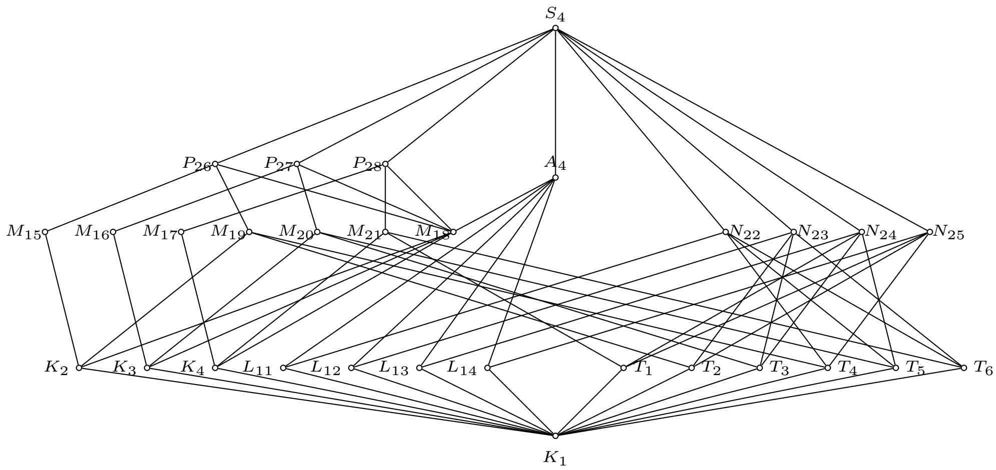

Figure 1.1. $L(S_4)$.

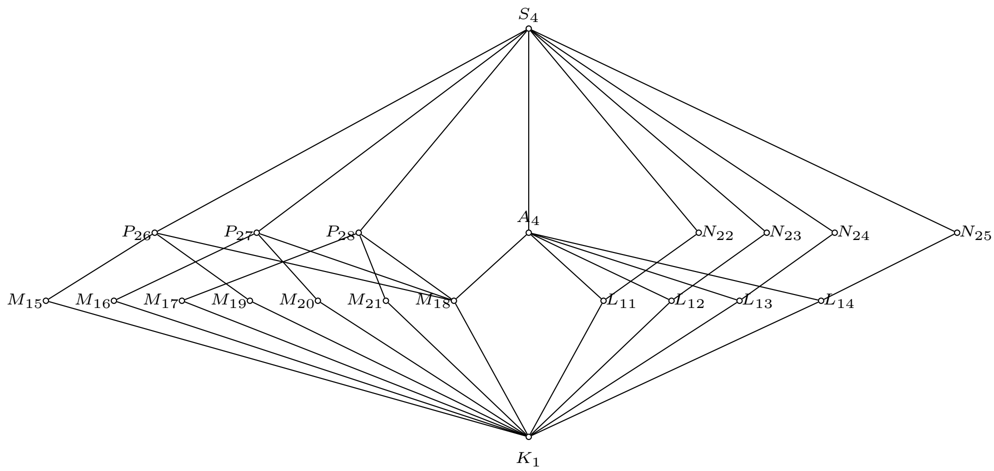

Figure 1.2. $LPrN(S_4)$.

2. PRONORMAL SUBGROUPS IN D_n

In this section, some properties of the collection of pronormal subgroups of D_n are investigated.

The following definition of a pronormal subgroup of a finite group is essentially due to Hall., see [7].

Definition 2.1. Let G be a group and H be a subgroup of G. Then H is said to be *pronormal*, if H and any given conjugate of H in G, say H^g, are also conjugates in $< H, H^g >$.

The definition and study of pronormal subgroups are primarily based on the conjugate(s) of a given subgroup in a group and as such, it is a study of conjugacy and existence of Sylow p-subgroups. In fact, the study of pronormal subgroups is primarily based on the pioneering work by Hall [7], Rose [13], Peng [11] and Mann [9].

We list some examples of pronormal subgroups in various groups as follows; see [13, 19].

- Every Hall subgroup (see Def. 3.1) of a finite solvable group is pronormal.
- Every normal subgroup of a group is pronormal.
- Every maximal subgroup of a group is pronormal.
- Every Sylow p-subgroup of a finite group is pronormal.
- Every Carter subgroup (i.e., nilpotent self-normalizing subgroup) of a finite solvable group is pronormal.

We recall the following Theorem, see [4].

Theorem 2.2. *Every subgroup of D_n is cyclic or dihedral. A complete listing of the subgroups is as follows:*

(1) $< a^d >$, *where* $d|n$, *with index* $2d$,

(2) $< a^k, a^i b >$, *where* $k|n$ *and* $0 \leq i \leq k-1$, *with index* k.

Every subgroup of D_n occurs exactly once in this listing.

Remarks.

1. A subgroup of D_n is said to be of Type (1) if it is cyclic subgroup as stated in (1) of Theorem 2.2.

2. A subgroup of D_n is said to be of Type (2) if it is dihedral subgroup as stated in (2) of Theorem 2.2.

The following Lemma shows an important property about dihedral groups, see [2, 4].

Lemma 2.3. *For n odd, the only proper normal subgroups of D_n are the subgroups of $< a >$. For n even, there are two additional proper normal subgroups, $< a^2, b >$ and $< a^2, ab >$, both of order n and isomorphic to $D_{\frac{n}{2}}$.*

Interestingly, a conjugate of a given subgroup of D_n is determined by some power of the generator a of D_n, as the following result shows.

Lemma 2.4. *For any subgroup H of D_n and for any k, there is a j such that $H^{a^k b} = H^{a^j}$.*

Proof. Let H be a Type (1) subgroup, then H is normal by Lemma 2.3. Now, let H be a Type (2) subgroup of the form $H =< a^m, a^ib >$ and accordingly $a^kbHa^kb = a^kb < a^m, a^ib > a^kb =< a^m, a^{2k-i}b >$. In this case, choose an element $a^{k-i} \in D_n$ and so the conjugate of H determined by a^{k-i} is $a^{k-i}Ha^{i-k} = a^{k-i} < a^m, a^ib > a^{i-k} =< a^m, a^{2k-i}b >$. Therefore, $a^{k-i}Ha^{i-k} = a^kbHa^kb$. ■

In what follows, we characterize the pronormal subgroups of D_n.

Theorem 2.5. *A subgroup of D_n is pronormal unless it is of the form $< a^m, a^ib >$ where $4|m|n$ and $0 \leq i \leq m-1$.*

Proof. Let H be a subgroup of D_n. If H of Type (1) then it is pronormal since it is normal by Lemma 2.3. We therefore assume the possibilities only when H is a subgroups of Type (2).

Claim 1. *If $H= < a^m, a^ib >$ is a subgroup of D_n of Type* (2), *with m is not divisible by* 4, *then H is pronormal.*

In view of Lemma 2.4, it is sufficient to consider H^x for an element $x = a^k$ of H. For $H^x =< a^m, a^{2k+i}b >$, we contend that $< H^x, H >=< a^g, a^ib >$ where $g = (m, 2k)$. Indeed, note that $< H^x, H >=< a^m, a^{2k}, a^ib >$, and as $g|m$ and $g|2k$, we have $< H^x, H >\subseteq< a^g, a^ib >$. Moreover, if $z \in< a^g, a^ib >$ then there is some $q \in \mathbb{N}$ such that $z = a^{gq+i}b = a^{(mt_1+2kt_2)q+i}b \in< a^m, a^{2k}, a^ib >=< H^x, H >$ for some t_1 $t_2 \in \mathbb{Z}$, and this proves $< H^x, H >=< a^g, a^ib >$.

We claim that H and H^x are conjugates in $< H^x, H >=< a^g, a^ib >$, i.e., there exists a $y \in< H^x, H >$ such that $H^x = H^y$ holds. We have $g = (m, 2k)$, so let $m = gm'$ and $2k = gk'$ for some m', $k' \in \mathbb{Z}$. Note that if m is even then $2|g$ and since $4 \nmid m$, we have $(m', 2) = 1$. Also if m is odd then $2 \nmid m'$ and so $(m', 2) = 1$. In both the cases we have $(m', 2) = 1$ and therefore there exist $d_1, d_2 \in \mathbb{Z}$ such that $1 = m'd_1 + 2d_2$. Now, $gk' = m'gd_1k' + 2gd_2k' = md_1k'+2gd_2k' = ms_1+2gs_2$, where $s_1 = d_1k'$ and $s_2 = d_2k'$, i.e., $2k = ms_1+2gs_2$. Put $y = a^{gs_2}$. Then $H^y =< a^m, a^{2gs_2+i}b >$ and so it contains an element $a^{2k+i}b$ of H^x and consquently, $H^x \subseteq H^y$. Therefore, $H^x = H^y$ since H^x and H^y are conjugates of H.

Claim 2. *If $H= < a^m, a^ib >$ is a subgroup of D_n of Type* (2), *with $m \geq 1$ is divisible by* 4, *then H is not pronormal.*

Note that this case only occurs if n is divisible by 4, since $4|m$ and $m|n$. In order to show that H is not pronormal in D_n, it is sufficient to find an element $g \in D_n$ such that H and H^g are not conjugates in $< H, H^g >$.

We have $< H, H^a >= < a^m, a^ib, a^2 >$. As m is even, we have $< H, H^a >=< a^2, b >$ if i is even and $< H, H^a >=< a^2, ab >$ if i is odd. As such, we have following two cases.

Case I. Suppose that i is odd. In this case $< H, H^a >=< a^2, ab >$, and if H and H^a are conjugates in $< H, H^a >$ then there must exist an element $x \in< H, H^a >$ such that $H^a = H^x$ and such x is of the form a^{2p} for some p or $a^{2p+1}b$ for some p.

Subcase I(1). If $x = a^{2p}$, then $H^x =< a^m, a^{4p+i}b >= H^a =< a^m, a^{2+i}b >$ and we must have $a^{mq}a^{4p+i}b = a^{2+i}b$ for some q. But then, $a^{mq+4p} = a^2$, i.e., $a^{mq+4p-2} = e$, where e is the identity element of D_n. Now, $o(a)$ is n and we have $n|4p+mq-2$ and so $4|4p+mq-2$. Also, $4|m$ and so we must have $4|-2$, which is not true and therefore no such x exists.

Subcase I(2). If $x = a^{2p+1}b$, then $H^x =< a^m, a^{4p+2-i}b >= H^a =< a^m, a^{2+i}b >$, and so we must have $a^{mq}a^{4p+2-i}b = a^{2+i}b$ for some q. As such, $a^{mq+4p+2-i} = a^{2+i}$, i.e., $a^{mq+4p-2i} = e$ where e is the identity element of D_n. Now, $o(a)$ is n and $4|n$, so we have $n|4p + mq - 2i$ and $4|4p + mq - 2i$. Also, $4|m$ and so we must have $4|2i$, this is not possible as i is odd, and so no such x exists.

Therefore, in this Case I, H and H^a are not conjugates in $< H, H^a >$.

Case II. Suppose that i is even. In this case $< H, H^a >=< a^2, b >$, and if H and H^a are conjugates in $< H, H^a >$ then there must exist an element $x \in< H, H^a >$ such that $H^a = H^x$ and such x is of the form a^{2p} or $a^{2p}b$ for some p.

Subcase II(1). If $x = a^{2p}$, then $H^x =< a^m, a^{4p+i}b >= H^a =< a^m, a^{2+i}b >$, and so we have $a^{mq}a^{4p+i}b = a^{2+i}b$ for some q. As such, $a^{mq+4p} = a^2$, i.e., $a^{mq+4p-2} = e$, where e is the identity element of D_n. Now, $o(a)$ is n and $4|n$ and so we have $n|4p + mq - 2$ and $4|4p + mq - 2$. Also, $4|m$ and so we must have $4| - 2$, which is not true and so no such x exists.

Subcase II(2). If $x = a^{2p}b$, then $H^x =< a^m, a^{4p-i}b >= H^a =< a^m, a^{2+i}b >$ and so we have $a^{mq}a^{4p-i}b = a^{2+i}b$ for some q. Accordingly, $a^{mq+4p-i} = a^{2+i}$, i.e., $a^{mq+4p-2i-2} = e$, where e is the identity element of D_n. Now, $o(a)$ is n and $4|n$, and so we have $n|4p + mq - 2i - 2$ and $4|4p + mq - 2i - 2$. Also, $4|m$ and so we must have $4|2$, which is not true and so no such x exists.

Therefore, in Case II also, H and H^a are not conjugates in $< H, H^a >$.

Consequently, in either of these cases, the subgroup H is not pronormal. ■

It is known that the number of subgroups of D_n for $n \geq 3$ is $|L(D_n)| =$ Number of divisors of n + Sum of divisors of n. Along the same line, we have the following formula for the number of pronormal subgroups of D_n, i.e., $|LPrN(D_n)|$.

Corollary 2.6. *For any* $n \geq 3$, $|LPrN(D_n)| = d(n) + \sum\limits_{d'|n \text{ and } 4\nmid d'} d'$, *where* $d(n)$ *is the number of divisors of* n.

Proof. From Theorem 2.5, every choice of a divisor m of n which is not divisible by 4 gives a dihedral pronormal subgroup $< a^m, a^i b >$ for every i. Moreover, every divisor m of n will determine a cyclic pronormal subgroup $< a^m >$ of D_n and these are the only pronormal subgroups of D_n. ■

We prove that the set of all pronormal subgroups of D_n forms a sublattice of the subgroup lattice of D_n, for any n.

Theorem 2.7. *$LPrN(D_n)$ is a sublattice of $L(D_n)$.*

Proof. We show that the intersection of two pronormal subgroups of D_n is again pronormal. Let H and K be two pronormal subgroups of D_n.

If one of these subgroups is of the form $< a^k >$ then by Lemma 2.3, we are through. So, let $H =< a^m, a^i b >$ and $K =< a^r, a^j b >$ for some $m, r \geq 1$, $m|n$, $r|n$, $0 \leq i \leq m-1$, $0 \leq j \leq r-1$, moreover $4 \nmid m$, $4 \nmid r$ by Theorem 2.5. Suppose that $a^k b \in H \cap K$ for some k, then $a^k b = a^{ms+i} b = a^{rt+j} b$ for suitable s and t therefore $ms + i = rt + j$ or $ms - rt = j - i$. Also, in this case $H \cap K = < a^{[m,r]}, a^k b >$. If no such $a^k b \in H \cap K$, then we have $H \cap K =< a^{[m,r]} >$. We prove in each of the following cases that $H \cap K$ is pronormal and note that, if $H \cap K =< a^{[m,r]} >$ then it is normal by Lemma 2.3 and so pronormal.

(i) If both m and r are even numbers and neither is a multiple of 4, then $[m, r]$ is also even and not a multiple of 4. And so, $H \cap K =< a^m, a^i b > \cap < a^r, a^j b >$ is pronormal.

(ii) If m is even and not a multiple of 4 and r is odd, then $[m, r]$ is an even number which is not a multiple of 4, and so $H \cap K =< a^m, a^i b > \cap < a^r, a^j b >$ is pronormal.

(iii) If both m and r are odd numbers, then $[m, r]$ is an odd number and consequently $H \cap K =< a^m, a^i b > \cap < a^r, a^j b >$ is pronormal.

Therefore the intersection of any two pronormal subgroups is a pronormal subgroup.

Next, we prove that the subgroup generated by the union of two pronormal subgroups is pronormal.

Let H and K be two pronormal subgroups of D_n.

Case I. Suppose that H and K are subgroups of Type (2), say $H =< a^m, a^i b >$ and $K =< a^r, a^j b >$ for some $m, r \geq 1$, $m|n$, $r|n$, $0 \leq i \leq m-1$, $0 \leq j \leq r-1$, moreover $4 \nmid m$, $4 \nmid r$ by Theorem 2.5.

We contend that $< H \cup K >=< a^g, a^i b >$, where $g = (m, r, i-j)$. Indeed, for $S =< a^g, a^i b >$ and $x \in S$, we have $x = a^{gk_1+i} b$, for some $k_1 \in \mathbb{Z}$. However, since $g = (m, r, i-j)$, there exist $p_1, p_2, p_3 \in \mathbb{Z}$ such that $g = mp_1 + rp_2 + (i-j)p_3$ and so $x = a^{(mp_1+rp_2+(i-j)p_3)k_1+i} b$, which is a finite product of elements of H and K, and so $x \in< H \cup K >$, therefore $S \subseteq< H \cup K >$. Now to show that

$S \supseteq < H \cup K >$ it is sufficient to show that $a^j b \in S$. We have $a^i b \in S$, $a^{j-i} \in S$ and so $a^j b \in S$. Consequently, $< a^m, a^i b, a^r, a^j b > \subseteq S$, i.e., $S \supseteq < H \cup K >$.

Now, since H and K are pronormal, we have $4 \nmid m$ and $4 \nmid r$, and so $4 \nmid g$, which implies that $< H \cup K >$ is pronormal.

Case II. Suppose that each one of H and K is a cyclic subgroup of Type (1), then obviously $< H \cup K >$ is also cyclic of Type (1) which is normal by Lemma 2.3 and so pronormal.

Case III. Suppose that one of H and K is a cyclic subgroup of Type (1) and the other one is of Type (2), say $H =< a^r >$ and $K =< a^m, a^i b >$. Then $< H \cup K >=< a^g, a^i b >$ where $g = (m, r)$. Now, $4 \nmid m$, so $4 \nmid g$, which implies that $< H \cup K >$ is pronormal.

We conclude that given pronormal subgroups H and K of a group D_n, we have that both $H \vee K =< H \cup K >$ and $H \wedge K = H \cap K$ are pronormal. Therefore $LPrN(D_n)$ is a sublattice of $L(D_n)$. ■

3. Hall subgroups of D_n, A_n and S_n

In this section, the properties of the collection of Hall subgroups of D_n, S_n and A_n are investigated.

The concept of a *Hall subgroup* of a finite group was introduced by Hall [7]; for more details, see [15, 19, 20].

Definition 3.1. A *Hall subgroup* of a finite group is a subgroup whose order is coprime to its index.

We prove in the following Theorem that the set of all Hall subgroups of D_n forms a lattice.

Theorem 3.2. *Let D_n be a dihedral group. The poset $LH(D_n)$ of all Hall subgroups of D_n is a lattice.*

Proof. We know that a subgroup of D_n is either cyclic or dihedral. $LH(D_n) \subseteq L(D_n)$, in order to show that $LH(D_n)$ is a lattice, we determine the meet $(\wedge_{LH})$ and join $(\vee_{LH})$ of two elements of $LH(D_n)$.

We write $n = 2^i p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_r^{\alpha_r} = 2^i \prod_{k=1}^{k=r} p_k^{\alpha_k}$ where each p_k is an odd prime.

Case I. Let $C_1 =< a^{n_1} >$ and $C_2 =< a^{n_2} >$, where $n_1 = \frac{n}{m_1}$ and $n_2 = \frac{n}{m_2}$ be two cyclic subgroups of D_n of Type (1) which are also Hall subgroups with orders m_1 and m_2 respectively. Since a cyclic subgroup of Type (1) which is also a Hall subgroup has to be of odd order, m_1 and m_2 are odd numbers.

Note that the subgroup $< a^l >$, where $l = \frac{n}{[m_1,m_2]}$, is the smallest subgroup containing both C_1 and C_2 in $L(D_n)$.

Claim I(1). *$< a^l >$ is a Hall subgroup and so $C_1 \vee C_2 = C_1 \vee_{LH} C_2 =< a^l >$.*

If $< a^l >$ is not a Hall subgroup, then $| < a^l > |$ contains a prime say p_x with power strictly less than α_x and stricly greater than 0, and as $| < a^l > |$ is $[m_1, m_2]$, we must have that the prime p_x is in one of the expansions of m_1 and m_2 with power strictly less than α_x and stricly greater than 0, both m_1 and m_2 being odd. Consequently, one of C_1 and C_2 is not a Hall subgroup, which is not true. Therefore, $< a^l >$ is a Hall subgroup.

Similarly, note that the subgroup $< a^g >$, where $g = \frac{n}{(m_1,m_2)}$, is the largest subgroup contained in both C_1 and C_2 in $L(D_n)$.

Claim I(2). *$< a^g >$ is a Hall subgroup and so $C_1 \wedge_{LH} C_2 = C_1 \wedge C_2 =< a^g >$.*

If $< a^g >$ is not a Hall subgroup, then $| < a^g > |$ contains a prime say p_x with power strictly less than α_x and stricly greater than 0, and as $| < a^g > |$ is (m_1, m_2), we must have that the prime p_x is in the expansions of both m_1 and m_2 with its power strictly less than α_x and stricly greater than 0. But this means, C_1 and C_2 are not Hall subgroups, which is not true. Therefore, $< a^g >$ is a Hall subgroup.

Case II. Let $C =< a^{n_1} >$ and $D =< a^{n_2}, a^j b >$ where $n_1 = \frac{n}{m_1}$, $n_2 = \frac{2n}{m_2}$ and $0 \leq j \leq n_2 - 1$ be two Hall subgroups of D_n with orders m_1 and m_2 respectively. Clearly, m_1 is odd and $2^{i+1} | m_2$ since C and D are Hall subgroups.

Consider the subgroups, $X =< a^l, a^j b >$, where $l = \frac{2n}{[m_1,m_2]}$ and $Y =< a^g >$, where $g = \frac{n}{(m_1,m_2)}$.

Claim II(1). *X is a Hall subgroup.*

Observe that the order of X is $[m_1, m_2]$ and its expansion contains primes with their complete powers along with 2^{i+1}, and as such, we have $\left([m_1, m_2], \frac{2n}{[m_1,m_2]}\right) = 1$. Consequently, X is a Hall subgroup.

Moreover, it is the smallest Hall subgroup containing both C and D having order $[m_1, m_2]$, therefore $X = C \vee_{LH} D = C \vee D$.

Claim II(2). *Y is a Hall subgroup.*

Since the expansion of $|Y| = (m_1, m_2)$ contains primes with their complete powers, we have $\left((m_1, m_2), \frac{2n}{(m_1,m_2)}\right) = 1$. Consequently, Y is a Hall subgroup.

Moreover, it is the largest Hall subgroup contained in both C and D having order (m_1, m_2), therefore $Y = C \wedge_{LH} D = C \wedge D$.

Case III. Let $X_1 =< a^{n_1}, a^k b >$ and $X_2 =< a^{n_2}, a^j b >$, where $n_1 = \frac{2n}{m_1}$, $n_2 = \frac{2n}{m_2}$, $0 \le k \le n_1 - 1$ and $0 \le j \le n_2 - 1$ be two Hall subgroups of D_n with orders m_1 and m_2 respectively. Clearly, $2^{i+1}|m_1, m_2$, since both X_1 and X_2 are Hall subgroups.

Consider, $X =< a^g, a^j b >$ where $g_1 = (n_1, n_2, k-j), r = \left(\frac{2n}{g_1}, g_1\right)$ & $g = \frac{g_1}{r}$.

Claim III(1). *X is a Hall subgroup.*

Subcase III(1)(i). Suppose that $r = 1$. As $r = 1$, we have $g = g_1$. Note that, $|X|$ is $2o(a^g)$, where $o(a^g) = \frac{n}{g}$. Therefore, the index of X in D_n is g and accordingly $\left(\frac{2n}{g}, g\right) = \left(\frac{2n}{g_1}, g_1\right) = r = 1$ and this shows that X is a Hall subgroup.

Subcase III(1)(ii). Suppose that $r \neq 1$. Since $r \neq 1$, we have $\left(\frac{2n}{g_1}, g_1\right) = r \neq 1$, but then $\left(\frac{2nr}{g_1}, \frac{g_1}{r}\right) = 1$, i.e., $\left(\frac{2n}{g}, g\right) = 1$. Therefore, the index of X in D_n is g and so $\left(\frac{2n}{g}, g\right) = 1$, therefore X is a Hall subgroup.

Moreover, X has the smallest possible order which is co-prime with its index in D_n, such that the factorization of its order contains m_1 and m_2. If $X' =< a^g, a^h b >$ for some g as above and $h \neq k, j$ is a Hall subgroup of D_n, containing both X_1 and X_2 then we must have $a^h b = a^{n_1 z_1 + n_2 z_2 + k z_3 + j z_4} b$ (as $ab = ba^{-1}$ is true in D_n) for some $z_1, z_2, z_3, z_4 \in \mathbb{Z}$. Since $g|(n_1, n_2, k-j)$, we have $a^h b \in < a^g, a^j b >$ which implies $X' \subseteq X$. Note that $|X| = |X'|$ since $X = X'$ and consequently, any subgroup other than X with the same order as that of X can not contain both X_1 and X_2, and therefore $X = X_1 \vee_{LH} X_2$.

Now consider

$$Y = \begin{cases} < a^s >, & \text{if } n_1 x + n_2 y = k - j \text{ has no integer solution where } s = \frac{2^{i+1} n}{(m_1, m_2)}. \\ < a^d, a^{k - n_1 x_0} b >, & \text{if } n_1 x + n_2 y = k - j \text{ has an integer solution where } d = \frac{2n}{(m_1, m_2)}. \end{cases}$$

Where (x_0, y_0) is an integer solution of an equation $n_1 x + n_2 y = k - j$.

Claim III(2). *Y is a Hall subgroup.*

Note that, Y is a subgroup of both X_1 and X_2 and so $Y \subseteq X_1 \wedge X_2$.

We have $X_1 =< a^{n_1}, a^k b >$ and $X_2 =< a^{n_2}, a^j b >$, where $n_1 = \frac{2n}{m_1}$, $n_2 = \frac{2n}{m_2}$, $0 \le k \le n_1 - 1$ and $0 \le j \le n_2 - 1$. Note that, if $a^v b$ is in Y, then $v = n_1 t + k = n_2 u + j$ for some $t, u \in \mathbb{Z}$, such that (t, u) is a solution of $n_1 x + n_2 y = k - j$.

Subcase III(2)(i). Suppose that $n_1 x + n_2 y = k - j$ has no integer solution. In this case, no such element $a^v b$ exists in Y. In order to show that Y is a Hall subgroup in this case, we consider $| < a^s > | = \frac{n(m_1, m_2)}{2^{i+1} n} = \frac{(m_1, m_2)}{2^{i+1}} = \prod_{l \in S \subseteq \{1, \ldots, r\}} p_l^{\alpha_l}$ where each prime p_l is odd and the index of $Y = \frac{2^{i+1} 2n}{(m_1, m_2)} =$

$\frac{2^{i+1}\prod_{k=1}^{k=r} p_k^{\alpha_k}}{\prod_{l\in S\subseteq\{1,\ldots,r\}} p_l^{\alpha_l}}$. But then, $\left(\prod_{l\in S\subseteq\{1,\ldots,r\}} p_l^{\alpha_l}, \frac{2^{i+1}\prod_{k=1}^{k=r} p_k^{\alpha_k}}{\prod_{l\in S\subseteq\{1,\ldots,r\}} p_l^{\alpha_l}}\right) = 1$, consequently, Y is a Hall subgroup.

Moreover, being a cyclic subgroup of D_n of Type (1), such subgroup Y is unique with order as above. Also, by definition of Y, it has the largest possible order which divides both m_1 and m_2 and such that this order is co-prime with the index of Y in D_n. Therefore, in this case $Y = X_1 \wedge_{LH} X_2$.

Subcase III(2)(ii). Suppose that, $n_1x + n_2y = k - j$ has an integer solution say (x_0, y_0) which implies $n_1x_0 + n_2y_0 = k - j$. We have $Y =< a^d, a^{k-n_1x_0}b >$ and so $|Y| = | < a^d, a^{k-n_1x_0}b > | = 2o(a^d)$. Observe that $o(a^d) = \frac{(m_1,m_2)}{2}$, and consequently, $|Y| = (m_1, m_2)$ and hence Y is a Hall subgroup.

If any subgroup Y' of order $|Y|$ is contained in both X_1 and X_2, say $Y' =< a^g, a^p b >$, then we must have $a^p b = a^{n_1t_1+k}b = a^{n_2t_2+j}b$ for some t_1 and t_2 which yields a solution to the equation $n_1x+n_2y = k-j$. Consequently, p is of the form $n_1t_1 + k$ and so $Y' = Y$. Therefore, any subgroup other than Y with order same as that of Y cannot be contained in both X_1 and X_2, and hence $Y = X_1 \wedge_{LH} X_2$.

We conclude that, with these newly observed meets and joins, the poset of all Hall subgroups of D_n, i.e., $LH(D_n)$ forms a lattice. ∎

Remark. We have proved that $LH(D_n)$ is a lattice. But in general, $LH(D_n)$ is not a sublattice of $L(D_n)$.

However, we have a characterization of $LH(D_n)$ to be a sublattice of $L(D_n)$ for some n in the following Theorem.

Theorem 3.3. *Let D_n be the dihedral group, $L(D_n)$ its lattice of subgroups and $LH(D_n)$ its lattice of Hall subgroups. Then, $LH(D_n)$ is a sublattice of $L(D_n)$ if and only if either n is some power of 2 or $2n$ is square free.*

Proof. For necessity, we prove that if n is not a power of 2 and $2n$ is not square free, then $LH(D_n)$ is not a sublattice of $L(D_n)$.

Case 1. Let $2n = 2p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}\cdots p_r^{\alpha_r}$ be such that each p_i is an odd prime and there exists $\alpha_x > 1$ for at least one x.

Note that every subgroup of order 2 is a Hall subgroup since $(2, p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}\cdots p_r^{\alpha_r}) = 1$. Now for two subgroups of order 2, say $H = \{e, b\}$ and $K = \{e, a^q b\}$ with $o(a^q) = p_x$, we observe that $< H\cup K >= S =< a^q, b >$. Note that $|S| = 2p_x$, therefore $(2p_x, \frac{2n}{2p_x}) = p_x$ and so S is not a Hall subgroup. This implies that the join of two Hall subgroups is not necessarily a Hall subgroup, and so in this case $LH(D_n)$ is not a sublattice of $L(D_n)$.

Case 2. Let $2n = 2^i p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}\cdots p_r^{\alpha_r}$, with $i \geq 2$ and all p_x are odd primes with at least one $\alpha_x \neq 0$.

Consider the subgroups $H =< a^q, b >$ and $K =< a^q, ab >$ where $q = p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}\cdots p_r^{\alpha_r}$ and $|H| = |K| = 2^i$. Also, each one of these is a Hall subgroup as $(2^i,\ p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}\cdots p_r^{\alpha_r}) = 1$. Now, $H \cap K =< a^q, b > \cap < a^q, ab >=< a^q >$, and $|H \cap K| = 2^{i-1}$ so $H \cap K$ is not a Hall subgroup as $\left(|H \cap K|,\ \frac{2n}{|H \cap K|}\right) = \left(2^{i-1}, \frac{2^i p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}\cdots p_r^{\alpha_r}}{2^{i-1}}\right) = 2$.

Consequently, in either case, $LH(D_n)$ is not a sublattice of $L(D_n)$.

Conversely, first suppose that n is some power of 2. Then any subgroup of D_n other than $\{e\}$ and the whole group, is also of order some power of 2 and the order of each such subgroup is not co-prime with $2n$ and as such, none of these subgroups is a Hall subgroup. So the only Hall subgroups are $\{e\}$ and D_n, therefore $LH(D_n)$ is a sublattice of $L(D_n)$.

Now, suppose that $2n$ is square-free so it has a representation $2n = 2p_1p_2p_3 \cdots p_r$, where each prime p_i is odd and all are distinct. Also, the order of any subgroup of D_n is either $\prod_{t \in S \subseteq \{1,2,\ldots,r\}} p_t$ or $2\prod_{t \in S \subseteq \{1,2,\ldots,r\}} p_t$, with index $\frac{2p_1p_2p_3\cdots p_r}{\prod_{t \in S \subseteq \{1,2,\ldots,r\}} p_t}$ or $\frac{2p_1p_2p_3\cdots p_r}{2\prod_{t \in S \subseteq \{1,2,\ldots,r\}} p_t}$ respectively, and as such, all such subgroups are Hall subgroups as $\left(\prod_{t \in S \subseteq \{1,2,\ldots,r\}} p_t, \frac{2p_1p_2p_3\cdots p_r}{\prod_{t \in S \subseteq \{1,2,\ldots,r\}} p_t}\right) = 1$ and $\left(2\prod_{t \in S \subseteq \{1,2,\ldots,r\}} p_t, \frac{2p_1p_2p_3\cdots p_r}{2\prod_{t \in S \subseteq \{1,2,\ldots,r\}} p_t}\right) = 1$. This proves that $LH(D_n)$ is a sublattice of $L(D_n)$. ■

Theorem 3.4 ([3], [8]).. *Let G be a finite group and H be a subgroup. Choose a prime p. Distinct Sylow p-subgroups of H do not lie in a common Sylow p-subgroup of G. In particular, $n_p(H) \leq n_p(G)$.*

Remark. Every Sylow p-subgroup of a finite group is a Hall subgroup.

We note that $LH(A_n)$ is a sublattice of $L(A_n)$ for $n \leq 5$. However, for $n \geq 6$, we have the following Theorem.

Theorem 3.5. *$LH(A_n)$ is not a sublattice of $L(A_n)$ for $n \geq 6$.*

Proof. We use induction on n. Firstly, we observe the following statement.

Claim. *$LH(A_6)$ is not a sublattice of $L(A_6)$.*

We have $|A_6| = 360$, so the order of its Sylow 2-subgroups is 8. Consider the Sylow 2-subgroups, $H =< (1\,2\,3\,4)(5\,6), (1\,3)(5\,6) >= \{e, (1\,2\,3\,4)(5\,6), (1\,3)(2\,4), (1\,4\,3\,2)(5\,6), (1\,3)(5\,6), (1\,4)(2\,3), (1\,2)(3\,4), (2\,4)(5\,6)\}$ and $K =< (1\,3\,2\,4)(5\,6), (1\,2)(5\,6) >= \{e, (1\,3\,2\,4)(5\,6), (1\,2)(3\,4), (1\,4\,2\,3)(5\,6), (1\,2)(5\,6), (1\,4)(2\,3), (1\,3)(2\,4), (3\,4)(5\,6)\}$, where e is the identity element of A_n. Now, $H \cap K = \{e, (1\,2)(3\,4), (1\,4)(2\,3), (1\,3)(2\,4)\}$, $|H \cap K| = 4$ and the index of $H \cap K$ in A_6 is 90. Therefore $H \cap K$ is not a Hall subgroup as $(4, 90) \neq 1$, so that $LH(A_6)$ is not a sublattice of $L(A_6)$.

Assume that $LH(A_k)$ is not a sublattice of $L(A_k)$, i.e., there exist two distinct Sylow 2-subgroups, say P_1 and P_2 such that $P_1 \cap P_2$ is nontrivial and not a Hall subgroup.

We claim that $LH(A_{k+1})$ is not a sublattice of $L(A_{k+1})$. There are Sylow 2-subgroups P_1 and P_2 of A_k for which $P_1 \cap P_2$ is not a Hall subgroup and by Theorem 3.4, their extensions say P_1' and P_2' are distinct Sylow 2-subgroups of A_{k+1}. Moreover, $P_1 \cap P_2 \subseteq P_1' \cap P_2'$, and so we get two distinct Sylow 2-subgroups of A_{k+1} with nontrivial intersection and which is not a Hall subgroup.

Consequently, $LH(A_n)$ is not a sublattice of $L(A_n)$ for $n \geq 6$. ■

We also note that, $LH(S_n) = L(S_n)$ for $n \leq 3$. The Hasse diagram of $LH(S_4)$ is depicted in Figure 3.1 in which the nomenclature of the subgroup is the same as that in the Hasse diagram of $L(S_4)$ depicted in Figure 1.1. Observe that $P_{28} \wedge P_{27} = M_{18}$ in $L(S_4)$ however $M_{18} \notin LH(S_4)$ and as such, $LH(S_4)$ is not a sublattice of $L(S_4)$.

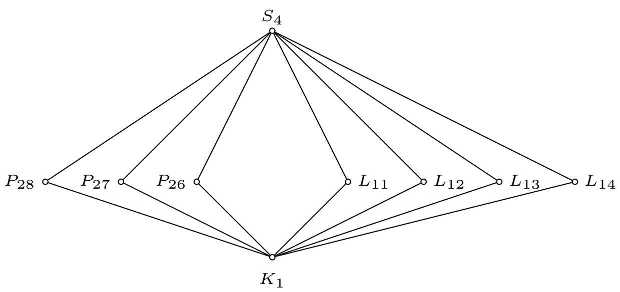

Figure 3.1. $LH(S_4)$.

In fact, for S_n $(n \geq 4)$, we have the following Theorem.

Theorem 3.6. *$LH(S_n)$ is not a sublattice of $L(S_n)$ for $n \geq 4$.*

Proof. We use induction on n. Firstly, we observe the following statement.

Claim. *$LH(S_4)$ is not a sublattice of $L(S_4)$.*

We have $|S_4| = 24$, and so the order of its Sylow 2-subgroups is 8. Consider two Sylow 2-subgroups $H = \ <(1\,2\,3\,4), (1\,3)> = \{e, (1\,2\,3\,4), (1\,3)(2\,4), (1\,4\,3\,2), (1\,3), (1\,4)(2\,3), (1\,2)(3\,4), (2\,4)\}$ and $K =<(1\,3\,2\,4), (1\,2)> = \{e, (1\,3\,2\,4), (1\,2), (1\,4\,2\,3), (1\,2)(3\,4), (1\,4)(2\,3), (1\,3)(2\,4), (3\,4)\}$, where e is the identity element of S_n. Now, $H \cap K = \{e, (1\,2)(3\,4), (1\,4)(2\,3), (1\,3)(2\,4)\}$. Observe that $|H \cap K| = 4$ and that the index of $H \cap K$ in S_4 is 6. Therefore $H \cap K$ is not a Hall subgroup as $(4, 24) \neq 1$; in fact, $H \wedge K = \{e\}$ in $LH(S_4)$, so that $LH(S_4)$ is not a sublattice of $L(S_4)$.

Assume that $LH(S_k)$ is not a sublattice of $L(S_k)$, for some k, i.e., there exist two distinct Sylow 2-subgroups and say P_1 and P_2 such that $P_1 \cap P_2$ is nontrivial and not a Hall subgroup.

Now, we need to show that $LH(S_{k+1})$ is not a sublattice of $L(S_{k+1})$. There are Sylow 2-subgroups P_1 and P_2 of S_k for which $P_1 \cap P_2$ is not a Hall subgroup and by Theorem 3.4, their extensions say P_1' and P_2' are distinct Sylow 2-subgroups of S_{k+1}. Moreover, $P_1 \cap P_2 \subseteq P_1' \cap P_2'$, and so we get two distinct Sylow 2-subgroups of S_{k+1} with their intersection a nontrivial subgroup which is not a Hall subgroup.

Consequently, $LH(S_n)$ is not a sublattice of $L(S_n)$ for $n \geq 4$. ■

4. Strongness in subgroup lattices

In this section, the concept of strongness is explored for the structures $L(D_n)$, $LH(D_n), L(A_n)$ and $L(S_n)$.

Faigle, *et al.* (see [5, 12, 16]) studied strong lattices of finite length in which the join-irreducible elements play a key role.

For the following definition and other relevant definitions in lattice theory we refer to Grätzer [6], Stern [16], Birkhoff [1].

Definition 4.1 ([16])**.** An element j of a lattice L is called *join-irreducible* if, for all $x, y \in L$, $j = x \vee y$ implies $j = x$ or $j = y$. For a lattice L of finite length $J(L)$ denotes the set of all non-zero join-irreducible elements.

We define join irreducible subgroups as follows.

Definition 4.2. A subgroup of a group G is said to be *join-irreducible* if it is a join-irreducible element of $L(G)$.

It is easy to observe that every cyclic subgroup of prime power order of a finite group is a join irreducible subgroup. From this fact and Lemma 2 of [21], the following Lemma follows.

Lemma 4.3. *A subgroup of a finite group is a join-irreducible subgroup if and only if it is a cyclic subgroup of prime power order.*

The following concept of a strong element was coined by Faigle; see [5, 16].

Definition 4.4. Let L be a lattice of finite length. A join-irreducible element $j \neq 0$ is called a *strong element* if the following condition holds for all $x \in L$: (St) $j \leq x \vee j^- \Longrightarrow j \leq x$, where j^- denotes the uniquely determined lower cover of j.

A lattice is said to be *strong* if every join-irreducible element of it is strong.

Remark. The condition (St) in the definition of a strong element is equivalent to the following; see [16] for more details. (St') For every $q < j \in J(L)$, $x \in L$, $j \leq x \vee q$ implies $j \leq x$.

The following characterization of strong lattices is due to Richter and Stern [12].

Theorem 4.5. *A lattice L of finite length is strong if and only if it does not contain a special pentagon sublattice with $j \in J(L)$.*

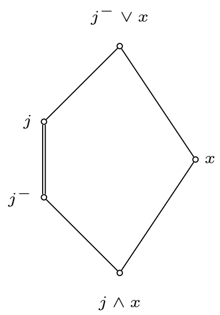

Figure 4.1. Special Pentagon.

The proof of the following Lemma is straightforward.

Lemma 4.6. *Let L be a finite lattice. If atoms are the only join-irreducible elements in L, then L is strong.*

Theorem 4.7. *$L(D_n)$ is a strong lattice.*

Proof. In order to show that $L(D_n)$ is a strong lattice, we need to show that every join-irreducible element is a strong element. We know that a subgroup of D_n is either cyclic or dihedral. Let H be a join-irreducible subgroup of D_n, then it is either a subgroup of the form $< a^i b >$ or a subgroup of the form $< a^m >$ of Type (1), where $| < a^m > |$ is divisible by exactly one prime by Lemma 4.3. Let $2n = p_0^{\alpha_0+1} \prod_{k=1}^{k=r} p_k^{\alpha_k}$ for some $i \geq 0$ and $p_0 = 2$. In view of Theorem 4.5, in order to show that $L(D_n)$ is strong, it is sufficient to show that the special pentagon as depicted in Figure 4.1 with j a join-irreducible element is not a sublattice of $L(D_n)$. Consider a join-irreducible subgroup H of D_n. Note that, if H is an atom, then there is no special pentagon containing H as depicted in Figure 4.1. Now, let $H =< a^m >$ where $m = p_0^{\alpha_0} p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_z^{x} \cdots p_r^{\alpha_r}$ for some $p_z \in \{p_0, p_1, \ldots, p_r\}$ and $2 \leq x < \alpha_z$, then $| < a^m > | = p_z^{\alpha_z - x}$. Also, the subgroup $H^- =< a^{mp_z} >$ is the unique lower cover of H with $| < a^{mp_z} > | = p_z^{\alpha_z - x - 1}$.

Suppose that $W =< a^t, a^j b >$ where $t|n$, $0 \leq j \leq t-1$ and a join-irreducible subgroup H are subgroups of D_n such that $H \subseteq H^- \vee W$, i.e., $< a^m > \subseteq < a^{mp_z} > \vee < a^t, a^j b >$ and $H \vee W = H^- \vee W$, i.e., $< a^m > \vee < a^t, a^j b > = < a^{mp_z} > \vee < a^t, a^j b >$. But then, we must have $(m,t) = (mp_z, t) = g$, then the factorization of g does not contain a power of p_z greater than x. As $m = p_0^{\alpha_0} p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_z^{x} \cdots p_r^{\alpha_r}$ and $t|n$ does not contain p_z with power more than x, we have $t | p_0^{\alpha_0} p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_z^{x} \cdots p_r^{\alpha_r}$ which implies $t|m$ and consequently, $H \subseteq W$. The case where W is of Type (1) is analogous to the case W is of Type (2).

Therefore, no such special pentagon as depicted in Figure 4.1 is in $L(D_n)$ for either choice of subgroup W and so $L(D_n)$ is a strong lattice. ■

Remark. Note that $LPrN(D_n)$ is strong since $LPrN(D_n)$ is a sublattice of $L(D_n)$. Though $LH(D_n)$ is not a sublattice of $L(D_n)$ in general, $LH(D_n)$ is a strong lattice as we now show.

Theorem 4.8. *In $LH(D_n)$ only atoms are join-irreducible elements.*

Proof. Let $|D_n| = 2n = 2^{i+1}\prod_{k=1}^{k=r} p_k^{\alpha_k}$ where the p_k's are odd primes and every $\alpha_k \neq 0$.

Case I. Suppose that H is a Hall subgroup of D_n of Type (1). Let $|H| = \prod_{t\in S\subseteq\{1,2,\ldots,k\}} p_t^{\alpha_t}$ such that $2 \nmid |H|$ since H is Hall subgroup. Moreover, if H is join-irreducible in $LH(D_n)$, then the factorization of $|H|$ contains exactly one prime, i.e., $|H| = p_z^{\alpha_z}$ where $z = 1, 2, \ldots, k$ and such a subgroup H is an atom in $LH(D_n)$.

Consequently, a Hall subgroup H of D_n of Type (1) is join-irreducible in $LH(D_n)$ if and only if H is an atom.

Case II. Suppose that K is a Hall subgroup of D_n of Type (2). Note that $|K| = 2^{i+1}\prod_{x\in S\subseteq\{1,2,\ldots,k\}} p_x^{\alpha x}$. Also, if K has order 2^{i+1} then it is a Hall subgroup. Note that nontrivial subgroups of K are not Hall subgroups and so K is join-irreducible in $LH(D_n)$ since it is an atom in $LH(D_n)$. If K has order $2^{i+1}\prod_{x\in S\subseteq\{1,2,\ldots,k\}} p_x^{\alpha x}$, i.e., order containing at least one prime other than 2, then it contains Hall subgroups of order $p_x^{\alpha_x}$ and also of order 2^{i+1}. Therefore, K is not join-irreducible in $LH(D_n)$ which implies that a Hall subgroup of D_n of Type (2) is join-irreducible if and only if its order is 2^{i+1}. Moreover, in $LH(D_n)$, it is an atom.

Therefore, in $LH(D_n)$ only atoms are join-irreducibles. ■

Corollary 4.9. *$LH(D_n)$ is a strong lattice.*

Proof. Immediate in view of Lemma 4.6 and Theorem 4.8. ■

Note that $L(S_2)$ and $L(S_3)$ are strong lattices as neither contains a sublattice isomorphic to a special pentagon as depicted in Figure 4.1 and described in Theorem 4.5. However, for $n \geq 4$ we have the following Theorem.

Theorem 4.10. *$L(S_n)$ is not a strong lattice for $n \geq 4$.*

Proof. In view of Theorem 4.5, in order to show that $L(S_4)$ is not a strong lattice, it is sufficient to show that it does contain a sublattice isomorphic to a special pentagon as depicted in Figure 4.1. Consider the subgroup $J =<(1\ 2\ 3\ 4)>= \{e, (1\ 2\ 3\ 4), (1\ 3)(2\ 4), (1\ 4\ 3\ 2)\}$ which is a join-irreducible subgroup by Lemma

4.3. For the subgroup $M =< (1\,2\,3), (1\,2) >= \{e, (1\,2\,3), (1\,3\,2), (1\,2), (1\,3), (2\,3)\}$ we have, $J \cap M = \{e\} < J^- \cup M >= S_4$ and the set $\{\{e\},\ J^-,\ J,\ M,\ S_4\}$ forms a sublattice isomorphic to a special pentagon as depicted in Figure 4.1 with J^- the unique lower cover of J, $J \vee M = S_4$, $J^- \wedge M = \{e\}$. Consequently, by Theorem 4.5, $L(S_4)$ is not a strong lattice.

Now, observe that S_4 is embedded in every S_n, $n \geq 5$. In view of Lemma 4.3, a subgroup of S_4 which is join-irreducible in $L(S_4)$ is also join-irreducible in every $L(S_n)$, $n \geq 5$. Consequently, $\{\{e\}, J^-, J, M, S_4\}$ also forms a sublattice isomorphic to a special pentagon as depicted in Figure 4.1 in each S_n, $n \geq 5$, and so $L(S_n)$ is not a strong lattice for $n \geq 4$. ■

Note that $L(A_3)$, $L(A_4)$ and $L(A_5)$ are strong lattices as none of these contains a sublattice isomorphic to a special pentagon as depicted in Figure 4.1 and described in Theorem 4.5. However, for $n \geq 6$ we have the following Theorem.

Theorem 4.11. *$L(A_n)$ is not a strong lattice for $n \geq 6$.*

Proof. Consider a subgroup $J =< (1\,2\,3\,4)(5\,6) >= \{e, (1\,2\,3\,4)(5\,6), (1\,3)(2\,4), (1\,4\,3\,2)(5\,6)\}$ of A_6 which is join-irreducible by Lemma 4.3. For the subgroup $M =< (1\,2\,3), (1\,2)(5\,6) >= \{e, (1\,2\,3), (1\,3\,2), (1\,2)(5\,6), (1\,3)(5\,6), (2\,3)(5\,6)\}$ of A_6 we have $J \cap M = \{e\}$, $< J^- \cup M >= S_4^*$, where S_4^* an isomorphic copy of S_4, and the set $\{\{e\},\ J^-, J,\ M,\ S_4^*\}$ forms a sublattice isomorphic to a special pentagon as depicted in Figure 4.1 with $J^- \prec J$, $J \vee M = S_4^*$, $J^- \wedge M = \{e\}$. Consequently, by Theorem 4.5, $L(A_6)$ is not a strong lattice.

Now, observe that A_6 is embedded in every A_n, $n \geq 7$. In view of the Lemma 4.3, a subgroup of A_7 that is join-irreducible in $L(A_6)$ is also a join-irreducible in every $L(A_n)$, $n \geq 7$. Consequently, $\{\{e\}, J^-, J, M, S_4^*\}$ also forms a sublattice isomorphic to a special pentagon as depicted in Figure 4.1 in each A_n, $n \geq 7$, and so $L(A_n)$ is not a strong lattice for $n \geq 6$. ■

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References

[1] G. Birkhoff, Lattice Theory, 3rd Ed. **25** (Amer. Math. Soc. Providence, R.I. 1967).

[2] S.K. Chebolu and K. Lockridge, *Fuchs' problem for dihedral groups*, J. Pure Appl. Alg. **221** (4) (2017) 971–982. https://doi.org/10.1016/j.jpaa.2016.08.015

[3] K. Conrad, Consequences of the Sylow Theorems. https://kconrad.math.uconn.edu/blurbs/grouptheory/sylowapp.pdf

[4] K. Conrad, Dihedral Groups II. https://kconrad.math.uconn.edu/blurbs/grouptheory/dihedral2.pdf

[5] U. Faigle, *Geometries on partially ordered sets*, J. Combin. Theory Ser. B **28** (1980) 26–51.
https://doi.org/10.1016/0095-8956(80)90054-4

[6] G. Grätzer, General Lattice Theory (Academic press, New York, 1978)

[7] P. Hall, *Theorems like Sylow's*, Proc. London Math. Soc. **22** (6) (1956) 286–304.
https://doi.org/10.1112/plms/s3-6.2.286

[8] I.S. Luthar and I.B.S. Passi, Algebra Volume 1: Groups (Narosa Publishing House Pvt. Ltd., New Delhi, 1999).

[9] A. Mann, *A criterion for pronormality*, J. London Math. Soc. **44** (1969) 175–176.
https://doi.org/10.1112/jlms/s1-44.1.175

[10] P.P. Pálfy, *Groups and lattices*, London Math. Soc. Lecture Note Ser. **305** (2003) 428–454.

[11] T.A. Peng, *Pronormality in finite groups*, J. London Math. Soc. **3** (1971) 301–306.
https://doi.org/10.1112/jlms/s2-3.2.301

[12] G. Richter and M. Stern, *Strongness in (semimodular) lattices of finite length*, Wiss. Z. Univ. Halle **33** (1984) 73–77.

[13] J.S. Rose, *Finite soluble groups with pronormal system normalizers*, Proc. London Math. Soc. **17** (3) (1969) 447–469.
https://doi.org/10.1112/plms/s3-17.3.447

[14] R. Schmidt, Subgroup Lattices of Groups, de Gruyter Expositions in Mathematics **14** (de Gruyter, Berlin, 1994).

[15] R. Soloman, *A brief history of the classification of the finite simple groups*, Bull. Amer. Math. Soc. **38** (3) (2001) 315–352.
https://doi.org/10.1090/S0273-0979-01-00909-0

[16] M. Stern, Semimodular Lattices (Cambridge University Press, 1999).

[17] R. Sulaiman, *Subgroups Lattice of symmetric group S_4*, Int. J. Algebra **6** (1) (2012) 29–35.

[18] M. Suzuki, Structure of Groups and Structures of its Lattice of Subgroups (Springer Verlag, Berlin, 1956).

[19] E.P. Vdovin and D.O. Revin, *Pronormality and strong pronormality of subgroups*, Algebra Log. **52** (2013) 15–23.
https://doi.org/10.1007/s10469-013-9215-z

[20] E.P. Vdovin and D.O. Revin, *On the pronormality of Hall subgroups*, Sib. Math. J. **54** (1) (2013) 22–28.
https://doi.org/10.1134/S0037446613010035

[21] P.M. Whitman, *Groups with a cyclic group as a lattice homomorph*, Ann. Math. **49** (2) (1948) 347–351.
https://doi.org/10.2307/1969283

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