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BINARY RELATIONS AND SUBMAXIMAL CLONES DETERMINED BY CENTRAL RELATION

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Abstract

Let ρ be an h-ary central relation ($h \geq 2$) and σ a binary relation on a finite set A such that $\sigma \neq \rho$. It is known from Rosenberg's classification theorem (1965) that the clone $\operatorname{Pol}\rho$ which consists of all operations on A that preserve ρ is a maximal clone on A. In this paper, we find all binary relations σ such that the clone $\operatorname{Pol}\{\rho, \sigma\}$ is a maximal subclone of $\operatorname{Pol}\rho$, where ρ is a fixed central relation.

Keywords: central relations, meet-reducible, meet-irreducible, submaximal, clones.

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1. Introduction

In 1941, Post presented the complete description of the countably many clones on 2 elements. It turned out that, all such clones are finitely generated and the lattice of these clones is countable. The structure of the lattice of clones on finitely many (but more than 2) elements is more complex and is of the cardinality $2^{\aleph_0}$. For $k \geq 3$, not much is known about the structure of the lattice of clones in spite of the efforts made by many researchers in this area. Therefore, every new piece of information is considered valuable. Indeed, it would be very interesting to know the clone lattice on the next level (below the maximal clones) and even a partial description will shed more light onto its structure. The complete description of all submaximal clones is known only for the 2-elements case and the 3-elements case (see [3, 4, 5]), however the result in [3] and many results in the literature on clones including those discussed in [4, 8, 10, 11], require intensive knowledge of submaximal clones (that sit below certain maximal clones) on arbitrary finite sets. Clone theory is considered to be very important because of its use to understand universal algebras.

In [4, Chapter 17], D. Lau presented all submaximal clones of the clone $\operatorname{Pol}\rho$ where ρ is a unary central relation on an arbitrary finite set. Recently in [12] we have characterized the five types of central relation σ such that the clone $\operatorname{Pol}\{\rho,\sigma\}$ is covered by $\operatorname{Pol}\rho$, where ρ is a fixed h-ary central relation ($h \geq 2$) on an arbitrary finite set. In this paper, we characterize the binary relations σ such that the clones $\operatorname{Pol}\{\sigma,\rho\}$ are covered by $\operatorname{Pol}\rho$, where ρ is a fixed h-ary central relation ($h \geq 2$) on a finite set. Moreover, we give a result which will help anyone to decide whether $\operatorname{Pol}\{\rho,\sigma\}$ is a submaximal clone where ρ and σ are as above.

This paper consists of five sections. After this introduction, in which we motivated this research and we announced the types of relations to be characterized in the paper, the second section provides the reader with necessary notions and notations. It is followed by the section dedicated to the description of the types of binary relations σ such that the clones $\operatorname{Pol}\{\sigma,\rho\}$ are covered by $\operatorname{Pol}\rho$. In Section 4, we show that the clones described in Section 3 are maximal and in Section 5, we show that the binary relations σ listed in Section 3 are the only binary relations such that the clones $\operatorname{Pol}\{\rho,\sigma\}$ are maximal below $\operatorname{Pol}\rho$.

2. Preliminaries

In this section, we provide the reader with some basic notions and notations; for more details the reader can see [4, 10, 11, 13].

Let A be a fixed finite set with k elements, n and h be two integers such that $1 \leq n,\ h$. An *n-ary operation* on A is a function $f : A^n \to A$. We will use the notation $O_A^{(n)}$ for the set of all n-ary operations on A, and O_A for the set of all finitary operations on A. For $\mathcal{C} \subseteq O_A$, $\mathcal{C}^{(n)}$ denoted the set $\mathcal{C} \cap O_A^{(n)}$. For $1 \leq i \leq n$, the *i-th projection* is the operation $\pi_i^{(n)} : A^n \to A, (a_1, \ldots, a_n) \mapsto a_i$. For arbitrary positive integers m and n, there is a one-to-one correspondence between the functions $f : A^n \to A^m$ and the m-tuples $\boldsymbol{f} = (f_1, \ldots, f_m)$ of functions $f_i : A^n \to A$ (for $i = 1, \ldots, m$) via $f \mapsto \boldsymbol{f} = (f_1, \ldots, f_m)$ with $f_i = \pi_i^{(m)} \circ f$ for all $i = 1, \ldots, m$. In particular, $\pi^{(n)} = (\pi_1^{(n)}, \ldots, \pi_n^{(n)})$ corresponds to the identity function $f : A^n \to A^n$. From now on, we will identify each function

$f : A^n \to A^m$ with the corresponding m-tuples $\boldsymbol{f} = (f_1, \ldots, f_m) \in (\mathcal{O}_A^{(n)})^m$ of n-ary operations. Using this convention, the composition of two functions $\boldsymbol{f} = (f_1, \ldots, f_m) : A^n \to A^m$ and $\boldsymbol{g} = (g_1, \ldots, g_p) : A^m \to A^p$ can be described as follows $\boldsymbol{g} \circ \boldsymbol{f} = (g_1 \circ \boldsymbol{f}, \ldots, g_p \circ \boldsymbol{f}) = (g_1(f_1, \ldots, f_m), \ldots, g_p(f_1, \ldots, f_m))$ where $g_i(f_1, \ldots, f_m)(\boldsymbol{a}) = g_i(f_1(\boldsymbol{a}), \ldots, f_m(\boldsymbol{a}))$ for all $\boldsymbol{a} \in A^n$ and $1 \leq i \leq p$.

A *clone* on A is a subset $\mathcal{C}$ of O_A that contains the projections and is closed under composition; that is $\pi_i^{(n)} \in \mathcal{C}$ for all $n \geq 1$ and $1 \leq i \leq n$, and $g \circ \boldsymbol{f} \in C^{(n)}$ whenever $g \in C^{(m)}$ and $\boldsymbol{f} \in (C^{(n)})^m$ (for $m, n \geq 1$). The clones on A form a complete lattice $\mathcal{L}_A$ under inclusion. Therefore, for each set $F \subseteq O_A$ of operations, there exists a smallest clone that contains F, which will be denoted by $\langle F \rangle$ and will be called *clone generated* by F.

For two clones C and D on A, we say that C is *maximal in* D if D covers C in $\mathcal{L}_A$. We also say that C is *submaximal* if C is maximal in a clone D and D is a maximal clone on A. For a maximal clone D, there are two types of clones C being maximal in D: C is *meet-reducible* if $C = D \cap F$ for a maximal clone F distinct from D (but not necessarily unique) and C is *meet-irreducible* if it is not meet-reducible.

Clones can be described via invariant relations. An *h-ary relation* on A is a subset of A^h. The set of finitary relations on A is denoted by R_A. For an n-ary operation $f \in O_A^{(n)}$ and an h-ary relation ρ on A, we say that f *preserves* ρ (or ρ is *invariant under* f, or f is a *polymorphism* of ρ) if for all $(a_{1,i}, \ldots, a_{h,i}) \in \rho, i = 1, \ldots, n$, $(f(a_{1,1}, \ldots, a_{1,n}), f(a_{2,1}, \ldots, a_{2,n}) \ldots, f(a_{h,1}, \ldots, a_{h,n})) \in \rho$. For any set $R \subseteq R_A$, $\mathrm{Pol}(R)$ is the set of operations on A preserving every relation on R, and for $F \subseteq O_A$ $\mathrm{Inv}(F)$ is the set of relations preserved by every operation on F. If $R = \{\rho\}$, we write $\mathrm{Pol}\,\rho$ for $\mathrm{Pol}\{\rho\}$. If A is finite, it is well known that Pol and Inv determine a Galois connection between the subsets of O_A and R_A, with closure operator $F \mapsto \mathrm{Pol}\,\mathrm{Inv}\,F$ on O_A and $R \mapsto [R] = \mathrm{Inv}\,\mathrm{Pol}(R)$ on R_A. The closed sets of operations are exactly the clones and the closed set of relations are called *relational algebras* or *relational clone* [9]. The set of relational algebras, ordered by inclusion is a lattice, which is dually isomorphic to the lattice L_A of clones on A. The relational algebras $[R]$ can be described in various ways ([4, 9]).

Let $\rho \subseteq A^h$; for an integer $m > 1$ and $\boldsymbol{a}_i = (a_{1,i}, \ldots, a_{m,i}) \in A^m, 1 \leq i \leq h$, we will write $(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_h) \in \rho$ if for all $j \in \{1, \ldots, m\}, (a_{j,1}, \ldots, a_{j,h}) \in \rho$. If A is finite, every clone on A other than O_A is contained in a maximal clone. An operation $g : A^3 \to A$ is called a *majority operation* if $g(a, a, b) = g(a, b, a) = g(b, a, a) = a$ for $a, b \in A$. We recall the following Baker-Pixley Theorem and the Rosenberg's list of maximal clones which will be used to prove some results.

Theorem 2.1 [1]**.** *For a finite algebra $\mathcal{A} = (A, F)$ with a majority term operation, an operation $f : A^n \to A$, is a term operation of A iff f preserves all subuniverses of $\mathcal{A}^2$.*

Theorem 2.2 [7]**.** *For each finite set A with $Card(A) \geq 3$, the maximal clones on A are the clones of the form* $\operatorname{Pol}\rho$ *where ρ is a relation of one of the following six types:*

(1) *a bounded partial order on A;*

(2) *a prime permutation on A;*

(3) *a prime affine relation on A;*

(4) *a nontrivial equivalence relation on A;*

(5) *a central relation on A;*

(6) *an h-regular relation on A.*

Here a partial order on A is called *bounded* if it has both a least and a greatest element. A *prime permutation* on A is (the graph of) a fixed point free permutation on A in which all cycles are of the same prime length, and a *prime affine relation* on A is the graph of the ternary operation $x - y + z$ for some elementary abelian p-group $(A; +, -, 0)$ on A (for p prime). An equivalence relation on A is *nontrivial* if it is neither the equality relation Δ_A on A nor the full relation A^2 on A.

To describe central relations and h-regular relations, we call an h-ary relation ρ on A *totally reflexive* (*reflexive* for $h = 2$) if ρ contains all h-tuples of A^h whose coordinates are not pairwise distinct, and *totally symmetric* (*symmetric* for $h = 2$) if ρ is invariant under any permutation of its coordinates. If ρ is totally reflexive and totally symmetric, we define the *center* of ρ, denoted by C_ρ, as follows

$$C_\rho = \{a \in A : (a, a_2, \ldots, a_h) \in \rho \text{ for all } a_2, \ldots, a_h \in A\}.$$

We say that ρ is a *central relation* if ρ is totally reflexive, totally symmetric and has a nonvoid center which is a proper subset of A. For an integer $h \geq 3$, a family $T = \{\theta_1, \ldots, \theta_r\}$ $(r \geq 1)$ of equivalence relations on A is called *h-regular* if each θ_i (for $1 \leq i \leq r$) has exactly h classes, and for arbitrary classes B_i of θ_i $(1 \leq i \leq r)$, the intersection $B_1 \cap B_2 \cap \ldots \cap B_r$ is nonempty. To each h-regular family $T = \{\theta_1, \ldots, \theta_r\}$ of equivalence relations on A, we associate an h-ary relation λ_T on A as follows

$$\lambda_T = \{(a_1, \ldots, a_h) \in A^h : (\forall i)(\exists\, p, q) p \neq q \text{ and } (a_p, a_q) \in \theta_i\}.$$

The relations of the form λ_T are called *h-regular* (or *h-generated*) relations. It is clear from the definition that h-regular relations are totally reflexive and totally symmetric. We recall the folowing classical construction. If α and β are two binary relation on A, the relational product of α and β, denoted by $\alpha \circ \beta$, is the set $\{(x, y) \in A^2 : (x, u) \in \alpha, (u, y) \in \beta \text{ for some } u \in A\}$. The relational product is an associative binary operation on the set $R_A^{(2)}$ of binary relations on A. For $n \geq 1$, we denote by α^n the n-th power $\alpha \circ \cdots \circ \alpha$ (n times) of α and by $tr(\alpha)$ the transitive closure of α. It is easy to see that $tr(\alpha) = \bigcup_{n \geq 1} \alpha^n$. We will denote by $\underline{h}$ the set $\{1, \ldots, h\}$.

Definition 2.3 ([4], Page 126)**.** Let $h \in \mathbb{N} \setminus \{0\}$. An h-ary relation $\sigma \in R_A^{(h)}$ is called *diagonal relation* if there exists an equivalence relation ϵ on $\{1, \ldots, h\}$ such that $\sigma := \{(a_1, \ldots, a_h) \in A^h : (i, j) \in \epsilon \implies a_i = a_j\}$.

The set of all diagonal relations on A is denoted by D_A and $D_A = \{\emptyset\} \cup \bigcup_{h \geq 1} D_A^{(h)}$, where $D_A^{(h)}$ is the set of all h-ary diagonal relation on A. In particular, A^h and $\delta^h = \{(x, x, \ldots, x) \in A^h : x \in A\}$ are diagonal relations. For more information on diagonal relations, see [4], Page 126.

Remark 2.4 ([4], Theorem 2.6.2, 2.6.3 Page 132)**.** Let $R \subseteq R_A$.

1. If $f \in \operatorname{Pol} R$, then $f \in \operatorname{Pol}([R])$.
2. If σ and σ' are relations such that $\sigma' \in [\{\sigma\}]$, then $\operatorname{Pol} \sigma \subseteq \operatorname{Pol} \sigma'$.

From now on, we assume that we are working on the set $E_k = \{0, 1, \ldots, k-1\}$, where k is an integer such that $k > 1$. We will denote by S_h the set of permutations on $\underline{h}$, where $h \geq 1$ is an integer and for $1 \leq i_1 < \cdots < i_h \leq k$, we denote by $S_{\{i_1, \ldots, i_h\}}$ the set of permutations on $\{i_1, \ldots, i_h\}$; $\gamma_{i_1, \ldots, i_h}$ the set $\{(\tau(i_1), \ldots, \tau(i_h)) : \tau \in S_{\{i_1, \ldots, i_h\}}\}$, and ι_k^h the set $\{(a_1, \ldots, a_h) \in E_k^h : \exists i \neq j, a_i = a_j\}$. It is well known (see [4]) that the Słupecki clone $\operatorname{Pol} \iota_k^k$ is a maximal clone. If σ is an equivalence relation, we denote by $[a]_\sigma$ the σ-class of a.

3. THE TYPES OF σ SUCH THAT THE CLONE $\operatorname{Pol}\{\rho, \sigma\}$ IS MAXIMAL IN $\operatorname{Pol} \rho$

In this section, we give the definition of the types of binary relations σ such that $\operatorname{Pol}\{\rho, \sigma\}$ is maximal in $\operatorname{Pol} \rho$ and the main result of this paper. Let k and h be two integers such that $k \geq 3$ and $h \geq 2$. For a prime permutation π of order p on E_k, we denote by σ_π the equivalence relation consisting of pairs $(a, b) \in E_k^2$ with $a = \pi^i(b)$ for some $0 \leq i < p$.

Definition 3.1. Let σ be a binary relation and ρ an h-ary central relation on $E_k (h \geq 2)$.

(i) A nonempty subset $B \subseteq E_k$ is called a *ρ-chain* if $B^h \subseteq \rho$. A ρ-chain B is called *maximal ρ-chain* if it is not contained in another ρ-chain.

(ii) We say that ρ is *σ-closed* if $(a_1, \ldots, a_h) \in \rho$ whenever $(u_1, \ldots, u_h) \in \rho$ for some $u_1, \ldots, u_h$ with $(a_i, u_i) \in \sigma, 1 \leq i \leq h$.

(iii) We suppose that ρ has t maximal ρ-chains $A_0, \ldots, A_{t-1}$ ($t \geq 2$). We say that σ is a central relation relative to maximal ρ-chains if σ is reflexive, symmetric, and for each $i \in E_t$, there exists a central element c_i of ρ such that for every $a \in A_i, (c_i, a) \in \sigma$.

(iv) If $h = 2$, then we say that ρ is the symmetric part of σ if $\rho = \sigma \cap \sigma^{-1}$.

For $h = 2$ and $\sigma \nsubseteq \rho$, we set $\lambda = \sigma \cap \rho$. For $h \geq 2$ we denote by σ_h(respectively σ'_h) the h-ary relation $\sigma_h = \{(a_1, \ldots, a_h) \in E_k^h : \exists\, u \in E_k, (a_1, u), \ldots, (a_h, u) \in \sigma\}$, $\sigma'_h = \{(a_1, \ldots, a_h) \in E_k^h : \exists\, u \in E_k, (u, a_1), \ldots, (u, a_h) \in \sigma\}$ and for every permutation π on $\underline{h}$, $(\rho)_\pi = \{(a_{\pi(1)}, \ldots, a_{\pi(h)}) : (a_1, \ldots, a_h) \in \rho\}$.

Here we state the main result of this paper.

Theorem 3.2. *Let k, h be two integers such that $k \geq 3$, $h \geq 2$; let ρ be an h-ary central relation with t maximal ρ-chains $A_0, \ldots, A_{t-1}$ and σ a binary relation on E_k. The clone* $\mathrm{Pol}\{\rho, \sigma\}$ *is maximal below* $\mathrm{Pol}\,\rho$ *if and only if σ fulfills one of the following eleven conditions:*

(I) *σ is a nontrivial equivalence relation and ρ is σ-closed;*

(II) *σ is a nontrivial equivalence relation and every σ-class contains a central element of ρ;*

(III) *σ is a bounded partial order with least element $\perp$, greatest element $\top$, $h = 2$, $\{\perp, \top\} \subseteq C_\rho$ and $tr(\sigma \cap \rho) = \sigma$;*

(IV) *σ is a bounded partial order with least element $\perp$, greatest element $\top$, $h \geq 3$ and $\{\perp, \top\} \subseteq C_\rho$;*

(V) *σ is a central relation, $h = 2$ and ρ and σ are comparable (i.e., $\rho \subsetneq \sigma$ or $\sigma \subsetneq \rho$);*

(VI) *σ is a central relation, $h \geq 3$ and $C_\rho \cap C_\sigma \neq \emptyset$;*

(VII) *σ is the graph of a prime permutation π and ρ is σ_π-closed;*

(VIII) *$\sigma_h = \rho$ and σ is a central relation relative to maximal ρ-chains;*

(IX) *$\rho \neq \sigma$ and ρ is the symmetric part of σ i.e., $\rho = \sigma \cap \sigma^{-1}$;*

(X) *σ is a partial order with a least element $\perp$ which is also a central element of ρ, $\sigma_2 = \rho$ and for every $i_1, \ldots, i_l \in \{0, \ldots, t-1\}$, $\bigcap\limits_{1 \leq j \leq l} A_{i_j}$ has a greatest element;*

(XI) *σ is a partial order with a greatest element $\top$ which is also a central element of ρ, $\sigma'_2 = \rho$ and for every $i_1, \ldots, i_l \in \{0, \ldots, t-1\}, \bigcap_{1 \leq j \leq l} A_{i_j}$ has a least element.*

Definition 3.3. Let $l \in \{I, \ldots, XI\}$. We say that σ is of type l if σ verifies the condition (l) of Theorem 3.2.

The proof of Theorem 3.2 is divided into two parts. The sufficiency of conditions in Propositions 4.1, 4.3, 4.8 and Corollary 4.5 and the necessity of conditions in Propositions 5.1, 5.9, 5.19 and Corollary 5.17. Since ρ has at least two maximal ρ-chains, the relation of type VIII exists only for $|C_\rho| \geq 2$. We need the following proposition for many proof in the sequel.

Proposition 3.4. *Let E_k be a finite set, ρ be an h-ary central relation, B a maximal ρ-chain and σ a diagonal relation on E_k, then* (1) $C_\rho \subseteq B$ *and* (2) $\mathrm{Pol}\,\sigma = O_{E_k}$.

Proof. (1) Let B be a maximal ρ-chain and $c \in C_\rho$. As $c \in C_\rho$, for any $a_1, \ldots, a_{h-1} \in B, (c, a_1, \ldots, a_{h-1}) \in \rho$. Hence $B \cup \{c\}$ is a ρ-chain and $B \subseteq B \cup \{c\}$. The maximality of B yields $c \in B$. Thus $C_\rho \subseteq B$. It is easy to check that (2) holds. ■

4. PROOF OF SUFFICIENCY CRITERION IN THEOREM 3.2

In this section we show that the clones listed in Theorem 3.2 are maximal below $\mathrm{Pol}\,\rho$. We distinguish four cases.

Case 1. σ is of type $l \in \{I, II, III, IV, VIII, X, XI\}$;

Case 2. σ is of type V or VI;

Case 3. σ is of type VII;

Case 4. σ is of type IX. We begin with Case 1.

Proposition 4.1. *Let k, h be two integers such that $k \geq 3$, $h \geq 2$, $l \in \{I, II, III, IV, VIII, X, XI\}$, ρ be an h-ary central relation and σ a binary relation on E_k. If σ is of type l, then the clone* $\mathrm{Pol}\{\rho, \sigma\}$ *is maximal in* $\mathrm{Pol}\,\rho$.

Before the proof of Proposition 4.1, we give some useful properties of σ. Let $g \in \mathrm{Pol}\,\rho \setminus \mathrm{Pol}\,\sigma$ be an n-ary operation. Then there exist $(a_1, b_1), \ldots, (a_n, b_n) \in \sigma$ such that $(g(\boldsymbol{a}), g(\boldsymbol{b})) \notin \sigma$, where $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$.

In the case when $l = III$, we may furthermore assume that $(a_1, b_1), \ldots, (a_n, b_n) \in \sigma \cap \rho$. This can be seen as follows. Write $\lambda := \sigma \cap \rho$. Since $tr(\lambda) = \sigma$ and λ is reflexive, there exists $q \geq 1$ such that $tr(\lambda) = \lambda^q = \sigma$. Moreover, for $i \geq 1$, $\mathrm{Pol}\,\lambda^i \subseteq \mathrm{Pol}\,\lambda^{i+1}$; in particular, $\mathrm{Pol}\,\lambda \subseteq \mathrm{Pol}\,\sigma$. Since $g \in \mathrm{Pol}\,\rho \setminus \mathrm{Pol}\,\sigma$, it follows that $g \notin \mathrm{Pol}\,\lambda$. Therefore there exist $(a_1, b_1), \ldots, (a_n, b_n) \in \lambda$ such that $(g(\boldsymbol{a}), g(\boldsymbol{a})) \notin \lambda$, where $\boldsymbol{a} := (a_1, \ldots, a_n), \boldsymbol{b} := (b_1, \ldots, b_n)$. Since $g \in \mathrm{Pol}\,\rho$, we have $(g(\boldsymbol{a}), g(\boldsymbol{b})) \in \rho$, so we most have $(g(\boldsymbol{a}), g(\boldsymbol{b})) \notin \sigma$.

Note that if $l \in \{I, II, VIII, X, XI\}$ and $h = 2$, then we get $\sigma \subsetneq \rho$ (by definition of σ_2 and σ'_2 and the reflexivity of σ). Note also that relation of type IV not occur with $h = 2$.

Lemma 4.2. *Let n, k be two integers such that $n \geq 1$ and $k \geq 3$, ρ be an h-ary central relation ($h \geq 2$) and σ be a binary relation on E_k. Furthermore let $g \in \operatorname{Pol}\rho \setminus \operatorname{Pol}\sigma$ be an n-ary operation and $l \in \{I, II, III, IV, VIII, X, XI\}$. If σ is of type l, then for all $c, d \in E_k$ such that $(c, d) \in \sigma$ and $c \neq d$, there exists a unary operation $f_{cd} \in \langle(\operatorname{Pol}\{\sigma, \rho\}) \cup \{g\}\rangle$ such that $(f_{cd}(c), f_{cd}(d)) \notin \sigma$.*

Proof. Let $c, d \in E_k$ such that $c \neq d$ and $(c, d) \in \sigma$; choose (a_i, b_i) as specified in the paragraph following Proposition 4.1. We will construct a unary operation $f_{cd} \in \langle(\operatorname{Pol}\{\sigma, \rho\}) \cup \{g\}\rangle$ such that $(f_{cd}(c), f_{cd}(d)) \notin \sigma$. If σ is of type I, II or VIII, then we consider the unary operations f^i_{cd}, $1 \leq i \leq n$ defined on E_k by $f^i_{cd}(x) = a_i$ if $x = c$ and $f^i_{cd}(x) = b_i$ otherwise. If σ is of type III, IV, X or XI, then for all $1 \leq i \leq n$, we consider the unary operations f^i_{cd} defined on E_k by $f^i_{cd}(x) = a_i$ if $(x, c) \in \sigma$ and $f^i_{cd}(x) = b_i$ otherwise. Using the observation below Proposition 4.1, and the (total) reflexivity of ρ, (total) symmetry of ρ, reflexivity and symmetry of σ (for type $I, II, VIII$) and reflexivity and transitivity of partial order (for types III, IV, X, XI) and the fact that $(a_i, b_i) \in \sigma \cap \rho$ for $h = 2$, we see that $f^i_{c,d} \in \operatorname{Pol}\{\sigma, \rho\}$. Setting $f_{cd}(x) = g(f^1_{cd}(x), \ldots, f^n_{cd}(x))$, we have $(f_{cd}(c), f_{cd}(d)) = (g(a_1, \ldots, a_n), g(b_1, \ldots, b_n)) \notin \sigma$ and $f_{cd} \in \langle(\operatorname{Pol}\{\rho, \sigma\}) \cup \{g\}\rangle$. ■

Now, we give the proof of Proposition 4.1.

Proof. Let $g \in \operatorname{Pol}\rho \setminus \operatorname{Pol}\{\sigma, \rho\}$ be an n-ary operation. We show that $\langle(\operatorname{Pol}\{\sigma, \rho\}) \cup \{g\}\rangle = \operatorname{Pol}\rho$. We have $\langle(\operatorname{Pol}\{\sigma, \rho\}) \cup \{g\}\rangle \subseteq \operatorname{Pol}\rho$. It remains to show that $\operatorname{Pol}\rho \subseteq \langle(\operatorname{Pol}\{\sigma, \rho\}) \cup \{g\}\rangle$. Let $f \in \operatorname{Pol}\rho$ be an m-ary operation on E_k. From Lemma 4.2, we can see that for $\boldsymbol{e}, \boldsymbol{d} \in E_k^m$ such that $(\boldsymbol{e}, \boldsymbol{d}) \in \sigma$ and $\boldsymbol{e} \neq \boldsymbol{d}$, there exists $1 \leq i \leq m$ such that $c_i \neq d_i$; the operation $f_{\boldsymbol{e},\boldsymbol{d}} := f_{c_i, d_i} \circ \pi^m_i$ where f_{c_i,d_i} is the unary operation provided by Lemma 4.2, is an m-ary operation belonging to $\langle(\operatorname{Pol}\{\sigma, \rho\}) \cup \{g\}\rangle$ such that $(f_{\boldsymbol{ed}}(\boldsymbol{e}), f_{\boldsymbol{ed}}(\boldsymbol{d})) = (g(a_1, \ldots, a_n), g(b_1, \ldots, b_n)) \notin \sigma$.

We set $S = \{f_{\boldsymbol{ed}} : \boldsymbol{e}, \boldsymbol{d} \in E_k^m, \boldsymbol{e} \neq \boldsymbol{d}, (\boldsymbol{e}, \boldsymbol{d}) \in \sigma\}$; for reason of simple notation we set $S = \{f_i : 1 \leq i \leq l\}$($l = \operatorname{Card} S$) and we consider the map $\operatorname{ext} : E_k^m \to E_k^{m+l}$ defined by $\operatorname{ext}(\boldsymbol{x}) = (\boldsymbol{x}, f_1(\boldsymbol{x}), \ldots, f_l(\boldsymbol{x}))$. Let $\boldsymbol{x}, \boldsymbol{y} \in E_k^m$ such that $\boldsymbol{x} \neq \boldsymbol{y}$. If $(\boldsymbol{x}, \boldsymbol{y}) \in \sigma$, then $(f_{\boldsymbol{xy}}(\boldsymbol{x}), f_{\boldsymbol{xy}})(\boldsymbol{y})) \notin \sigma$, if $(\boldsymbol{x}, \boldsymbol{y}) \notin \sigma$, then by definition of ext we have $(\operatorname{ext}(\boldsymbol{x}), \operatorname{ext}(\boldsymbol{y})) \notin \sigma$. Thus for $\boldsymbol{x} \neq \boldsymbol{y}$, $(\operatorname{ext}(\boldsymbol{x}), \operatorname{ext}(\boldsymbol{y})) \notin \sigma$. Furthermore we define an operation H on the range $\{\operatorname{ext}(\boldsymbol{x}) : \boldsymbol{x} \in E_k^m\}$ of ext by $H(\operatorname{ext}(\boldsymbol{x})) = f(\boldsymbol{x})$.

Now, taking into account the different values of l, we construct an extension $\tilde{H}$ of H on E_k^{m+l} belonging to $\operatorname{Pol}\{\rho, \sigma\}$.

(i) If σ is of type I, we choose and fix $T = \{e_1, \ldots, e_q\}$ (where q is the number of σ-classes) such that $(e_i, e_j) \notin \sigma$ for $1 \leq i < j \leq q$, and we define α from E_k to $\{1, \ldots, q\}$ by $\alpha(a) = i$ if $(a, e_i) \in \sigma$. Hence we construct an extension $\tilde{H}$ of H on E_k^{m+l} as follows. Let $\boldsymbol{y} = (y_1, \ldots, y_{m+l}) \in E_k^{m+l}$, set

$$\tilde{H}(\boldsymbol{y}) = \begin{cases} f(\boldsymbol{z}) & \text{if } \exists \boldsymbol{z} \in E_k^m, \boldsymbol{y} = \operatorname{ext}(\boldsymbol{z}); \\ f(\boldsymbol{u}) & \text{if } \forall \boldsymbol{z} \in E_k^m, \operatorname{ext}(\boldsymbol{z}) \neq \boldsymbol{y} \wedge \exists \boldsymbol{u} \in E_k^m, \\ & \quad (\operatorname{ext}(\boldsymbol{u}), \boldsymbol{y}) \in \sigma; \\ f(e_{\alpha(y_1)}, \dots, e_{\alpha(y_m)}) & \text{elsewhere.} \end{cases}$$

The function $\tilde{H}$ is well defined. We will show that $\tilde{H} \in \operatorname{Pol}\{\sigma, \rho\}$. First we show that $\tilde{H} \in \operatorname{Pol}\sigma$. Using reflexivity, symmetry and transitivity of σ it is easy to see that $\tilde{H} \in \operatorname{Pol}\sigma$. Second we show that $\tilde{H} \in \operatorname{Pol}\rho$. We can see that for any $\boldsymbol{y} \in E_k^{m+l}$, there exists $\boldsymbol{v} \in E_k^m$ such that $\tilde{H}(\boldsymbol{y}) = f(\boldsymbol{v})$ and $((y_1, \dots, y_m), \boldsymbol{v}) \in \sigma$ $(*)$. Let $\boldsymbol{x}_i = (x_{i,1}, \dots, x_{i,h}) \in \rho$, $1 \le i \le m + l$. For $j = 1, \dots, h$, we set $\boldsymbol{d}_j = (x_{1,j}, \dots, x_{m+l,j})$ and $\boldsymbol{d}'_j = (x_{1,j}, \dots, x_{m,j})$. It is clear that $(\boldsymbol{d}_1, \dots, \boldsymbol{d}_h)$, $(\boldsymbol{d}'_1, \dots, \boldsymbol{d}'_h) \in \rho$. From $(*)$, there exist $\boldsymbol{v}_j = (v_{1,j}, \dots, v_{m,j})$, such that $\tilde{H}(\boldsymbol{d}_j) = f(\boldsymbol{v}_j)$ and $((x_{1,j}, \dots, x_{m,j}), \boldsymbol{v}_j) = (\boldsymbol{d}'_j, \boldsymbol{v}_j) \in \sigma$ for $1 \le j \le h$. Hence $\boldsymbol{u}_1 = (v_{1,1}, \dots, v_{1,h}), \dots, \boldsymbol{u}_m = (v_{m,1}, \dots, v_{m,h}) \in \rho$ (due to ρ σ-closed, $(\boldsymbol{d}'_j, \boldsymbol{v}_j) \in \sigma, 1 \le j \le h$ and $(\boldsymbol{d}'_1, \dots, \boldsymbol{d}'_h) \in \rho$). Since $f \in \operatorname{Pol}\rho$, we have $f(\boldsymbol{u}_1, \dots, \boldsymbol{u}_h) = (f(\boldsymbol{v}_1), \dots, f(\boldsymbol{v}_h)) \in \rho$. Therefore $(\tilde{H}(\boldsymbol{d}_1), \dots, \tilde{H}(\boldsymbol{d}_h)) = (f(\boldsymbol{v}_1), \dots, f(\boldsymbol{v}_h)) = f(\boldsymbol{u}_1, \dots, \boldsymbol{u}_m) \in \rho$. Thus $\tilde{H} \in \operatorname{Pol}\rho$. Hence $\tilde{H} \in \operatorname{Pol}\{\rho, \sigma\}$.

(ii) If σ is of type II, for every $a \in E_k$, we set $c_{[a]_\sigma} = \min(C_\rho \cap [a]_\sigma)$ (where E_k is ordered by the natural order of $\mathbb{N}$). Set

$$\tilde{H}(\boldsymbol{y}) = \begin{cases} f(\boldsymbol{u}) & \text{if } \exists \boldsymbol{u} \in E_k^m,\ \boldsymbol{y} = \operatorname{ext}(\boldsymbol{u}); \\ c_{[f(\boldsymbol{u})]_\sigma} & \text{if } \forall \boldsymbol{z} \in E_k^m, \operatorname{ext}(\boldsymbol{z}) \neq \boldsymbol{y} \wedge \exists \boldsymbol{u} \in E_k^m, \\ & \quad (\operatorname{ext}(\boldsymbol{u}), \boldsymbol{y}) \in \sigma; \\ c_{[f(c_{[y_1]_\sigma}, \dots, c_{[y_m]_\sigma})]_\sigma} & \text{elsewhere.} \end{cases}$$

The function $\tilde{H}$ is well defined. Using the reflexivity and the transitivity of partial order we can show that $\tilde{H} \in \operatorname{Pol}\sigma$. It remains to show that $\tilde{H} \in \operatorname{Pol}\rho$. Using the fact that $f \in \operatorname{Pol}\rho$, $(\operatorname{ext}(\boldsymbol{u}), \operatorname{ext}(\boldsymbol{v})) \in \sigma$ iff $\boldsymbol{u} = \boldsymbol{v}$, and $c_{[f(\boldsymbol{u})]_\sigma}$ a central element of ρ for $\boldsymbol{u} \in E_k^m$, we obtain $\tilde{H} \in \operatorname{Pol}\rho$. Thus $\tilde{H} \in \operatorname{Pol}\{\rho, \sigma\}$.

(iii) If σ is of type III or IV, then we set

$$\tilde{H}(\boldsymbol{y}) = \begin{cases} f(\boldsymbol{u}) & \text{if } \exists \boldsymbol{u} \in E_k^m, \boldsymbol{y} = \operatorname{ext}(\boldsymbol{u}); \\ \top & \text{if } \forall \boldsymbol{z} \in E_k^m, \operatorname{ext}(\boldsymbol{z}) \neq \boldsymbol{y} \wedge \exists \boldsymbol{u} \in E_k^m, (\operatorname{ext}(\boldsymbol{u}), \boldsymbol{y}) \in \sigma; \\ \bot & \text{elsewhere.} \end{cases}$$

The function $\tilde{H}$ is well defined. Using the reflexivity and transitivity of partial order one can show that $\tilde{H} \in \operatorname{Pol}\sigma$. Since $\{\bot, \top\} \subseteq C_\rho$ and $f \in \operatorname{Pol}\rho$, it is easy to show that $\tilde{H} \in \operatorname{Pol}\rho$. Thus $\tilde{H} \in \operatorname{Pol}\{\rho, \sigma\}$.

(iv) If σ is of type VIII, then for $\boldsymbol{y} \in E_k^{m+l} \setminus \mathrm{ext}(E_k^m)$ set $D_{\boldsymbol{y}} = \{f(\boldsymbol{x}) : (\mathrm{ext}(\boldsymbol{x}), \boldsymbol{y}) \in \sigma\}$. If $\boldsymbol{b}_1, \dots, \boldsymbol{b}_h \in D_{\boldsymbol{y}}$, then there exist $\boldsymbol{x}_1, \dots, \boldsymbol{x}_h \in E_k^m$ such that $\boldsymbol{b}_i = f(\boldsymbol{x}_i)$ and $(\mathrm{ext}(\boldsymbol{x}_i), \boldsymbol{y}) \in \sigma$ for all $i \in \underline{h}$. Consequently, $(\boldsymbol{x}_1, \dots, \boldsymbol{x}_h) \in \sigma_h = \rho$. Since $f \in \mathrm{Pol}\,\rho$, it follows that $(\boldsymbol{b}_1, \dots, \boldsymbol{b}_h) = (f(\boldsymbol{x}_1), \dots, f(\boldsymbol{x}_h)) \in \rho$. Therefore $D_{\boldsymbol{y}}^h \subseteq \rho$; so $D_{\boldsymbol{y}}$ is a ρ-chain. Hence we set $\tau(\boldsymbol{y}) = \min\{j : D_{\boldsymbol{y}} \subseteq A_j\}$ for $\boldsymbol{y} \in E_k^{m+l} \setminus \mathrm{ext}(E_k^m)$ and we fix $c_i \in A_i \cap C_\rho$ such that for every $a \in A_i$ $(a, c_i) \in \sigma$. We set

$$\tilde{H}(\boldsymbol{y}) = \begin{cases} f(\boldsymbol{u}) & \text{if } \exists \boldsymbol{u} \in E_k^m, \boldsymbol{y} = \mathrm{ext}(\boldsymbol{u}); \\ c_{\tau(\boldsymbol{y})} & \text{if } \forall \boldsymbol{z} \in E_k^m, \mathrm{ext}(\boldsymbol{z}) \neq \boldsymbol{y} \wedge \exists \boldsymbol{u} \in E_k^m, (\mathrm{ext}(\boldsymbol{u}), \boldsymbol{y}) \in \sigma; \\ c_0 & \text{elsewhere.} \end{cases}$$

The function $\tilde{H}$ is well defined. We show that $\tilde{H} \in \mathrm{Pol}\,\sigma$. Let $\boldsymbol{y}_1 = (u_1, \dots, u_{m+l})$, $\boldsymbol{y}_2 = (v_1, \dots, v_{m+1}) \in E_k^{m+l}$ such that $(\boldsymbol{y}_1, \boldsymbol{y}_2) \in \sigma$. We show that $(\tilde{H}(\boldsymbol{y}_1), \tilde{H}(\boldsymbol{y}_2)) \in \sigma$. If $\boldsymbol{y}_1 = \boldsymbol{y}_2$ we are done, because σ is reflexive. Assume that $\boldsymbol{y}_1 \neq \boldsymbol{y}_2$. We distinguish two cases.

Case 1. $\boldsymbol{y}_1 = \mathrm{ext}(\boldsymbol{u}_1)$ and for all $\boldsymbol{z} \in E_k^m$, $\mathrm{ext}(\boldsymbol{z}) \neq \boldsymbol{y}_2$, then the definition of $\tilde{H}$ yields that $\tilde{H}(\boldsymbol{y}_2) = c_{\tau(\boldsymbol{y}_2)}$; so $(\tilde{H}(\boldsymbol{y}_1), \tilde{H}(\boldsymbol{y}_2)) = (f(\boldsymbol{u}_1), c_{\tau(\boldsymbol{y}_2)}) \in \sigma$.

Case 2. $(\tilde{H}(\boldsymbol{y}_1), \tilde{H}(\boldsymbol{y}_2)) \in \{(c_0, c_0), (c_{\tau(\boldsymbol{y}_1)}, c_{\tau(\boldsymbol{y}_2)}), (c_{\tau(\boldsymbol{y}_1)}, c_0), (c_0, c_{\tau(\boldsymbol{y}_2)})\} \subseteq \sigma$ because $C_\rho \subseteq A_i, 0 \leq i \leq t-1$. Thus $\tilde{H} \in \mathrm{Pol}\,\sigma$.

(v) If σ is of type X, then for $\boldsymbol{y} \in E_k^{m+l} \setminus \mathrm{ext}(E_k^m)$ set also $D_{\boldsymbol{y}} = \{f(\boldsymbol{x}) : (\mathrm{ext}(\boldsymbol{x}), \boldsymbol{y}) \in \sigma\}$. If $\boldsymbol{b}_1, \boldsymbol{b}_2 \in D_{\boldsymbol{y}}$, then there exist $\boldsymbol{x}_i \in E_k^m, i = 1, 2$ such that $\boldsymbol{b}_i = \mathrm{ext}(\boldsymbol{x}_i)$ and $(\mathrm{ext}(\boldsymbol{x}_i), \boldsymbol{y}) \in \sigma$ for $i = 1, 2$. Consequently, $(\boldsymbol{x}_1, \boldsymbol{x}_2) \in \sigma_2 = \rho$. Since $f \in \mathrm{Pol}\,\rho$, it follows that $(\boldsymbol{b}_1, \boldsymbol{b}_2) = (f(\boldsymbol{x}_1), f(\boldsymbol{x}_2)) \in \rho$. Therefore $D_{\boldsymbol{y}}^2 \subseteq \rho$, so $D_{\boldsymbol{y}}$ is a ρ – chain. Furthermore, we set $A(\boldsymbol{y}) = \cap\ \{A_j : D_{\boldsymbol{y}} \subseteq A_j\}$ for $\boldsymbol{y} \in E_k^{m+l} \setminus \mathrm{ext}(E_k^m)$ and we denote by $\top_{A(\boldsymbol{y})}$ the greatest element of $A(\boldsymbol{y})$. We extend H on E_k^{m+l} by setting for all $\boldsymbol{y}$ not in the range of ext,

$$\tilde{H}(\boldsymbol{y}) = \begin{cases} \top_{A(\boldsymbol{y})} & \text{if } \exists \boldsymbol{x} \in E_k^m, (\mathrm{ext}(\boldsymbol{x}), \boldsymbol{y}) \in \sigma, \\ \perp & \text{elsewhere.} \end{cases}$$

Since σ is reflexive and transitive, then one can easily show that $\tilde{H} \in \mathrm{Pol}\,\sigma$. Due to $\perp \in C_\rho$ it is easy to see that $\tilde{H} \in \mathrm{Pol}\,\rho$.

(vi) If σ is of type XI, then σ^{-1} is of type X. The same argument above show that the extension $\tilde{H}$ of H defined by

$$\tilde{H}(\boldsymbol{y}) = \begin{cases} \perp_{A(\boldsymbol{y})} & \text{if } \exists \boldsymbol{x} \in E_k^m, (\mathrm{ext}(\boldsymbol{x}), \boldsymbol{y}) \in \sigma; \\ \top & \text{elsewhere} \end{cases}$$

belongs to $\mathrm{Pol}\{\rho, \sigma\}$. We have shown that $\tilde{H}$ belongs to $\mathrm{Pol}\{\rho, \sigma\}$ for $l \in \{I, II, III, IV, VIII, X, XI\}$. Therefore $f(\boldsymbol{x}) = \tilde{H}(\boldsymbol{x}, f_1(\boldsymbol{x}), \dots, f_l(\boldsymbol{x}))$ and $f \in \langle \mathrm{Pol}\{\rho, \sigma\} \cup \{g\} \rangle$ as desired. ∎

Now, we look at Case 2 ($l \in \{V, VI\}$). In this case, the maximality of $\mathrm{Pol}\{\rho, \sigma\}$ below $\mathrm{Pol}\,\rho$ is given by the following known result.

Proposition 4.3 ([12], Theorem 3.2)**.** *Let $k \geq 3$, ρ be an h-ary central relation on E_k with $h \geq 2$ and σ be a binary central relation on E_k such that $\sigma \neq \rho$. The clone* $\mathrm{Pol}\{\rho, \sigma\}$ *is maximal below* $\mathrm{Pol}\,\rho$ *if and only if σ fulfills one of the following two conditions:*

(V) *ρ and σ are comparable (i.e., $\rho \subsetneq \sigma$ or $\sigma \subsetneq \rho$).*

(VI) *$h \geq 3$ and $C_\rho \cap C_\sigma \neq \emptyset$.*

We continue with Case 3 ($l = VII$). Here we use the following result due to Rosenberg and Szendrei. Recall that $\Delta_{E_k} = \{(x, x) : x \in E_k\}$.

Proposition 4.4 ([8], Proposition 4.3)**.** *Let $k \geq 3$, π be a fixed point free permutation on E_k with $\pi^p = id$ (p prime) and ρ be an h-ary σ_π-closed central relation ($h \geq 2$). The relational subalgebras of $[\{\pi^\circ, \rho\}]$ form a 4-element boolean lattice consisting of $[\{\pi^\circ, \rho\}], [\{\pi^\circ\}], [\{\rho\}]$ and $[\{\Delta_{E_k}\}]$.*

The next Corollary gives the maximality of $\mathrm{Pol}\{\pi^\circ, \rho\}$ in $\mathrm{Pol}\,\rho$.

Corollary 4.5. *Let $k \geq 3$, π be a fixed point free permutation on E_k with $\pi^p = id$ (p prime) and ρ be an h-ary σ_π-closed central relation ($h \geq 2$). Then the clone* $\mathrm{Pol}\{\rho, \pi^\circ\}$ *is maximal below* $\mathrm{Pol}\,\rho$*.*

Proof. It follows from Proposition 4.4. ∎

We finish our investigation with Case 4 ($l = IX$).

Lemma 4.6. *Let $k \geq 3$, ρ be a binary central relation and σ a binary relation such that $\rho = \sigma \cap \sigma^{-1}$. A binary relation τ on E_k is preserved by all operations in* $\mathrm{Pol}\,\sigma$ *if and only if $\tau \in \{\emptyset, \Delta_{E_k}, \sigma, \sigma^{-1}, \rho, E_k^2\}$.*

Proof. It is clear that if $\tau \in \{\emptyset, \Delta_{E_k}, \rho, \sigma, \sigma^{-1}, E_k^2\}$, then $\mathrm{Pol}\,\sigma \subseteq \mathrm{Pol}\,\tau$. Now, let τ be a binary relation such that $\mathrm{Pol}\,\sigma \subseteq \mathrm{Pol}\,\tau$. If $\tau = \emptyset$, we are done. Otherwise $\emptyset \subsetneq \tau$. Since σ is reflexive, $\mathrm{Pol}\,\sigma$ contains constant unary operations; therefore $\Delta_{E_k} \subseteq \tau$. If $\tau = \Delta_{E_k}$, we are done. Otherwise $\Delta_{E_k} \subsetneq \tau$ and there exists $(u, v) \in \tau$ such that $u \neq v$. If $(u, v) \in \rho$, then for $(a, b) \in \rho$, the unary operation f defined by $f(x) = a$ if $x = u$ and $f(x) = b$ otherwise, preserves σ (due to $\mathrm{Im}(f) = \{a, b\}$ and $\{a, b\}^2 \subseteq \rho \subseteq \sigma$); so f preserves τ and $(a, b) = (f(u), f(v)) \in \tau$. Hence $\rho \subseteq \tau$. If $\rho = \tau$, we are done. Otherwise, $\rho \subsetneq \tau$ and there exists $(u, v) \in \tau \setminus \rho$. We have the following three cases: (i) $(u, v) \in \sigma$, (ii) $(v, u) \in \sigma$, (iii) $(u, v) \notin \sigma \cup \sigma^{-1}$. We fix $c \in C_\rho$. If $(u, v) \notin \sigma \cup \sigma^{-1}$, then for $(a, b) \in E_k^2$, the unary operation g defined by $g(x) = \begin{cases} a & \text{if } x = u, \\ b & \text{if } x = v, \\ c & \text{elsewhere} \end{cases}$ preserves σ (due to $\rho \subseteq \sigma$ and $(u, v) \notin \sigma \cup \sigma^{-1}$)). Therefore $(a, b) = (g(u), g(v)) \in \tau$. Hence $\tau = E_k^2$.

If $(u,v) \in \sigma$, then for $(a,b) \in \sigma$, the unary operation g above preserves σ (due to $(v,u) \notin \sigma$). Thus $(a,b) = (g(u), g(v)) \in \tau$ and $\sigma \subseteq \tau$. If $\sigma = \tau$, we are done; otherwise $\sigma \subsetneq \tau$ and there exists $(u,v) \in \tau \setminus \sigma$. We choose $(m,n) \in \sigma \setminus \rho$. For $(a,b) \in E_k^2$, the binary operation h defined by $h(x,y) = \begin{cases} a & \text{if } (x,y) = (u,m), \\ b & \text{if } (x,y) = (v,n), \\ c & \text{elsewhere} \end{cases}$ preserves σ; hence $(a,b) = (h(u,m), h(v,n)) \in \tau$. Therefore $\tau = E_k^2$. If $(v,u) \in \sigma$, then using the same argument as above, we can show that $\tau \in \{\sigma^{-1}, E_k^2\}$. ■

Lemma 4.7. *If σ is of type IX, then* $\operatorname{Pol}\sigma$ *contains a majority operation.*

Proof. Let $c \in C_\rho$ and m be the ternary operation defined on E_k by

$$m(x_1, x_2, x_3) = \begin{cases} x_i & \text{if } x_i = x_j \text{ for some } 1 \leq i < j \leq 3, \\ c & \text{elsewhere.} \end{cases}$$

From the definition m is a majority operation. It is easy to see that $m \in \operatorname{Pol}\sigma$. ■

Proposition 4.8. *If σ is of type IX, then* $\operatorname{Pol}\sigma$ *is meet-irreducible and maximal below* $\operatorname{Pol}\rho$.

Proof. Let F be a clone such that $\operatorname{Pol}\sigma \subsetneq F$ and $F \neq O_{E_k}$. We will prove that $F = \operatorname{Pol}\rho$. From Lemma 4.7, $\operatorname{Pol}\sigma$ contains a majority operation m; hence $m \in F$ and by Baker-Pixley Theorem 2.1 we get $F = \bigcap_{\tau \in R} \operatorname{Pol}\tau$ for a set R of binary relations on E_k. Since $\operatorname{Pol}\sigma \subsetneq F$, we get from Lemma 4.6 that $R \subseteq \{\emptyset, \Delta_{E_k}, \rho, \sigma, \sigma^{-1}, E_k^2\}$. By assumptions, there exists an operation f such that $f \in F$ and $f \notin \operatorname{Pol}\sigma = \operatorname{Pol}\sigma^{-1}$; therefore $\sigma, \sigma^{-1} \notin R$. Thus $R \subseteq \{\emptyset, \Delta_{E_k}, \rho, E_k^2\}$, which implies that $F = \operatorname{Pol}\rho$. ■

Remark 4.9. *In fact, one can prove that if σ is of type VIII, X or XI, then* $\operatorname{Pol}\sigma$ *is meet-irreducible below* $\operatorname{Pol}\rho$.

The more difficult part of this work is the proof of necessity in Theorem 3.2 discussed in the next section.

5. PROOF OF NECESSITY IN THEOREM 3.2

In this section, we show that the relations of type I–XI are the only binary relations σ such that the clones $\operatorname{Pol}\{\rho, \sigma\}$ are maximal in $\operatorname{Pol}\rho$. For an arbitrary h-ary central relation ρ ($h \geq 2$) the submaximal clones of $\operatorname{Pol}\rho$ are divided into two types, the meet-reducible submaximal clones of the form $\operatorname{Pol}\rho \cap \operatorname{Pol}\sigma$ where σ is one of the six types listed in Theorem 2.2 and the meet-irreducible submaximal clones $\operatorname{Pol}\sigma$ where σ is not in the list of Theorem 2.2. Following this observation, our investigation is about binary relation σ of Theorem 2.2 and binary relation σ such that $\operatorname{Pol}\sigma$ is only covered by $\operatorname{Pol}\rho$. Since the case of binary central relation is fully described by Proposition 4.3, we share our investigation into four cases: (i) σ is a nontrivial equivalence relation, (ii) σ is a bounded partial order, (iii) σ is the graph of a prime permutation and (iv) $\operatorname{Pol}\sigma$ is meet-irreducible below $\operatorname{Pol}\rho$.

5.1. Case (i): σ is a nontrivial equivalence relation

Proposition 5.1. *Let $k \geq 3$, ρ be an h-ary central relation ($h \geq 2$) and σ be a nontrivial equivalence relation on E_k with t classes. If* $\operatorname{Pol}\{\rho,\sigma\}$ *is a maximal subclone of* $\operatorname{Pol}\rho$*, then σ is of type I or II.*

The proof of Proposition 5.1 is divided into Lemmas 5.3–5.8. We set

$$\sigma_j = \big\{(a_1,\dots,a_h) \in E_k^h : \exists\, u_1 \in [a_1]_\sigma, \dots, \exists\, u_j \in [a_j]_\sigma,\\ (u_1,\dots,u_j,a_{j+1},\dots,a_h) \in \rho\big\}$$

and $\delta_j = \bigcap_{s\in S_h}(\sigma_j)_s$, $j \in \underline{h}$. For $j \in \underline{h}$ we have $\rho \subseteq \sigma_j$ and $\operatorname{Pol}\{\sigma,\rho\} \subseteq \operatorname{Pol}\{\rho,\sigma_j\}$ (due to $\sigma_j \in [\{\rho,\sigma\}]$). If $h = 2$, then $\rho \nsubseteq \sigma$ (due to ρ is a central relation); we choose $(a,b) \in \rho \setminus \sigma$. If $h \geq 3$, then we choose $(a,b) \in E_k^2 \setminus \sigma$. Let $(e,d) \in \sigma$ such that $e \neq d$. Consider the unary operation f_1 defined on E_k by $f_1(x) = a$ if $x = e$ and $f_1(x) = b$ otherwise. Since $(e,d) \in \sigma$ and $(f_1(e), f_1(d)) = (a,b) \notin \sigma$, then $f_1 \notin \operatorname{Pol}\sigma$; but, $f_1 \in \operatorname{Pol}\{\rho,\sigma_j\}$ (due to $\operatorname{Im} f_1 = \{a,b\}$, $\{a,b\}^2 \subseteq \rho \subseteq \sigma_j$ for $h = 2$; and ρ and σ_j are totally reflexive for $h \geq 3$). Therefore, for all $j \in \underline{h}$, $\operatorname{Pol}\{\rho,\sigma\} \subsetneq \operatorname{Pol}\{\rho,\sigma_j\}$ $(*_1)$. From definition, σ_h is totally reflexive, totally symmetric and $\rho \subseteq \sigma_h \subseteq E_k^h$; so we have the following three cases: (1) $\rho = \sigma_h$, (2) $\rho \subsetneq \sigma_h \subsetneq E_k^h$ and (3) $\sigma_h = E_k^h$.

Lemma 5.2. *If the assumptions of Proposition* 5.1 *are satisfied, then the case $\rho \subsetneq \sigma_h \subsetneq E_k^h$ is impossible.*

Proof. Suppose that $\rho \subsetneq \sigma_h \subsetneq E_k^h$. Since σ_h is totally reflexive (reflexive if $h = 2$), totally symmetric (symmetric if $h = 2$) and $\rho \subsetneq \sigma_h \subsetneq E_k^h$, then σ_h is an h-ary central relation. From $(*_1)$ we have $\operatorname{Pol}\{\rho,\sigma\} \subsetneq \operatorname{Pol}\{\rho,\sigma_h\}$. Furthermore, $\operatorname{Pol}\rho$ and $\operatorname{Pol}\sigma_h$ are two distinct maximal clones, so $\operatorname{Pol}\{\rho,\sigma_h\} \subsetneq \operatorname{Pol}\rho$. Thus $\operatorname{Pol}\{\rho,\sigma\} \subsetneq \operatorname{Pol}\{\rho,\sigma_h\} \subsetneq \operatorname{Pol}\rho$, contradicting the maximality of $\operatorname{Pol}\{\rho,\sigma\}$ in $\operatorname{Pol}\rho$. ■

Lemma 5.3. *If the assumptions of Proposition* 5.1 *are satisfied and $\rho = \sigma_h$, then σ is of type I.*

Proof. Assume that $\rho = \sigma_h$. It is easy to check that σ_h is σ-closed. Hence $\rho = \sigma_h$ and σ is of type I. ■

We continue our investigation with the Case (3) $\sigma_h = E_k^h$. We recall that σ has t classes ($t \geq 2$). For $i \geq 2$ we denote by ξ_i the i-ary relation

$$\xi_i = \big\{(a_1, \ldots, a_i) \in E_k^i : \exists\, a_1' \in [a_1]_\sigma, \ldots, a_i' \in [a_i]_\sigma,\ \{a_1', \ldots, a_i'\}^h \subseteq \rho\big\}.$$

We have $\sigma_h = \xi_h = E_k^h$ and ξ_t satisfies one of the following two conditions:

(3.1) $\xi_t = E_k^t$, (3.2) $\xi_t \neq E_k^t$. In the case $\xi_t \neq E_k^t$, we denote by n the least integer N such that $\xi_N \neq E_k^N$. We have $n > h$ (due to $\xi_h = E_k^h$).

Lemma 5.4. *If the assumptions of Proposition* 5.1 *are satisfied and* $\sigma_h = E_k^h$, *then* $\xi_t = E_k^t$.

Proof. Assume that $\xi_t \neq E_k^t$. The minimality of n yields that $\xi_{n-1} = E_k^{n-1}$. It is easy to check that ξ_n is totally symmetric and totally reflexive. Let $c \in C_\rho$ and $(a_1, \ldots, a_{n-1}) \in E_k^{n-1} = \xi_{n-1}$, we have, $(a_1, \ldots, a_{n-1}, c) \in \xi_n$ (due to σ is reflexive). So ξ_n is an n-ary central relation ($n > h$), and ξ_n and ρ are two distinct central relations; therefore $\operatorname{Pol}\{\rho, \xi_n\} \subsetneq \operatorname{Pol}\rho$. Since $\rho,\ \xi_n \in [\{\sigma, \rho\}]$, we have $\operatorname{Pol}\{\rho, \sigma\} \subseteq \operatorname{Pol}\{\rho, \xi_n\}$ and the previous unary operation f_1 preserves ρ and ξ_n and does not preserve σ. Thus $\operatorname{Pol}\{\rho, \sigma\} \subsetneq \operatorname{Pol}\{\rho, \xi_n\} \subsetneq \operatorname{Pol}\rho$; contradicting the maximality of $\operatorname{Pol}\{\rho, \sigma\}$ in $\operatorname{Pol}\rho$. Hence $\xi_t = E_k^t$. ∎

Now we assume that $\xi_t = E_k^t$. Therefore there exist $u_1, \ldots, u_t \in E_k$ such that $(u_i, u_j) \notin \sigma$ for $1 \leq i < j \leq t$ and $\{u_1, \ldots, u_t\}^h \subseteq \rho$. We set $T = \{u_1, u_2 \ldots, u_t\}$, T is called a *transversal* of σ and ρ. Furthermore, we assume that there is a transversal T of σ and ρ such that $T^h \subseteq \rho$. Recall that for all $j \in \underline{h}$, $\delta_j = \bigcap_{s \in S_h} (\sigma_j)_s$ is totally reflexive (or reflexive if $h = 2$) and totally symmetric (symmetric if $h = 2$). We have $\delta_h = E_k^h$. For all $1 \leq j \leq h-1$, $\rho \subseteq \delta_j \subseteq E_k^h$ and we have the following three subcases: (4.1) $\rho = \delta_j$, (4.2) $\rho \subsetneq \delta_j \subsetneq E_k^h$ or (4.3) $\delta_j = E_k^h$.

First, we study the subcase $\rho \subsetneq \delta_j \subsetneq E_k^h$ for some $1 \leq j \leq h-1$.

Lemma 5.5. *If the assumptions of Proposition* 5.1 *are satisfied and there is a transversal* T *of the* σ*-classes such that* $T^h \subseteq \rho$, *then there is no* $1 \leq j \leq h-1$ *such that* $\rho \subsetneq \delta_j \subsetneq E_k^h$.

Proof. Let $1 \leq j \leq h-1$ such that $\rho \subsetneq \delta_j \subsetneq E_k^h$; it is clear that δ_j is an h-ary central relation distinct from ρ and a similar argument as in the proof of Lemma 5.2 shows that $\operatorname{Pol}(\{\rho, \sigma\}) \subsetneq \operatorname{Pol}(\{\rho, \delta_j\}) \subsetneq \operatorname{Pol}(\rho)$. ∎

Lemma 5.6. *If the assumptions of Proposition* 5.1 *are satified and there exists a transversal* T *such that* $T^h \subseteq \rho$, *then there is no* $1 \leq j \leq h-1$ *such that* $\rho = \delta_j$.

Proof. Let $1 \leq j \leq h-1$ such that $\rho = \delta_j$. Since $\delta_i \subseteq \delta_l$ for all $1 \leq i \leq l \leq h-1$, we can suppose that j is the greatest integer N such that $\rho = \delta_N$. Thus $\rho = \delta_j \subsetneq \delta_{j+1}$. Therefore $\delta_{j+1} = E_k^h$. So $\delta_1 = \delta_j = \rho$. Recall that $\delta_1 = \bigcap_{s \in S_h} (\sigma_1)_s$ and

$$\sigma_1 := \{(a_1, \dots, a_h) \in E_k^h : \exists\, u \in E_k, (a_1, u) \in \sigma \wedge (u, a_2, \dots, a_h) \in \rho\}.$$

Thus $\sigma_1 \neq E_k^h$. We have the following possibilities

$$\text{(i) } \rho = \sigma_1 \text{ or (ii) } \rho \subsetneq \sigma_1 \subsetneq E_k^h.$$

Assume that (i) $\rho = \sigma_1$ holds. Let $(a_1, \dots, a_h) \in E_k^h \setminus \rho$; since $\sigma_h = E_k^h$, there exist $u_1, \dots, u_h \in E_k$ such that $(u_1, \dots, u_h) \in \rho$ and $(a_1, u_1) \in \sigma, \dots, (a_h, u_h) \in \sigma$ $(*_3)$. Since $(u_1, u_2, \dots, u_h) \in \rho = \sigma_1$, there exists $v_1 \in E_k$ such that $(u_1, v_1) \in \sigma$ and $(v_1, u_2, \dots, u_h) \in \rho$. Thus $(a_1, u_1) \in \sigma$ and $(u_1, v_1) \in \sigma$. So $(a_1, v_1) \in \sigma$ and $(a_1, u_2, \dots, u_h) \in \sigma_1 = \rho$ (due to $(v_1, u_2, \dots, u_h) \in \rho$). By total symmetry of ρ we have $(u_2, \dots, u_h, a_1) \in \rho = \sigma_1$; so there exists $v_2 \in E_k$ such that $(u_2, v_2) \in \sigma$ and $(v_2, u_3, \dots, u_h, a_1) \in \rho$. By transitivity of σ and $(*_3)$ we obtain $(a_2, v_2) \in \sigma, (v_2, u_3, \dots, u_h, a_1) \in \rho$ and we deduce that $(a_2, u_3, \dots, u_h, a_1) \in \sigma_1 = \rho$. Therefore by induction we can show that $(a_1, a_2, \dots, a_h) \in \rho$; contradicting the choice of $(a_1, \dots, a_h)$. Therefore $\rho \subsetneq \sigma_1 \subsetneq E_k^h$.

Since $\rho = \bigcap_{s \in S_h} (\sigma_1)_s$, and we have $\operatorname{Pol} \sigma_1 \subseteq \operatorname{Pol} \rho$; in addition $\sigma_1 \in [\{\sigma, \rho\}]$, so $\operatorname{Pol}\{\rho, \sigma\} \subseteq \operatorname{Pol} \sigma_1$. It follows that $\operatorname{Pol}\{\rho, \sigma\} \subseteq \operatorname{Pol} \sigma_1 \subseteq \operatorname{Pol} \rho$. The unary operation f_1 defined above preserves ρ and σ_1 and does not preserves σ, therefore $\operatorname{Pol}\{\rho, \sigma\} \subsetneq \operatorname{Pol} \sigma_1$. We will show that $\operatorname{Pol} \sigma_1 \subsetneq \operatorname{Pol} \rho$. From $\rho \subsetneq \sigma_1 \subsetneq E_k^h$ we choose $(b_1, \dots, b_h) \in E_k^h \setminus \sigma_1$ and $(u_1, \dots, u_h) \in \sigma_1 \setminus \rho$ and $c \in C_\rho$. Consider the unary operation f defined on E_k by $f(x) = b_i$ if $x = u_i$ for some $1 \leq i \leq h$ and $f(x) = c$ otherwise. The function f is well defined (because $|\{u_1, \dots, u_h\}| = h$ and ρ totally reflexive). We have $(u_1, \dots, u_h) \in \sigma_1$ and $(f(u_1), \dots, f(u_h)) = (b_1, \dots, b_h) \notin \sigma_1$, so $f \notin \operatorname{Pol} \sigma_1$. It is easy to check that $f \in \operatorname{Pol} \rho$. Hence $\operatorname{Pol}\{\rho, \sigma\} \subsetneq \operatorname{Pol} \sigma_1 \subsetneq \operatorname{Pol} \rho$; contradicting the maximality of $\operatorname{Pol}\{\rho, \sigma\}$ in $\operatorname{Pol} \rho$. ■

From Lemmas 5.5–5.6, we conclude that for all $1 \leq j \leq h-1$, $\delta_j = E_k^h$. Therefore $\delta_1 = E_k^h = \bigcap_{s \in S_h} (\sigma_1)_s$. Hence $E_k^h = \sigma_1 = (\sigma_1)_s$ for all $s \in S_h$. We set $F = \{\{x_1, \dots, x_{h-1}\} \subseteq E_k : \operatorname{Card}(\{x_1, \dots, x_{h-1}\}) = h-1, \{x_1, \dots, x_{h-1}\} \cap C_\rho = \emptyset\}$. Let $m = \operatorname{Card}(F)$, then $m \geq 2$ (because $k \geq 3$ and $h \geq 2$) and set

$$\gamma_{m(h-1)+1} = \big\{(a_1, a_{1,1}, \dots, a_{1,h-1}, \dots, a_{m,1}, \dots, a_{m,h-1}) \in E_k^{m(h-1)+1} : \\ \exists\, u_1 \in [a_1]_\sigma : \{(u_1, a_{i,1}, \dots, a_{i,h-1}),\ 1 \leq i \leq m\} \subseteq \rho\big\}.$$

We have two subcases:

$$\text{(i) } \gamma_{m(h-1)+1} \neq E_k^{m(h-1)+1} \text{ and (ii) } \gamma_{m(h-1)+1} = E_k^{m(h-1)+1}.$$

Lemma 5.7. *If the assumptions of Proposition* 5.1 *are satisfies and* $\sigma_1 = E_k^h$*, then* $\gamma_{m(h-1)+1} = E_k^{m(h-1)+1}$.

Proof. Assume $\sigma_1 = E_k^h$ and $\gamma_{m(h-1)+1} \neq E_k^{m(h-1)+1}$. We will show that $\mathrm{Pol}(\{\rho,\sigma\}) \subsetneq \mathrm{Pol}(\{\rho,\gamma_{m(h-1)+1}\}) \subsetneq \mathrm{Pol}(\rho)$. Since $\gamma_{m(h-1)+1} \in [\{\sigma,\rho\}]$, we have $\mathrm{Pol}(\{\rho,\sigma\}) \subseteq \mathrm{Pol}(\{\rho,\gamma_{m(h-1)+1}\}) \subseteq \mathrm{Pol}(\rho)$. The above unary operation f_1 preserves ρ and $\gamma_{m(h-1)+1}$ and does not preserve σ; therefore $\mathrm{Pol}(\{\rho,\sigma\}) \subsetneq \mathrm{Pol}(\{\rho,\gamma_{m(h-1)+1}\})$. It remains to show that $\mathrm{Pol}(\{\rho,\gamma_{m(h-1)+1}\}) \subsetneq \mathrm{Pol}(\rho)$. Let $c \in C_\rho$ and $(a_1,\dots,a_h) \in E_k^h \setminus \rho$, we set $W = \{(i_1,\dots,i_h) : 1 \leq i_1 < \cdots < i_h \leq m(h-1)+1\}$, denoted simply by $W = \{(i_1^j,\dots,i_h^j) : 1 \leq j \leq q\}$ where $q = |W|$. For $1 \leq j \leq q$, we set $\boldsymbol{y}_j = (x_{j,1},\dots,x_{j,m(h-1)+1})$ such that for all $p, 1 \leq p \leq m(h-1)+1$, $x_{j,p} = a_l$ if $p = i_l^j$ for some $1 \leq l \leq h$ and $x_{j,p} = c$ otherwise. For $1 \leq i \leq m(h-1)+1$, we set $\boldsymbol{x}_i = (x_{1,i},\dots,x_{q,i})$. Let $\boldsymbol{v} = (v_1,\dots,v_{m(h-1)+1}) \in E_k^{m(h-1)+1} \setminus \gamma_{m(h-1)+1}$ and f be the q-ary operation defined on E_k by

$$f(\boldsymbol{x}) = \begin{cases} v_i, & \text{if } \boldsymbol{x} = \boldsymbol{x}_i \text{ for some } 1 \leq i \leq m(h-1)+1, \\ c & \text{otherwise.} \end{cases}$$

The operation f is well defined, because $|\{\boldsymbol{x}_i : 1 \leq i \leq m(h-1)+1\}| = m(h-1)+1$. From construction, for all $1 \leq i_1 < \cdots < i_h \leq m(h-1)+1$, $(\boldsymbol{x}_{i_1},\dots,\boldsymbol{x}_{i_h}) \notin \rho (*_4)$. We have $f \in \mathrm{Pol}\,\rho$ because $c \in C_\rho$ and $(*_4)$ holds. Using $(\sigma_1)_s = E_k^h$ for all $s \in S_h$, the total symmetry, total reflexivity of ρ ($h \geq 3$) and $c \in C_\rho$, we can show that $\{\boldsymbol{y}_1,\dots,\boldsymbol{y}_q\} \subseteq \gamma_{m(h-1)+1}$. Thus $f(\boldsymbol{y}_1,\dots,\boldsymbol{y}_q) = (f(\boldsymbol{x}_1),\dots,f(\boldsymbol{x}_{m(h-1)+1})) = (v_1,\dots,v_{m(h-1)+1}) \notin \gamma_{m(h-1)+1}$; so $f \notin \mathrm{Pol}(\gamma_{m(h-1)+1})$. Thus $\mathrm{Pol}(\{\rho,\sigma\}) \subsetneq \mathrm{Pol}(\{\rho,\gamma_{m(h-1)+1}\}) \subsetneq \mathrm{Pol}(\rho)$; contradicting the maximality of $\mathrm{Pol}\{\sigma,\rho\}$ below $\mathrm{Pol}\,\rho$. ∎

From Lemma 5.7, we have $\gamma_{m(h-1)+1} = E_k^{m(h-1)+1}$.

Lemma 5.8. *If the assumptions of Proposition* 5.1 *are satisfied, there exists a transversal T of σ-classes such that $T^h \subseteq \rho$ and $\gamma_{m(h-1)+1} = E_k^{m(h-1)+1}$, then every equivalence class of σ contains a central element of ρ.*

Proof. Let $a \in E_k$, we set

$$\boldsymbol{u} = (a, x_{1,1},\dots,x_{1,h-1},x_{2,1},\dots,x_{2,h-1},\dots,x_{m,1},\dots,x_{m,h-1})$$

such that $\{x_{j,1},\dots,x_{j,h-1}\} \in F, 1 \leq j \leq m$. Since $\boldsymbol{u} \in E_k^{m(h-1)+1} = \gamma_{m(h-1)+1}$, there exists $v \in [a]_\sigma$ such that for all $j \in \{1,\dots,m\}$, $(v,x_{j,1},\dots,x_{j,h-1}) \in \rho$. Hence $v \in C_\rho$. Thus every equivalence class of σ contains a central element of ρ. ∎

Under the assumptions of Lemma 5.8, σ is of type II.

Proof. (Proof of Proposition 5.1.) It follows from Lemmas 5.3–5.8. ■

5.2. Case (ii): σ is a bounded partial order

For $a \in E_k$ we set $[a]_\sigma = \{x \in E_k : (a, x) \in \sigma\}$. The following proposition characterizes σ under the maximality of $\operatorname{Pol}\{\rho, \sigma\}$ in $\operatorname{Pol}\rho$.

Proposition 5.9. *Let $k \geq 3$, ρ be an h-ary central relation ($h \geq 2$) and σ be a bounded partial order with least element $\bot$ and greatest element $\top$. If $\operatorname{Pol}\{\rho, \sigma\}$ is maximal below $\operatorname{Pol}\rho$, then σ is of type III or IV.*

We assume that $\operatorname{Pol}\{\rho, \sigma\}$ is maximal below $\operatorname{Pol}\rho$ and we consider the following h-ary relations

$$\delta = \{(a_1, \dots, a_h) \in E_k^h : \exists\, u \in E_k, (a_1, u) \in \sigma, (u, a_2, \dots, a_h) \in \rho\},$$

$$\delta' = \{(a_1, \dots, a_h) \in E_k^h : \exists\, u \in E_k, (a_1, \dots, a_{h-1}, u) \in \rho, (u, a_h) \in \sigma\}.$$

We can see that if $h = 2$, then $\delta = \sigma \circ \rho$ and $\delta' = \rho \circ \sigma$. We have $\rho \subseteq \delta \subseteq E_k^h$ and $\rho \subseteq \delta' \subseteq E_k^h$. Hence δ (respectively δ') satisfies one of the following conditions

$$\text{(a) } \rho = \delta;\ \text{(b) } \rho \subsetneq \delta \subsetneq E_k^h \text{ or (c) } \delta = E_k^h$$
$$\text{(resp. (a) } \rho = \delta';\ \text{(b) } \rho \subsetneq \delta' \subsetneq E_k^h \text{ or (c) } \delta' = E_k^h).$$

Lemma 5.10. *If the assumptions of Proposition* 5.9 *are satisfied, then $\rho \neq \delta$ (respectively $\rho \neq \delta'$).*

Proof. Assume that $\rho = \delta$. Let $a_1, \dots, a_h \in E_k$, we have $(a_1, \top) \in \sigma$ and $(\top, a_2, \dots, a_{h-1}, \top) \in \rho$. Thus $(a_1, a_2, \dots, a_{h-1}, \top) \in \delta = \rho$. Hence $\top \in C_\rho$. Furthermore $(a_h, \top) \in \sigma$ and $(\top, a_1, a_2, \dots, a_{h-1}) \in \rho$. Therefore $(a_h, a_1, \dots, a_{h-1}) \in \rho$ and $\rho = E_k^h$, contradiction. A similar argument solves the case $\rho = \delta'$. ■

It follows that $\rho \subsetneq \delta \subsetneq E_k^h$ or $\delta = E_k^h$ and $\rho \subsetneq \delta' \subsetneq E_k^h$ or $\delta' = E_k^h$.

Lemma 5.11. *If the assumptions of Proposition* 5.9 *are satified, then the Case $\rho \subsetneq \delta \subsetneq E_k^h$ (respectively $\rho \subsetneq \delta' \subsetneq E_k^h$) is impossible.*

Proof. Assume that $\rho \subsetneq \delta \subsetneq E_k^h$. First, we show that $\operatorname{Pol}(\{\rho, \sigma\}) \subsetneq \operatorname{Pol}(\{\rho, \delta\})$. It is easy to see that $\operatorname{Pol}\{\rho, \sigma\} \subseteq \operatorname{Pol}\{\rho, \delta\}$ (due to $\rho, \delta \in [\{\rho, \sigma\}]$). If ρ is binary, then $\rho \not\subseteq \sigma$. Hence there exists $(u, v) \in \rho$ such that $(u, v) \notin \sigma$. Let $(a, b) \in \sigma$ such that $a \neq b$. We consider the unary operation f defined by $f(x) = u$ if $x = a$ and $f(x) = v$ otherwise. The operation f does not preserve σ because $(a, b) \in \sigma$ and $(f(a), f(b)) = (u, v) \notin \sigma$. Since $\rho \subseteq \delta$ and $(u, v) \in \rho$, we have $\{u, v\}^2 \subseteq \rho \subseteq \delta$. Thus f preserves ρ and δ. So $\operatorname{Pol}\{\rho, \sigma\} \subsetneq \operatorname{Pol}\{\rho, \delta\}$. If the arity of ρ is greater than 2, then δ is totally reflexive. Let $(a,b) \in \sigma$ such that $a \neq b$. The operation h defined by $h(x) = b$ if $x = a$ and $h(x) = a$ otherwise, preserves ρ and δ and does not preserve σ(due to $(a,b) \in \sigma$ and $(h(a), h(b)) = (b,a) \notin \sigma$). Hence $\mathrm{Pol}\{\rho,\sigma\} \subsetneq \mathrm{Pol}\{\rho,\delta\}$.

Second, we show that $\mathrm{Pol}\{\rho,\delta\} \subsetneq \mathrm{Pol}\,\rho$. Let $(u_1,\dots,u_h) \in \delta \setminus \rho$ and $c \in C_\rho$, $(a_1,\dots,a_h) \in E_k^h \setminus \delta$. The unary operation l defined on E_k by $l(x) = a_i$ if $x = u_i$ for some $1 \leq i \leq h$ and $l(x) = c$ otherwise, is well defined (because $|\{u_1,\dots,u_h\}| = h$ and ρ totally reflexive) and preserves ρ. Since $(u_1,\dots,u_h) \in \delta$ and $(l(u_1),\dots,l(u_h)) = (a_1,\dots,a_h) \notin \delta$, l does not preserve δ. Therefore $\mathrm{Pol}(\{\rho,\delta\}) \subsetneq \mathrm{Pol}(\rho)$. We conclude that $\mathrm{Pol}\{\rho,\sigma\} \subsetneq \mathrm{Pol}\{\rho,\delta\} \subsetneq \mathrm{Pol}\,\rho$, contradicting the maximality of $\mathrm{Pol}\{\rho,\sigma\}$ in $\mathrm{Pol}\,\rho$. A similar argument solves the case $\rho \subsetneq \delta' \subsetneq E_k^h$. ∎

Lemma 5.12. *If the assumptions of Proposition* 5.9 *are satified and* $\delta = E_k^h$, *then* $\{\perp,\top\} \subseteq C_\rho$.

Proof. Let $a_1, a_2, \dots, a_{h-1} \in E_k$. Since $(\top, a_1,\dots,a_{h-1}) \in E_k^h = \delta$, there exists $u \in E_k$ such that $(\top, u) \in \sigma$ and $(u, a_1,\dots,a_{h-1}) \in \rho$. Hence $u = \top$ and $(\top, a_1,\dots,a_{h-1}) \in \rho$ (due to $\top$ is the greatest element of σ). Therefore $\top \in C_\rho$. Using the previous argument replacing δ and $\top$ by δ' and $\perp$ respectively we can show that $\perp \in C_\rho$. ∎

We have shown that $\{\perp,\top\} \subseteq C_\rho$ and we will use the arity of ρ to conclude.

Lemma 5.13. *If the assumptions of Proposition* 5.9 *are satified and* $h \geq 3$, *then* σ *is a bounded partial order of type IV.*

Proof. From Lemmas 5.10–5.12, σ satisfies condition (IV) of Theorem 3.2. ∎

From now on we suppose that $\{\perp,\top\} \subseteq C_\rho$ and ρ is a binary central relation. We set $\lambda = \sigma \cap \rho$. We have $(\perp,\top) \in \lambda$, so $\Delta_{E_k} \subsetneq \lambda \subseteq \sigma$. Let $tr(\lambda)$ be the transitive closure of λ. Since σ is transitive, we have $tr(\lambda) \subseteq \sigma$. Hence (1) $tr(\lambda) \subsetneq \sigma$ or (2) $tr(\lambda) = \sigma$.

Lemma 5.14. *If the assumptions of Proposition* 5.9 *are satified,* $\{\perp,\top\} \subseteq C_\rho$ *and* ρ *being binary, then* $tr(\lambda) = \sigma$.

Proof. Assume that (1) holds. Since $tr(\lambda) \in [\{\sigma,\rho\}]$, we have $\mathrm{Pol}\{\rho,\sigma\} \subseteq \mathrm{Pol}\{\rho, tr(\lambda)\} \subseteq \mathrm{Pol}\,\rho$. Let $(a,b) \in \sigma$ such that $(a,b) \notin tr(\lambda)$, then $(\top,\perp) \notin \sigma$ and the unary operation f defined on E_k by $f(x) = \top$ if $(a,x) \in tr(\lambda)$ and $f(x) = \perp$ otherwise does not preserve σ (because $(a,b) \in \sigma$ and $(f(a), f(b)) = (\top,\perp) \notin \sigma$). But, using reflexivity and transitivity of $tr(\lambda)$ one can check that f preserves $tr(\lambda)$. Thus $f \in \mathrm{Pol}\,\rho$ because $\{\perp,\top\} \subseteq C_\rho$; therefore $\mathrm{Pol}\{\rho,\sigma\} \subsetneq \mathrm{Pol}\{\rho, tr(\lambda)\}$. Let $(u,v) \in tr(\lambda)$ such that $u \neq v$ and $(a,b) \in \rho \setminus \sigma$; then $(a,b) \notin tr(\lambda)$. Let

g be the unary operation defined on E_k by $g(x) = a$ if $x = u$ and $g(x) = b$ otherwise. The operation g does not preserve $tr(\lambda)$ (due to $(u, v) \in tr(\lambda)$ and $(g(u), g(v)) = (a, b) \notin tr(\lambda)$). Since $\operatorname{Im} g = \{a, b\}$ and $(a, b) \in \rho$, we obtain $g \in \operatorname{Pol}\rho$. So $\operatorname{Pol}\{\rho, tr(\lambda)\} \subsetneq \operatorname{Pol}\rho$. Hence $\operatorname{Pol}\{\rho, \sigma\} \subsetneq \operatorname{Pol}\{\rho, tr(\lambda)\} \subsetneq \operatorname{Pol}\rho$; contradicting the maximality of $\operatorname{Pol}\{\rho, \sigma\}$ in $\operatorname{Pol}\rho$. ■

Lemma 5.15. *If the assumptions of Proposition* 5.9 *are satisfied,* $\{\perp, \top\} \subseteq C_\rho$*,* ρ *being binary and* $tr(\lambda) = \sigma$*, then* σ *is of type III.*

Proof. It is easy to observe that σ is of type III. ■

Proof. (Proof of Proposition 5.9) It follows from Lemmas 5.10–5.15. ■

5.3. Case (iii): σ is the graph of a prime permutation

In this case, the characterization of σ is given by the following known result.

Proposition 5.16 ([8], Page 37)**.** *Let* $k \geq 3$*,* ρ *be an* h*-ary central relation* $(h \geq 2)$ *and* π *a fixed point free permutation on* E_k *with* $\pi^p = id$ *(*p *prime). The relational algebra* $[\{\pi^\circ, \rho\}]$ *contains one of the following relations:*

(1) *a nontrivial unary relation,*

(2) *a nontrivial* σ_π*-closed equivalence relation,*

(3) *a* σ_π*-closed central relation,*

(4) *a* σ_π*-closed regular relation.*

Corollary 5.17. *Let* $k \geq 3$*,* π *be a fixed point free permutation of* E_k *with* $\pi^p = id$ *(*p *prime) and* ρ *be an* h*-ary central relation* $(h \geq 2)$*. If* $\operatorname{Pol}\{\rho, \pi^\circ\}$ *is maximal below* $\operatorname{Pol}\rho$*, then* ρ *is* σ_π*-closed.*

Proof. Assume that $\operatorname{Pol}\{\rho, \pi^\circ\}$ is maximal below $\operatorname{Pol}\rho$. From Proposition 5.16 $[\{\rho, \pi^\circ\}]$ contains a relation γ satisfying (1), (2), (3) or (4). Thus $\operatorname{Pol}\{\rho, \pi^\circ\} \subseteq \operatorname{Pol}\{\gamma, \rho\} \subseteq \operatorname{Pol}\rho$. From Theorem 2.2, $\operatorname{Pol}\gamma$ is a maximal clone. Assume that $\gamma \neq \rho$, then $\operatorname{Pol}\{\rho, \gamma\} \subsetneq \operatorname{Pol}\rho$ (because $\operatorname{Pol}\rho$ and $\operatorname{Pol}\gamma$ are two different maximal clones). Thus $\operatorname{Pol}\{\rho, \pi^\circ\} \subseteq \operatorname{Pol}\{\gamma, \rho\} \subsetneq \operatorname{Pol}\rho$. Let $a \in E_k$ such that $a \in \gamma$ whenever γ is unary. Consider the constant unary operation c_a with value a. It is easy to see that c_a preserves ρ and γ, and c_a does not preserve π°. Hence $\operatorname{Pol}\{\rho, \pi^\circ\} \subsetneq \operatorname{Pol}\{\gamma, \rho\} \subsetneq \operatorname{Pol}\rho$, contradicting the choice of π. Therefore $\gamma = \rho$ and we conclude that ρ is σ_π-closed. ■

5.4. Case (iv): $\operatorname{Pol}\sigma$ is maximal and meet-irreducible below $\operatorname{Pol}\rho$

In this subsection, we will show that relations of type VIII, IX, X, and XI are the only binary relations σ such that $\operatorname{Pol}\sigma$ is maximal and meet – irreducible below $\operatorname{Pol}\rho$. First we prove the following important lemma useful for some proofs.

Lemma 5.18. *Let $k \geq 3$, ρ be an h-ary central relation $(h \geq 2)$ and γ an h-ary nonempty relation on E_k. If* $\operatorname{Pol}\rho = \operatorname{Pol}\gamma$, *then $\rho = \gamma$.*

Proof. Assume that γ is a nonempty h-ary relation and ρ an h-ary central relation $(h \geq 2)$ such that $\operatorname{Pol}\rho = \operatorname{Pol}\gamma$. Our aim is to show that $\rho = \gamma$.

Let $a \in E_k$, the constant unary operation c_a with value a preserves ρ (due to ρ (totally) reflexive), hence c_a preserves γ. Therefore $(\underbrace{a, \ldots, a}_{h \text{ times}})$ belongs to γ (due to $\gamma \neq \emptyset$). Thus $\delta^h \subseteq \gamma$. Since $\operatorname{Pol}\delta^h = O_{E_k}$, we obtain $\delta^h \subsetneq \gamma$. So there exist $\boldsymbol{a} = (a_1, \ldots, a_h) \in \gamma$ and $\alpha, \beta \in \{1, \ldots, h\}$ such that $a_\alpha \neq a_\beta$. We have the following statement.

Claim 1. $\forall 1 \leq i < j \leq h, \exists\, \boldsymbol{a}^{ij} = (a_1^{ij}, \ldots, a_h^{ij}) \in \gamma$ *such that* $a_i^{ij} \neq a_j^{ij}$.

In fact, if $h = 2$, then Claim 1 is true; if $h > 2$, then assume that Claim 1 is false. Therefore there exist $1 \leq i_0 < j_0 \leq h$ such that for all $\boldsymbol{a} = (a_1, \ldots, a_h) \in \gamma$, we have $a_{i_0} = a_{j_0}$. Set

$$\theta = \big\{(i, j) \in \underline{h}^2 : a_i = a_j \ \forall \boldsymbol{a} \in \gamma\big\}.$$

It is easy to see that θ is a nontrivial equivalence relation on $\underline{h}$ (due to $(\alpha, \beta) \notin \theta$ and $(i_0, j_0) \in \theta$). Set also

$$\delta_\theta = \big\{(a_1, \ldots, a_h) \in E_k^h : (i, j) \in \theta \implies a_i = a_j\big\}.$$

An easy check shows that $\operatorname{Pol}\delta_\theta = O_{E_k}$ and $\gamma \subseteq \delta_\theta$. Let $\boldsymbol{b} = (b_1, \ldots, b_h) \in \delta_\theta$. For all $1 \leq i < j \leq h$ such that $(i, j) \notin \theta$, $b_i \neq b_j$ and there exists $\boldsymbol{e}^{ij} = (e_1, \ldots, e_h) \in \gamma$ such that $e_i \neq e_j$. Set $E = \{\boldsymbol{e}^{ij} : 1 \leq i < j \leq h \text{ and } (i, j) \notin \theta\}$. For reason of simple notation we set $E = \{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_q\}$ with $q = |E|$. We set also $\boldsymbol{x}_i = (e_{1,i}, \ldots, e_{q,i})$ for each $i \in \underline{h}$. By construction of $(\boldsymbol{x}_i)_{i \in \underline{h}}$, we have $\boldsymbol{x}_i \neq \boldsymbol{x}_j$ for all $1 \leq i < j \leq h$ such that $(i, j) \notin \theta$. The q-ary function f defined on E_k^q by

$$f(\boldsymbol{x}) = \begin{cases} b_i & \text{if } \boldsymbol{x} = \boldsymbol{x}_i \text{ for some } 1 \leq i \leq h, \\ b_1 & \text{elsewhere} \end{cases}$$

preserves ρ (due to $\operatorname{Im}(f) = \{b_1, \ldots, b_h\}, |\{b_1, \ldots, b_h\}| \leq h - 1$ and ρ totally reflexive and totally symmetric). Hence f preserves γ and $\boldsymbol{b} \in \gamma$ (due to $E \subseteq \gamma$ and $\boldsymbol{b} = f(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_q) = (f(\boldsymbol{x}_1), \ldots, f(\boldsymbol{x}_h))$. Thus $\delta_\theta \subseteq \gamma$ and $\gamma = \delta_\theta$. So $O_{E_k} = \operatorname{Pol}\delta_\theta = \operatorname{Pol}\gamma = \operatorname{Pol}\rho \neq O_{E_k}$ which is a contradiction. So Claim 1 is true.

Claim 2. $\rho \subseteq \gamma$.

In fact, from Claim 1, for all $1 \leq i < j \leq h$ there exists $\boldsymbol{a}^{ij} = (a_1^{ij}, \ldots, a_h^{ij}) \in \gamma$ such that $a_i^{ij} \neq a_j^{ij}$. Set $F = \{\boldsymbol{a}^{ij} : 1 \leq i < j \leq h\}$, for reason of simple notation we set $F = \{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_q\}$ with $q = |F|$. Let $\boldsymbol{b} = (b_1, \ldots, b_h) \in \rho$. Setting

$(\boldsymbol{x}_i)_{i\in\underline{h}}$ as above, the function f defined above preserves ρ (due to $\boldsymbol{b} \in \rho$, ρ is (totally) reflexive and (totally) symmetric); hence f preserves γ. Therefore $\boldsymbol{b} = f(\boldsymbol{e}_1, \dots, \boldsymbol{e}_q) = (f(\boldsymbol{x}_1), \dots, f(\boldsymbol{x}_h)) \in \gamma$. So $\rho \subseteq \gamma$.

Claim 2. *Yields the following statement.*

Claim 3. γ *is* (*totally*) *symmetric.*

In fact, let $(a_1, \dots, a_h) \in \gamma$ and $\pi \in S_h$, we will show that $(a_{\pi(1)}, \dots, a_{\pi(h)}) \in \gamma$. If $(a_1, \dots, a_h) \in \rho$, then $(a_{\pi(1)}, \dots, a_{\pi(h)}) \in \rho \subseteq \gamma$ (due to ρ (totally) symmetric). Suppose now that $(a_1, \dots, a_h) \notin \rho$. Let $c \in C_\rho$. The unary operation g defined on E_k by

$$g(x) = \begin{cases} a_{\pi(i)} & \text{if } x = a_i \text{ for some } 1 \le i \le h, \\ c & \text{elsewhere} \end{cases}$$

preserves ρ (due to $(a_1, \dots, a_h) \notin \rho$ and $c \in C_\rho$); hence g preserves γ and $(a_{\pi(1)}, \dots, a_{\pi(h)}) = (g(a_1), \dots, g(a_h)) \in \gamma$. Therefore γ is (totally) symmetric.

We end this proof with the following statement.

Claim 4. $\rho = \gamma$.

From Claim 2, we have $\rho \subseteq \gamma$. It remains to show that $\gamma \subseteq \rho$. Since ρ satisfies the Claim 2 and γ is (totally) reflexive and (totally) symmetric (due to $\rho \subseteq \gamma$ and Claim 3), a similar argument as in the proof of Claim 2 shows that $\gamma \subseteq \rho$. Therefore $\rho = \gamma$. ■

Proposition 5.19. *Let $k \ge 3$, σ be a binary relation and ρ an h-ary central relation on E_k with t distinct maximal ρ-chains $A_0, A_1, \dots, A_{t-1}$ ($h \ge 2$). If* $\operatorname{Pol}\sigma$ *is meet-irreducible and maximal below* $\operatorname{Pol}\rho$, *then σ is of type VIII, IX, X or XI.*

The proof of Proposition 5.19 is shared into the following lemmas. Let $\sigma \subseteq E_k^2$ such that $\operatorname{Pol}\sigma$ is meet-irreducible and maximal below $\operatorname{Pol}\rho$. Set

$$\sigma_1 = \{x \in E_k : \exists\, u \in E_k, (x,u) \in \sigma\},\ \sigma_1' = \{x \in E_k : \exists\, u \in E_k, (u,x) \in \sigma\},$$
$$\sigma_2 = \sigma \circ \sigma^{-1} \text{ and } \sigma_2' = \sigma^{-1} \circ \sigma.$$

Since $\sigma \neq \emptyset$, we have $\sigma_1 \neq \emptyset$ and $\sigma_1' \neq \emptyset$. It follows that (a) $\emptyset \subsetneq \sigma_1 \subsetneq E_k$ or (b) $\sigma_1 = E_k$ and (a') $\emptyset \subsetneq \sigma_1' \subsetneq E_k$ or (b') $\sigma_1' = E_k$.

Lemma 5.20. *If the assumptions of Proposition* 5.19 *are satisfied, then we have* (a) $\sigma_1 = \sigma_1' = E_k$ *and* (b) $\sigma_2, \sigma_2' \in \{\Delta_{E_k}, \rho, E_k^2\}$.

Proof. (a) Clearly $\operatorname{Pol}\sigma \subseteq \operatorname{Pol}\sigma_1$. If $\sigma_1 \neq E_k$, then σ_1 is a relation of type (5) in Theorem 2.2; hence $\operatorname{Pol}\sigma$ is not meet-irreducible, contradicting the choice of σ. Therefore $\sigma_1 = E_k$. A similar argument shows that $\sigma_1' = E_k$.

(b) We have also $\operatorname{Pol}\sigma \subseteq \operatorname{Pol}\sigma_2$. Naturally we have the following cases:

(i) σ is not reflexive, (ii) σ is reflexive and not symmetric and (iii) σ is reflexive and symmetric. Now we discuss the cases (i)–(iii) and show that in any case $\operatorname{Pol}\sigma \subsetneq \operatorname{Pol}\sigma_2$.

(i) If σ is not reflexive, then there exists $u \in E_k$ such that $(u,u) \notin \sigma$; from (a) there exists $v \in E_k$ such that $(u,v) \in \sigma$; therefore $(u,u) \in \sigma_2$ and the constant unary function on E_k with value u preserves σ_2 and does not preserve σ. Therefore $\operatorname{Pol}\sigma \subsetneq \operatorname{Pol}\sigma_2$.

(ii) If σ is reflexive and not symmetric, there exists $(x,y) \in \sigma$ such that $(y,x) \notin \sigma$; the unary operation g defined on E_k by $g(w) = y$ if $w = x$ and $g(w) = x$ otherwise, preserves σ_2 because $(x,y),(y,x) \in \sigma_2$ and σ_2 is reflexive, and does not preserve σ. Hence $\operatorname{Pol}\sigma \subsetneq \operatorname{Pol}\sigma_2$.

(iii) If σ is reflexive and symmetric, then σ is not transitive (due to σ is not an equivalence relation). Let $(c,d) \in \sigma_2 \backslash \sigma$, then $(c,u),(d,u) \in \sigma$ for some $u \in E_k$ (due to $\sigma_2 = \sigma \circ \sigma$). The unary function g defined on E_k by $g(x) = c$ if $x = c$ and $g(x) = d$ otherwise, preserves σ_2 and does not preserve σ. Thus $\operatorname{Pol}\sigma \subsetneq \operatorname{Pol}\sigma_2$.

If $\operatorname{Pol}\sigma_2 = O_{E_k}$, then σ_2 is a diagonal relation. Hence $\sigma_2 \in \{\Delta_{E_k}, E_k^2\}$. Now assume that $\operatorname{Pol}\sigma_2 = \operatorname{Pol}\rho$. If $h \geq 3$, then we choose $(a,b) \in E_k^2 \setminus \sigma_2$ and $(u,v) \in \sigma_2$ such that $u \neq v$ (due to $\sigma_2 \notin \{\Delta_{E_k}, E_k^2\}$). Let f be the unary operation on E_k defined by $f(x) = a$ if $x = u$ and $f(x) = b$ otherwise. From $(u,v) \in \sigma_2$ and $f(u,v) = (f(u), f(v)) = (a,b) \notin \sigma_2$, f does not preserve σ_2; but f preserves ρ (due to ρ totally reflexive and $\operatorname{Im}(f) = \{a,b\}$). Hence $f \in \operatorname{Pol}\rho \backslash \operatorname{Pol}\sigma_2$, contradiction. Therefore $h = 2$. From Lemma 5.18, we obtain $\rho = \sigma_2$. In conclusion, we have $\sigma_2 \in \{\Delta_{E_k}, \rho, E_k^2\}$.

A similar argument shows that $\sigma_2' \in \{\Delta_{E_k}, \rho, E_k^2\}$. Therefore (b) holds. ∎

From Lemma 5.20, we have $\sigma_1 = E_k = \sigma_1'$. We set $\eta = \{x \in E_k : (x,x) \in \sigma\}$; therefore η satisies one of the following two conditions

$$\text{(i) } \emptyset \subsetneq \eta \subsetneq E_k, \text{ (ii) } \emptyset = \eta \text{ or } \eta = E_k.$$

Lemma 5.21. *If the assumptions of Proposition* 5.19 *are satisfied, then the subcase* (i) *is impossible.*

Proof. Assume that (i) holds, then the unary relation η is a unary central relation, so $\operatorname{Pol}\eta$ is a maximal clone distinct from $\operatorname{Pol}\rho$. Since $\eta \in [\{\sigma\}]$ and $\operatorname{Pol}\eta$ is a maximal clone, we get $\operatorname{Pol}\sigma \subsetneq \operatorname{Pol}\eta$. Therefore $\operatorname{Pol}\sigma$ is not meet-irreducible; contradiction. ∎

Hence σ is reflexive or irreflexive and Lemma 5.20 yields the following nine cases:

(1) $\sigma_2 = \sigma_2' = \Delta_{E_k}$; (2) $\sigma_2 = \Delta_{E_k}$ and $\sigma_2' = E_k^2$; (3) $\sigma_2 = E_k^2$ and $\sigma_2' = \Delta_{E_k}$; (4) $\sigma_2 = \Delta_{E_k}$ and $\sigma_2' = \rho$; (5) $\sigma_2 = \rho$ and $\sigma_2' = \Delta_{E_k}$; (6) $\sigma_2 = \sigma_2' = \rho$; (7) $\sigma_2 = \rho$ and $\sigma_2' = E_k^2$; (8) $\sigma_2 = E_k^2$ and $\sigma_2' = \rho$; (9) $\sigma_2 = \sigma_2' = E_k^2$. We will study these cases in the following lines. First we look at the Case (1): $\sigma_2 = \sigma_2' = \Delta_{E_k}$.

Lemma 5.22. *If the assumptions of Proposition* 5.19 *are satisfied, then the Case* (1) $\sigma_2 = \sigma_2' = \Delta_{E_k}$ *is impossible.*

Proof. The function s defined on E_k by $s(x) = y$, if $(x, y) \in \sigma$, is a permutation on E_k (due to E_k being finite, $\sigma_2' = \Delta_{E_k}$ and $\sigma_1 = E_k$). Let m be the order of s; for $1 \leq r < m$, set $F_r = \{x \in E_k : s^r(x) = x\}$. Let $f \in \operatorname{Pol}\sigma \cap O_{E_k}^{(n)}$ and $x_1, \dots, x_n \in E_k$. Since σ is the graph of s, $(x_1, s(x_1)), \dots, (x_n, s(x_n)) \in \sigma$. Therefore $(f(x_1, \dots, x_n), f(s(x_1), \dots, s(x_n)) \in \sigma = s^\circ$; so $s(f(x_1, \dots, x_n)) = f(s(x_1), \dots, s(x_n))$ and we can show by induction on r, $1 \leq r \leq m-1$, that $s^r(f(x_1, \dots, x_n)) = f(s^r(x_1), \dots, s^r(x_n))$ $(*_5)$.

Now we show that $\operatorname{Pol}\sigma \subseteq \operatorname{Pol} F_r$. Let $f \in \operatorname{Pol}\sigma$ be an n-ary operation and $x_1, \dots, x_n \in F_r$. Then $s^r(x_i) = x_i, i = 1, \dots, n$. We get $f(x_1, \dots, x_n) = f(s^r(x_1), \dots, s^r(x_n))$ (due to $x_i \in F_r, i \in \underline{n}$) $= s^r(f(x_1, \dots, x_n))$ (by $(*_5)$); therefore $f(x_1, \dots, x_n) \in F_r$ and $f \in \operatorname{Pol} F_r$. Consequently $\operatorname{Pol}\sigma \subseteq \operatorname{Pol} F_r$. Since F_r is a unary relation and $\operatorname{Pol}\sigma$ is meet-irreducible, we must have $F_r \in \{\emptyset, E_k\}$. Hence, for each $r \in \{1, \dots, m-1\}$, $F_r = \emptyset$ and s^r is a fixed point free permutation on E_k. Let p be a prime divisor of m; then $s^{\frac{m}{p}}$ is a fixed point free permutation on E_k in which all cycles are of length p. Thus, $(s^{\frac{m}{p}})^\circ$ is a relation of type (2) in Theorem 2.2 and $\operatorname{Pol}\sigma \subsetneq \operatorname{Pol}(s^{\frac{m}{p}})^\circ$, contradicting the fact that $\operatorname{Pol}\sigma$ is meet-irreducible below $\operatorname{Pol}\rho$. ∎

Second, we study the Cases (2) and (3).

Lemma 5.23. *If the assumptions of Propositions* 5.19 *are satisfied, then the Cases* (2) *and* (3) *are impossible.*

Proof. For Case (2), since $\sigma_1' = E_k$, for each $a \in E_k$ there exists $u \in E_k$ such that $(u, a) \in \sigma$. If $(u_1, a) \in \sigma$ and $(u_2, a) \in \sigma$, then $(u_2, u_1) \in \sigma_2 = \Delta_{E_k}$. Thus $u_1 = u_2$. Consider the unary operation f defined on E_k by $f(x) = y$ if $(y, x) \in \sigma$. Let $(a, b) \in E_k^2 = \sigma_2'$, then there exists $u \in E_k$ such that $(u, a), (u, b) \in \sigma$; so $f(a) = f(b) = u$. If follows that f is a constant unary function. Let $a \in E_k$; $(a, a) \in \Delta_{E_k} = \sigma_2$; so there exists $u \in E_k$ such that $(a, u) \in \sigma$, i.e., $f(u) = a$. Hence f is a surjective function on E_k, contradiction with the fact that f is a constant function. We deduce that the Case (2) is impossible.

The Case (3) is also impossible (use the unary operation g define by $g(x) = y$ iff $(x, y) \in \sigma$). ∎

Third, we study the Cases (4) and (5).

Lemma 5.24. *If the assumptions of Proposition* 5.19 *are satified, then the Cases* (4) *and* (5) *are impossible.*

Proof. For Case (4), define the unary operation f on E_k by $f(x) = y$ iff $(y, x) \in \sigma$. Let c be a central element of ρ. For $x \in E_k$, $(x, c) \in \rho = \sigma^{-1} \circ \sigma$; thus $f(x) = f(c)$. Therefore f is constant on E_k. By definition, σ^{-1} is the graph of f and σ_1 is the image of f. From (1) of Lemma 5.20, we get $E_k = \sigma_1 = \mathrm{Im}(f)$. Thus $\mathrm{Im}(f) = E_k$, contradiction with the fact that f is a constant function.

A similar argument proves that Case (5) is impossible. ■

Fourth, we investigate the Case (6) $\sigma_2 = \sigma'_2 = \rho$. For $l = 2, \ldots, k$, we set $\sigma_l = \{(a_1, \ldots, a_l) : \exists\, u \in E_k : (a_1, u), \ldots, (a_l, u) \in \sigma\}$ and $\sigma'_l = \{(a_1, \ldots, a_l) : \exists\, u \in E_k : (u, a_1), \ldots, (u, a_l) \in \sigma\}$ (for $l = 2$, this coincides with the definitions of σ_2 and σ'_2 given earlier). By definition, σ_l and σ'_l are totally symmetric. In addition, $\sigma_l \subseteq \bigcup_{0 \le j \le t-1} A^l_j$ and $\sigma'_l \subseteq \bigcup_{0 \le j \le t-1} A^l_j$. If $(a, b) \in \rho$, then $\{a, b\}^l \subseteq \sigma_l \cap \sigma'_l$.

Lemma 5.25. *Under the assumptions of Proposition* 5.19, *we have the following statements:*

(1) *If $\sigma_2 = \rho$, then for every maximal ρ-chain B there exists $\top_B \in E_k$ such that $(x, \top_B) \in \sigma$ for all $x \in B$.*

(2) *If $\sigma'_2 = \rho$, then for every maximal ρ-chain B there exists $\bot_B \in E_k$ such that $(\bot_B, x) \in \sigma$ for all $x \in B$.*

Proof. For (1), let $l \in \{2, \ldots, k\}$; since $\sigma_l \in [\{\sigma\}]$, we have $\mathrm{Pol}\,\sigma \subseteq \mathrm{Pol}\,\sigma_l$. If $\rho \subseteq \sigma$, then σ is reflexive; hence $\sigma \subseteq \sigma \circ \sigma^{-1} = \rho$, therefore $\rho = \sigma$, contradicting the choice of σ. Thus $\rho \nsubseteq \sigma$. Let $(a, b) \in \rho \backslash \sigma$, and assume that $\sigma_k \subsetneq \bigcup_{0 \le j \le t-1} A^k_j$. Clearly $\mathrm{Pol}\,\sigma \subseteq \mathrm{Pol}\,\sigma_k \subseteq \mathrm{Pol}\,\rho$. The unary operation defined on E_k by $l(x) = b$ if $(a, x) \in \sigma$ and $l(x) = a$ otherwise preserves σ_k (due to $(a, b) \in \rho$ and $\mathrm{Im}(l) = \{a, b\}$) and does not preserve σ (because there exists $u \in E_k$ such that $(a, u) \in \sigma$ and $(u, b) \in \sigma^{-1}$; and $(l(b), l(u)) = (a, b) \notin \sigma$); hence $\mathrm{Pol}\,\sigma \subsetneq \mathrm{Pol}\,\sigma_k$. To see that $\mathrm{Pol}\,\sigma_k \subsetneq \mathrm{Pol}\,\rho$, we choose $(c_1, \ldots, c_k) \in (\bigcup_{0 \le j \le t-1} A^k_j) \setminus \sigma_k$ and consider the k-tuples $\boldsymbol{w}_1 = (b, a, \ldots, a), \boldsymbol{w}_2 = (a, b, a \ldots, a), \ldots, \boldsymbol{w}_k = (a, \ldots, a, b)$ (recall that $(a, b) \in \rho \backslash \sigma$). The k-ary operation on E_k defined by $g(\boldsymbol{x}) = c_i$ if $x = \boldsymbol{w}_i$ for some $1 \le i \le k$ and $g(\boldsymbol{x}) = c_1$ elsewhere is well defined (because $|\{\boldsymbol{w}_i : 1 \le i \le k\}| = k$), preserves ρ (because $\{c_1, \ldots, c_k\}$ is a ρ-chain) and does not preserve σ_k because $\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_k\} \subseteq \sigma_k$ and $g(\boldsymbol{w}_1, \ldots, \boldsymbol{w}_k) = (g(\boldsymbol{w}_1), \ldots, g(\boldsymbol{w}_k)) = (c_1, \ldots, c_k) \notin \sigma_k$. Therefore $\mathrm{Pol}\,\sigma$ is not maximal in $\mathrm{Pol}\,\rho$; contradiction. We conclude that $\sigma_k = \bigcup_{0 \le j \le t-1} A^k_j$ giving the existence of $\top_B$ for each maximal ρ-chain B.

The Case (2) is obtained with a similar argument as above. ■

Let B, D be two maximal ρ-chains, then $C_\rho \subseteq B \cap D$ (due to (1) of Proposition 3.4). So $(\bot_B, c), (\bot_D, c) \in \sigma$ and $(\bot_B, \bot_D) \in \sigma \circ \sigma^{-1} = \rho$. We conclude that

$\bot = \{\bot_{A_0}, \dots, \bot_{A_{t-1}}\}$ is a ρ-chain. Let B be a maximal ρ-chain containing $\bot$, then $\bot_B \in B$ and $(\bot_B, \bot_B) \in \sigma$. Thus the set $U = \{x \in E_k : (x,x) \in \sigma\}$ is not empty and $\operatorname{Pol}\sigma \subseteq \operatorname{Pol} U$. Hence $U = E_k$ and σ is reflexive. Thus $\sigma \subsetneq \rho$ (due to $\sigma \subseteq \sigma \circ \sigma^{-1} = \rho$ and $\rho \nsubseteq \sigma$). Therefore, $\{\bot_B, \top_B\} \subseteq B$ for every maximal ρ-chain B.

Lemma 5.26. *If the assumptions of Proposition* 5.19 *are satisfied and σ being transitive, then the case $\sigma_2 = \sigma_2' = \rho$ is impossible.*

Proof. Assume that σ is transitive and $\sigma_2' = \sigma_2 = \rho$. Since σ is reflexive, $\gamma = \sigma \cap \sigma^{-1}$ is an equivalence relation and $\gamma \neq E_k^2$. If $\gamma \neq \Delta_{E_k}$, then γ is a nontrivial equivalence relation and $\operatorname{Pol}\sigma \subseteq \operatorname{Pol}\gamma$; contradiction with our assumption on σ. Thus $\gamma = \Delta_{E_k}$ and σ is a partial order. Since $\{\bot_{A_0}, \dots, \bot_{A_{t-1}}\}$ and $\{\top_{A_0}, \dots, \top_{A_{t-1}}\}$ are contained in maximal ρ-chains (due to $\sigma_2 = \sigma_2' = \rho$), there are $u, v \in E_k$ such that $(u, \bot_{A_0}), \dots, (u, \bot_{A_{t-1}}), (\top_{A_0}, v), \dots, (\top_{A_{t-1}}, v) \in \sigma$. Hence by transitivity of σ, u is the least element of σ and v the greatest element of σ. Therefore σ is a bounded partial order, contradiction. ∎

Lemma 5.27. *If the assumptions of Proposition* 5.19 *are satisfied, σ is not transitive and $\sigma_2 = \sigma_2' = \rho$, then σ is symmetric.*

Proof. Assume that $\sigma_2 = \sigma_2' = \rho$ and σ is not transitive. Set $\gamma = \sigma \cap \sigma^{-1}$. Then we have $\operatorname{Pol}\sigma \subseteq \operatorname{Pol}\gamma$. Suppose that $\operatorname{Pol}\sigma \subsetneq \operatorname{Pol}\gamma$. Since $\operatorname{Pol}\sigma$ is meet-irreducible and maximal below $\operatorname{Pol}\rho$, we have $\operatorname{Pol}\gamma = O_{E_k}$ or $\operatorname{Pol}\gamma = \operatorname{Pol}\rho$. Hence $\gamma \in \{\emptyset, \Delta_{E_k}, E_k^2\}$ or $\gamma = \rho$ (due to Lemma 5.18).

From discussion preceding Lemma 5.26, σ is reflexive and $\gamma \subseteq \sigma \subsetneq \rho$, thus $\gamma = \Delta_{E_k}$. Thus σ is antisymmetric. In addition, $\{\top_{A_0}, \dots, \top_{A_{t-1}}\}$ is a ρ-chain; so there exists a maximal ρ-chain D such that $\{\top_{A_0}, \dots, \top_{A_{t-1}}\} \subseteq D$; hence for every $i \in E_t$, we have $(\top_{A_i}, \top_D) \in \sigma$. Since $\sigma \subseteq \rho$ and D is a maximal ρ-chain, we have $\bot_D, \top_D \in D$ and $(\bot_D, \top_D) \in \rho$. If $(\top_D, \bot_D) \in tr(\sigma)$, then there exist $u_1, \dots, u_n \in E_k$ such that $(\top_D, u_1), (u_1, u_2), \dots, (u_{n-1}, u_n), (u_n, \bot_D) \in \sigma$.

Since $\{u_1, \top_D\}$ is also a ρ-chain, there exists a maximal ρ-chain B such that $\{u_1, \top_D\} \subseteq B$; hence $\top_D = \top_B = u_1$ (due to σ antisymmetric); by induction we show that $u_i = \top_D$, $1 \leq i \leq n$. Hence $\top_D = \bot_D$, $\top_{A_0} = \dots = \top_{A_{t-1}}$ and E_k is a ρ-chain; contradiction. Thus $(\top_D, \bot_D) \notin tr(\sigma)$, and $tr(\sigma) \neq \rho$. Since $(\bot_D, c), (c, \top_D) \in \sigma$, we get $(\bot_D, \top_D) \in tr(\sigma)$ and $\operatorname{Pol}\sigma = \operatorname{Pol}(tr(\sigma))$ (due to $\operatorname{Pol}\sigma \subseteq \operatorname{Pol}(tr(\sigma))$), $\operatorname{Pol}\sigma$ is meet-irreducible in $\operatorname{Pol}\rho$ and $\operatorname{Pol}\rho \neq \operatorname{Pol}(tr(\sigma))$.

If $tr(\sigma)$ is antisymmetric, then $tr(\sigma)$ is a partial order on E_k and $\sigma \subsetneq tr(\sigma)$. Let $(a,b) \in tr(\sigma)$ such that $(a,b) \notin \sigma$ and $(u,v) \in \sigma$ such that $u \neq v$. Then the unary operation h defined on E_k by $h(x) = a$ if $(x,u) \in tr(\sigma)$ and $h(x) = b$ otherwise, preserves $tr(\sigma)$ because $(a,b) \in tr(\sigma)$ and $tr(\sigma)$ is a partial order, and does not preserve σ (due to $(u,v) \in \sigma$ and $(h(u), h(v)) = (a,b) \notin \sigma$); contradiction. Hence $tr(\sigma)$ is not antisymmetric; so there exist $a, b \in E_k$ such that

$(a,b),(b,a) \in tr(\sigma)$ and $a \neq b$. Since σ is antisymmetric, we suppose that $(a,b) \notin \sigma$. Let $(u,v) \in \sigma$ such that $u \neq v$. The unary operation h' defined on E_k by $h'(x) = a$ if $x = u$ and $h'(x) = b$ otherwise, preserves $tr(\sigma)$ because $(a,b),(b,a) \in tr(\sigma)$ and $tr(\sigma)$ is reflexive, and does not preserve σ (due to $(u,v) \in \sigma$ and $(h'(u),h'(v)) = (a,b) \notin \sigma$); contradiction. Therefore $\operatorname{Pol}\sigma = \operatorname{Pol}\gamma$.

Since $\gamma \subsetneq \gamma \circ \gamma$ (due to γ reflexive and symmetric), it can be shown that $\operatorname{Pol}\sigma = \operatorname{Pol}\gamma \subsetneq \operatorname{Pol}(\gamma \circ \gamma)$. Since $\operatorname{Pol}\sigma$ is meet-irreducible and maximal below $\operatorname{Pol}\rho$, we get $\operatorname{Pol}(\gamma \circ \gamma) = O_{E_k}$ or $\operatorname{Pol}(\gamma \circ \gamma) = \operatorname{Pol}\rho$. Therefore $\gamma \circ \gamma = E_k^2$ or $\gamma \circ \gamma = \rho$ (by Lemma 5.18). On account of $\gamma \circ \gamma \subseteq \sigma \circ \sigma^{-1} = \rho$, we conclude that $\gamma \circ \gamma = \rho$. Hence γ fulfills the assumptions of Lemma 5.25; thus for every maximal ρ-chain B there exists $u_B \in B$ such that $(a, u_B) \in \gamma$ for all $a \in B$.

If $\gamma \subsetneq \sigma$, then there exists $(a,b) \in \sigma$ such that $(b,a) \notin \sigma$. Let B be a maximal ρ-chain containing a and b. The unary operation defined on E_k by $f(a) = b, f(b) = a$ and $f(x) = u_B$ if $x \notin \{a,b\}$, preserves γ since $(a,b) \notin \gamma$ and γ is reflexive and symmetric. But $(f(a),f(b)) = (b,a) \notin \sigma$ and $(a,b) \in \sigma$; so f does not preserve σ. Hence $\operatorname{Pol}\sigma \subsetneq \operatorname{Pol}\gamma \subsetneq \operatorname{Pol}\rho$, contradicting the fact that $\operatorname{Pol}\sigma$ is maximal in $\operatorname{Pol}\rho$. Therefore $\gamma = \sigma$ and σ is reflexive and symmetric. ■

Lemma 5.28. *If the assumptions of Proposition* 5.19 *are satisfied, σ is not transitive and $\rho \circ \sigma \neq E_k^2$, then the case $\sigma_2 = \sigma_2' = \rho$ is impossible.*

Proof. Assume that $\sigma_2 = \sigma_2' = \rho$, $\rho \circ \sigma \neq E_k^2$ and σ is not transitive. Since $\sigma_2 = \sigma_2' = \rho$ and σ is not transitive, by Lemma 5.27 σ is reflexive and symmetric. Therefore $\rho = \sigma \circ \sigma$ (due to $\sigma^{-1} = \sigma$ and σ is reflexive). Since σ and ρ are reflexive, σ and ρ are subsets of $\rho \circ \sigma$. If $\rho = \rho \circ \sigma$, then, from $\sigma \circ \sigma = \rho$, we have $\rho \circ \sigma = (\rho \circ \sigma) \circ \sigma = \rho \circ (\sigma \circ \sigma) = \rho \circ \rho = E_k^2$ contradicting the assumption $\rho \circ \sigma \neq E_k^2$. Therefore $\rho \subsetneq \rho \circ \sigma$. Furthermoe, $\sigma \subseteq \sigma \circ \sigma = \rho \subsetneq \rho \circ \sigma$ (due to σ is reflexive and $\sigma^{-1} = \sigma$). In addition, $(\rho \circ \sigma)^{-1} = \sigma^{-1} \circ \rho^{-1} = \sigma \circ \rho = \sigma \circ \sigma \circ \sigma = \rho \circ \sigma$; hence $\rho \circ \sigma$ is reflexive and symmetric, and $\sigma \subseteq \rho \subsetneq \rho \circ \sigma$. Let $(u,v) \in \sigma \setminus \Delta_{E_k}$ and $(a,b) \in \rho \circ \sigma \setminus \sigma$, then the unary operation f defined on E_k by $f(u) = a$ and $f(x) = b$ otherwise preserves $\rho \circ \sigma$ (due to $\rho \circ \sigma$ is reflexive and symmetric, $(a,b) \in \rho \circ \sigma$ and $\operatorname{Im}(f) = \{a,b\}$) and does not preserve σ (due to $(u,v) \in \sigma$ and $(f(u),f(v)) = (a,b) \notin \sigma$); therefore $\operatorname{Pol}\sigma \subsetneq \operatorname{Pol}(\rho \circ \sigma)$ (due to $\rho \circ \sigma \in [\{\sigma\}]$). Since $\rho \subsetneq \rho \circ \sigma \neq E_k^2$ and $\rho \circ \sigma$ is symmetric, then ρ and $\rho \circ \sigma$ are two distinct central relations; so $\operatorname{Pol}\rho \neq \operatorname{Pol}\rho \circ \sigma$. As $\operatorname{Pol}\sigma$ is meet-irreducible below $\operatorname{Pol}\rho$ we have a contradiction. ■

Lemma 5.29. *If the assumptions of Propositions* 5.19 *are satisfied, σ is not transitive, $\sigma_2 = \sigma_2' = \rho$ and $\rho \circ \sigma = E_k^2$, then we obtain a relation of type VIII.*

Proof. Assume that $\rho \circ \sigma = E_k^2$, $\sigma_2 = \sigma_2' = \rho$ and σ is not transitive; using Lemma 5.27, σ is reflexive and symmetric. It remains to show that for every maximal ρ-chain B, there exists a central element c_B of ρ such that for every $a \in B$, $(a, c_B) \in \sigma$.

Let B be a maximal ρ-chain of cardinality m, and set $\gamma = \{(a_1, \ldots, a_k) \in E_k^k : \exists u \in E_k$ such that $(a_i, u) \in \sigma$ for all $1 \le i \le m$ and $(a_i, u) \in \rho$ for all $m+1 \le i \le k\}$ and

$$\beta = \{(a_1, \ldots, a_k) \in E_k^k : (a_i, a_j) \in \rho \text{ for all } 1 \le i < j \le m\}.$$

It is easy to see that $\gamma \subseteq \beta$. We will show that $\operatorname{Pol}\gamma \subseteq \operatorname{Pol}\rho$. Let $f \in \operatorname{Pol}\gamma$ be an n-ary operation and let $(a_i, b_i) \in \rho$, $1 \le i \le n$, then there exist $u_i \in E_k$, $1 \le i \le n$, such that (a_i, u_i), $(u_i, b_i) \in \sigma$ (due to $\sigma \circ \sigma = \rho$). Thus $(a_i, b_i, \underbrace{u_i \ldots, u_i}_{k-2 \text{ times}}) \in \gamma$ $(*_7)$. Hence $f((a_1, b_1, u_1, \ldots, u_1), \ldots, (a_n, b_n, u_n, \ldots, u_n)) = (f(a_1, \ldots, a_n),\ f(b_1, \ldots, b_n),\ f(u_1, \ldots, u_n), \ldots, f(u_1, \ldots, u_n)) \in \gamma$. Therefore $(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)) \in \sigma \circ \sigma = \rho$. So $\operatorname{Pol}\gamma \subseteq \operatorname{Pol}\rho$. Now suppose that $\gamma \subsetneq \beta$. Let $(b_1, \ldots, b_k) \in \beta \setminus \gamma$, $(a, b) \notin \rho$ and c be a central element of ρ. For $1 \le i \le k$, set $\boldsymbol{x}_i = (a_{1,i}, \ldots, a_{k,i})$ with
$a_{ji} = \begin{cases} c & \text{if } j = i, \\ a & \text{otherwise} \end{cases}$ for $1 \le i \le m$ and $a_{ji} = \begin{cases} b & \text{if } j = i, \\ a & \text{otherwise} \end{cases}$ for $m+1 \le i \le k$.

It is easy to see that $(\boldsymbol{x}_i, \boldsymbol{x}_j) \in \rho$ if and only if i and j are elements of $\{1, \ldots, m\}$ or $i = j$. The k-ary function f defined by

$$f(\boldsymbol{x}) = \begin{cases} b_i & \text{if } \boldsymbol{x} = \boldsymbol{x}_i \text{ for some } 1 \le i \le k; \\ c & \text{otherwise} \end{cases}$$

is well define (due to $|\{\boldsymbol{x}_i : 1 \le i \le k\}| = k$) and preserves ρ. In addition, by construction, we have $\{\boldsymbol{x}_1 \ldots, \boldsymbol{x}_k\} \subseteq \gamma$, and $f(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k) = (f(\boldsymbol{x}_1), \ldots, f(\boldsymbol{x}_k)) = (b_1, \ldots, b_k) \notin \gamma$; hence $\operatorname{Pol}\gamma \subsetneq \operatorname{Pol}\rho$. From $\sigma \subsetneq \rho$, there exists $(a, b) \in \rho \setminus \sigma$. Let $(u, v) \in \sigma$ such that $u \ne v$, then the unary function f' defined by $f'(x) = a$ if $x = u$ and $f'(x) = b$ otherwise, preserves γ (due to $(a, b) \in \rho$, $\{a, b\}^k \subseteq \gamma$ and $\operatorname{Im}(f') = \{a, b\}$), and does not preserve σ (due to $(u, v) \in \sigma$ and $(f'(u), f'(v)) = (a, b) \notin \sigma$). Thus $\operatorname{Pol}\sigma \subsetneq \operatorname{Pol}\gamma \subsetneq \operatorname{Pol}\rho$, contradicting our assumption on σ. Therefore $\gamma = \beta$.

Let $B = \{a_1, \ldots, a_n\}$ be a maximal ρ-chain and $a_{n+1}, \ldots, a_k \in E_k$ such that $E_k = \{a_1, \ldots, a_k\}$. Since $(a_1, \ldots, a_k) \in \beta = \gamma$, there exists $u \in E_k$ such that $(a_i, u) \in \sigma$ for, $1 \le i \le m$, and $(a_i, u) \in \rho$, for $m+1 \le i \le k$; so $u \in C_\rho$ and $u \in B$ from Proposition 3.4.

We conclude that σ is of type VIII. ■

The following lemma will be useful for the remaining cases.

Lemma 5.30. *Under the assumptions of Proposition* 5.19, *we have the following statements:*

(1) *If* $\sigma_h = E_k^h$, *then there exists* $\top \in E_k$ *such that for any* $x \in E_k, (x, \top) \in \sigma$.

(2) *If* $\sigma'_h = E_k^h$, *then there exists* $\bot \in E_k$ *such that for any* $x \in E_k$, $(\bot, x) \in \sigma$.

Proof. For $2 \leq l \leq k$, σ_l and σ'_l are relations defined below. We give the proof of (1); and (2) is obtained using a similar argument.

For (1), assume that $\sigma_h = E_k^h$ and $\sigma_k \neq E_k^k$. Let $n \geq h$ be the least integer N such that $\sigma_N \neq E_k^N$, then $n > h \geq 2$. We will show that $\operatorname{Pol}\sigma \subsetneq \operatorname{Pol}\sigma_n \nsubseteq \operatorname{Pol}\rho$. By definition σ_n is totally symmetric. Furthermore, using $\sigma_{n-1} = E_k^{n-1}$ one can easily see that σ_n is totally reflexive. Since $\sigma_n \in [\{\sigma\}]$, we get $\operatorname{Pol}\sigma \subseteq \operatorname{Pol}\sigma_n$. Let $(a, b) \in E_k^2 \setminus \sigma$ and $(u, v) \in \sigma \setminus \Delta_{E_k}$, then the unary operation f defined on E_k by $f(x) = a$ if $x = u$ and $f(x) = b$ otherwise, preserves σ_n (due to $n > h \geq 2$, σ_n totally reflexive, $\operatorname{Im}(f) = \{a, b\}$ and $\{a, b\}^n \subseteq \sigma_n$) and does not preserve σ (due to $(u, v) \in \sigma$ and $(f(u), f(v)) = (a, b) \notin \sigma$); therefore $\operatorname{Pol}\sigma \subsetneq \operatorname{Pol}\sigma_n$. Let $(a_1, \ldots, a_h) \in E_k^h \setminus \rho$ and $(u_1, \ldots, u_h) \in \rho \setminus \iota_k^h$, then the unary operation g defined on E_k by $g(x) = a_i$ if $x = u_i$, for some $1 \leq i \leq h$ and $g(x) = a_1$ otherwise preserves σ_n (due to $\operatorname{Im}(g) = \{a_1, \ldots, a_h\}$, σ_n totally reflexive and $h \leq n - 1$) and does not preserve ρ (due to $(u_1, \ldots, u_h) \in \rho$ and $(g(u_1), \ldots, g(u_h)) = (a_1, \ldots, a_h) \notin \rho$. Therefore $\operatorname{Pol}\sigma \subsetneq \operatorname{Pol}\sigma_n \nsubseteq \operatorname{Pol}\rho$, contradicting the meet-irreducibility of $\operatorname{Pol}\sigma$ below $\operatorname{Pol}\rho$. Thus $\operatorname{Pol}\sigma_n = O_{E_k}$; since σ_n is totally reflexive and totally symmetric, it follows that $\sigma_n = E_k^n$, contradiction with $\sigma_n \neq E_k^n$. Thus $\sigma_k = E_k^k$ and there exists $\top \in E_k$ such that $(x, \top) \in \sigma$ for all $x \in E_k$ and (1) holds. ∎

From Lemma 5.21, as $(\top, \top) \in \sigma$ (resp $(\bot, \bot) \in \sigma$) whenever $\sigma_h = E_k^h$ (resp. $\sigma'_h = E_k^h$), we can claim that σ is reflexive. We continue the investigation with Cases (7) ($\sigma_2 = \rho$ and $\sigma'_2 = E_k^2$) and (8) ($\sigma_2 = E_k^2$ and $\sigma'_2 = \rho$). Let $(a, b) \in E_k^2 \setminus \sigma$ and $(u, v) \in \sigma$ such that $u \neq v$, consider the unary operation f_0 defined by $f_0(x) = a$ if $x = u$ and $f_0(x) = b$ otherwise.

Lemma 5.31. *If the assumptions of Proposition* 5.19 *are satisfied,* $\sigma_2 = \rho$ *and* $\sigma'_2 = E_k^2$, *then* σ *is of type X.*

Proof. Assume that $\sigma_2 = \rho$, $\sigma'_2 = E_k^2$. Since $\sigma'_2 = E_k^2$, from Lemma 5.30, there exists $\bot \in E_k$ such that for all $a \in E_k$ $(\bot, a) \in \sigma$. So $(\bot, \bot) \in \sigma$, and using the discussion preceding Lemma 5.26, we get that σ is reflexive. As $\sigma_2 \neq \sigma'_2$, we get that σ is not symmetric (Because if σ were symmetric, then it would hold that $\sigma_2 = \sigma \circ \sigma^{-1} = \sigma \circ \sigma = \sigma^{-1} \circ \sigma = \sigma'_2$). It follows that σ is reflexive and not symmetic. Let $a \in E_k$, then there exists $v \in E_k$ such that $(a, v) \in \sigma$ (see (a) of Lemma 5.20). Therefore $(a, v), (\bot, v) \in \sigma$ and consequently $(a, \bot) \in \sigma_2 = \rho$. Thus $\bot \in C_\rho$. Set $\gamma = \sigma \cap \sigma^{-1}$. We have $\gamma \subsetneq \sigma \subseteq \sigma \circ \sigma^{-1} = \rho$ and $\operatorname{Pol}\sigma \subseteq \operatorname{Pol}\gamma$. Our discussion is divided into two cases: (i) $\operatorname{Pol}\sigma \subsetneq \operatorname{Pol}\gamma$ and (ii) $\operatorname{Pol}\sigma = \operatorname{Pol}\gamma$.

(i) If $\operatorname{Pol}\sigma \subsetneq \operatorname{Pol}\gamma$, then $\operatorname{Pol}\gamma = O_{E_k}$ or $\operatorname{Pol}\gamma = \operatorname{Pol}\rho$. If $\operatorname{Pol}\gamma = O_{E_k}$, then as γ is reflexive and $\gamma \subsetneq \sigma$, we get that $\gamma = \Delta_{E_k}$. If $\operatorname{Pol}\gamma \neq O_{E_k}$, then $\operatorname{Pol}\gamma = \operatorname{Pol}\rho$ and by Lemma 5.18, we get that $\gamma = \rho$. Therefore $\gamma \in \{\Delta_{E_k}, \rho\}$;

furthermore $\gamma \subsetneq \sigma \subseteq \rho$, so $\gamma = \Delta_{E_k}$ and σ is antisymmetric. Let $b \in E_k$ such that $b \neq \perp$, then $(\perp, b) \in \sigma$ and $(b, \perp) \notin \sigma$. Therefore $(b, \perp) \notin tr(\sigma)$. Thus $tr(\sigma) \notin \{\rho, E_k^2\}$. If $\sigma \subsetneq tr(\sigma)$, then the unary function h defined by $h(x) = a$ if $(x, u) \in tr(\sigma)$ and $h(x) = b$ otherwise where $(a, b) \in tr(\sigma) \setminus \sigma$ and $(u, v) \in \sigma$, $u \neq v$, preserves $tr(\sigma)$ and does not preserve σ. Hence $\operatorname{Pol}\sigma \subsetneq \operatorname{Pol}(tr(\sigma))$, $tr(\sigma) \notin \{\rho, E_k^2\}$ and $\sigma \subsetneq tr(\sigma)$; contradiction with the fact that $\operatorname{Pol}\sigma$ is meet-irreducible and maximal below $\operatorname{Pol}\rho$. Thus $\sigma = tr(\sigma)$ and σ is a partial order.

From Lemma 5.25, for every maximal ρ-chain B there exists $\top_B$ such that for all $x \in B$ $(x, \top_B) \in \sigma \subseteq \rho$. Hence $\top_B \in B$ and $\top_B$ is the greatest element of B. It remains to show that every intersection of maximal ρ-chains has a greatest element.

Let $B_1, \dots, B_n$ be n maximal ρ-chains ($n \geq 2$). Let l be the cardinality of $\bigcap_{1 \leq i \leq n} B_i$ and m be the cardinality of $\bigcap_{1 \leq i \leq n} B_i$. If $\bigcup_{1 \leq i \leq n} B_i$ is a maximal ρ-chain, we are done, because in that case $B_1 = B_i$, $i = 2, \dots, n$. If $\bigcup_{1 \leq i \leq n} B_i$ is not a maximal ρ-chain and $\bigcap_{1 \leq i \leq n} B_i = \{\perp\}$, then $\perp$ is the greatest element of $\bigcap_{1 \leq i \leq n} B_i$. If $\bigcap_{1 \leq i \leq n} B_i$ is not a maximal ρ-chain and $\bigcap_{1 \leq i \leq n} B_i \neq \{\perp\}$, then let $a, v, w \in E_k$ such that $a \in \bigcap_{1 \leq i \leq n} B_i$, $a \neq \perp$ and $v, w \in \bigcap_{1 \leq i \leq n} B_i$, $(v, w) \notin \rho$. Let λ and β be the sets defined by $\lambda := \{(a_1, \dots, a_m) \in E_k^m : (a_i, a_j) \in \rho \ \ \forall 1 \leq i \leq l, \ \forall\ 1 \leq j \leq m\}$ and

$$\beta := \{(a_1, \dots, a_m) \in E_k^m : \forall 1 \leq i \leq l, \forall 1 \leq j \leq m, (a_i, a_j) \in \rho, \exists\, u \in E_k,\\ (a_i, u) \in \sigma\ \forall\ 1 \leq i \leq l \wedge (a_i, u) \in \rho\ \forall\ l+1 \leq i \leq m\}.$$

We have $\beta \subseteq \lambda$. Now suppose that $\beta \neq \lambda$. We will show that $\operatorname{Pol}\beta \subseteq \operatorname{Pol}\rho$. Let $f \in \operatorname{Pol}\beta$ be an n-ary operation and $(a_i, b_i) \in \rho$, $1 \leq i \leq n$, then there exist u_i, $i = 1, \dots, u_n$, such that $(a_i, u_i), (b_i, u_i) \in \sigma, 1 \leq i \leq n$. Set $\boldsymbol{y}_i = (a_i, b_i, u_i, \dots, u_i), 1 \leq i \leq n$ and $\boldsymbol{x}_1 = (a_1, \dots, a_n)$, $\boldsymbol{x}_2 = (b_1, \dots, b_n)$, $\boldsymbol{x}_3 = (u_1, \dots, u_n), \dots, \boldsymbol{x}_m = (u_1, \dots, u_n)$. Then $\{\boldsymbol{y}_1, \dots, \boldsymbol{y}_n\} \subseteq \beta$ (due to $\rho = \sigma_2$ and $(a_i, b_i) \in \rho, 1 \leq i \leq n\}$); as $f \in \operatorname{Pol}\beta$, we deduce that $f(\boldsymbol{y}_1, \dots, \boldsymbol{y}) = (f(\boldsymbol{x}_1), \dots, f(\boldsymbol{x}_m)) \in \beta$ and by definition of β, we get that $(f(\boldsymbol{x}_1), f(\boldsymbol{x}_2)) \in \rho$ (due to $l \geq 2$); therefore $f \in \operatorname{Pol}\rho$ and $\operatorname{Pol}\beta \subseteq \operatorname{Pol}\rho$. It is easy to show that $\operatorname{Pol}\sigma \subseteq \operatorname{Pol}\beta$ (using $\sigma_2 = \rho$); therefore $\operatorname{Pol}\sigma \subseteq \operatorname{Pol}\beta \subseteq \operatorname{Pol}\rho$.

Let $(b_1, \dots, b_m) \in \lambda \setminus \beta$. For $1 \leq i \leq m$, set $\boldsymbol{x}_i = (a_{i,1}, \dots, a_{i,k})$ with

$$a_{i,j} = \begin{cases} a & \text{if } i = j, \\ \perp & \text{otherwise} \end{cases} \quad \text{for} \ \ 1 \leq i \leq l \text{ and } a_{i,j} = \begin{cases} \perp & \text{if } 1 \leq j \leq l, \\ w & \text{if } i = j, \\ v & \text{elsewhere} \end{cases} \quad \text{for}$$

$l+1 \leq i \leq m$. It is easy to see that $(\boldsymbol{x}_i, \boldsymbol{x}_j) \in \rho$ if and only if i or $j \in \{1, \dots, l\}$ $(*)$. Consider the k-ary function f defined by $f(\boldsymbol{x}) = b_i$ if $\boldsymbol{x} = \boldsymbol{x}_i$, for some $1 \leq i \leq m$ and $f(\boldsymbol{x}) = \perp$ elsewhere. The operation f is well defined (because $|\{\boldsymbol{x}_1, \dots, \boldsymbol{x}_m\}| = m$). We will show that f does not preserve β. Let $\boldsymbol{y}_j = (a_{1,j}, \dots, a_{m,j}), 1 \leq j \leq k$, from the definition of $\boldsymbol{x}_i, 1 \leq i \leq m$ and $(*)$ we have that $\{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_k\} \subseteq \beta$, and $f(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_k) = (f(\boldsymbol{x}_1), \ldots, f(\boldsymbol{x}_m)) = (b_1, \ldots, b_m) \notin \beta$, so $f \notin \operatorname{Pol}\beta$. In addition, we can show that $f \in \operatorname{Pol}\rho$. Hence $\operatorname{Pol}\beta \subsetneq \operatorname{Pol}\rho$. Since $\sigma \subsetneq \rho$ (due to $\operatorname{Pol}\sigma \subsetneq \operatorname{Pol}\rho$), then there exists $(e,d) \in \rho \setminus \sigma$. Let $(x,y) \in \sigma$ with $x \neq y$ and $(e,d) \in \rho \setminus \sigma$; there exists $u \in E_k$ such that $(e,u),(d,u) \in \sigma$. The unary operation f defined on E_k by $f(x) = e$, $f(y) = d$ and $f(t) = u$ elsewhere preserves β (due to $(e,d) \in \rho$ and $\rho = \sigma_2$) and does not preserve σ (due to $(x,y) \in \sigma$ and $(f(x), f(y)) = (e,d) \notin \sigma$). Therefore $\operatorname{Pol}\sigma \subsetneq \operatorname{Pol}\beta \subsetneq \operatorname{Pol}\rho$, contradicting the fact that $\operatorname{Pol}\sigma$ is meet-irreducible below $\operatorname{Pol}\rho$. Hence $\beta = \lambda$, and $\bigcap_{1\leq i\leq n} B_i = \{a_1, \ldots, a_l\}$ is such that $(a_1, \ldots, a_l, \underbrace{a_l \ldots, a_l}_{m-l \text{ times}}) \in \beta$; from the definition of β, we deduce that $\bigcap_{1\leq i\leq n} B_i$ has a greatest element. Therefore σ is of type X.

(ii) If $\operatorname{Pol}\sigma = \operatorname{Pol}\gamma$, then $\gamma \neq \Delta_{E_k}$, γ is reflexive and symmetric. We will show that $\gamma = \sigma$. In fact, if $\gamma \subsetneq \sigma$, then there exists $(a,b) \in \sigma$ such that $(a,b) \notin \gamma$. So $(b,a) \notin \sigma$ (due to $\gamma = \sigma \cap \sigma^{-1}$). Furthermore, $\gamma \subseteq \gamma \circ \gamma$ and $\operatorname{Pol}\sigma \subseteq \operatorname{Pol}(\gamma \circ \gamma)$.

If $\gamma \circ \gamma = \gamma$, then γ is a nontrivial equivalence relation and $\operatorname{Pol}\sigma \subseteq \operatorname{Pol}\gamma$, contradiction. Hence $\gamma \subsetneq \gamma \circ \gamma$ and $\operatorname{Pol}\sigma = \operatorname{Pol}\gamma \subseteq \operatorname{Pol}\gamma \circ \gamma$, as $\operatorname{Pol}\sigma$ is maximal and meet-irreducle below $\operatorname{Pol}\rho$, we get that $\operatorname{Pol}\gamma \circ \gamma = O_{E_k}$ or $\operatorname{Pol}\gamma \circ \gamma = \operatorname{Pol}\rho$. By our assumption, $\sigma_2 = \rho$; furthermore $\gamma \subseteq \sigma \cap \sigma^{-1} \subseteq \sigma$ and γ is symmetric; so $\Delta_{E_k} \subsetneq \gamma \subseteq \gamma \circ \gamma \subseteq \sigma \circ \sigma^{-1} = \sigma_2 = \rho \neq E_k^2$. Therefore $\gamma \circ \gamma$ is not a diagonal relation; so the equality $\operatorname{Pol}\gamma \circ \gamma = O_{E_k}$ is impossible. Therefore $\operatorname{Pol}\gamma \circ \gamma = \operatorname{Pol}\rho$ and by Lemma 5.18, $\rho = \gamma \circ \gamma$. It follows that γ fulfills the assumptions of Lemma 5.25; so for any maximal ρ-chain B there exists $u_B \in E_k$ such that for any $x \in B$, $(x, u_B) \in \gamma$. Let B be a maximal ρ-chain containing a and b, the unary operation defined on E_k by $f(a) = b$, $f(b) = a$ and $f(x) = u_B$ if $x \notin \{a,b\}$ preserves γ (because $(a,b) \notin \gamma$, γ is reflexive, symmetric and $(a,u_B),(b,u_B) \in \gamma$) and does not preserve σ (because $(a,b) \in \sigma$ and $(f(a), f(b)) = (b,a) \notin \sigma$). Therefore, $\operatorname{Pol}\sigma \neq \operatorname{Pol}\gamma$, contradiction with the assumption $\operatorname{Pol}\sigma = \operatorname{Pol}\gamma$. Hence $\gamma = \sigma$ and σ is symmetric, contradicting the fact that $\sigma_2 \neq \sigma_2'$. ∎

Lemma 5.32. *If the assumptions of Proposition* 5.19 *are satisfied,* $\sigma_2 = E_k^2$ *and* $\sigma_2' = \rho$, *then* σ *is of type XI.*

Proof. Note that σ^{-1} fulfills the assumptions of Lemma 5.31 and $\operatorname{Pol}\sigma = \operatorname{Pol}\sigma^{-1}$. Therefore σ^{-1} is a relation of type X. Hence σ is the relation of type XI. ∎

Now, we finish our discussion with Case 9: $\sigma_2 = \sigma_2' = E_k^2$. We have two subcases $\sigma_h = \sigma_h' = E_k^h$ or ($\sigma_h \neq E_k^h$ or $\sigma_h' \neq E_k^h$). We begin with subcase $\sigma_h = \sigma_h' = E_k^h$. Using Lemmas 5.30 and 5.21, it is easy to see that, σ is reflexive. Naturally, σ can be transitive or not.

Lemma 5.33. *If the assumptions of Proposition* 5.19 *are satisfied and* $\sigma_h = \sigma_h' = E_k^h$, *then the case* σ *transitive is impossible.*

Proof. Assume that σ is transitive. If σ is symmetric, then $\sigma = \sigma \circ \sigma = \sigma_2 = E_k^2$; contradiction. Hence σ is not symmetric. Set $\gamma = \sigma \cap \sigma^{-1}$. We have $\operatorname{Pol} \sigma \subseteq \operatorname{Pol} \gamma$ (due to $\gamma \in [\{\sigma\}]$). If $\operatorname{Pol} \sigma \subsetneq \operatorname{Pol} \gamma$, then $\gamma \in \{\Delta_{E_k}, \rho\}$ because $\operatorname{Pol} \sigma$ is meet-irreducible and $\Delta_{E_k} \subseteq \gamma \subseteq \sigma \subsetneq E_k^2$. Suppose that $\gamma = \Delta_{E_k}$, then by Lemma 5.30 σ is a bounded partial order, contradiction. So $\gamma = \rho$. Therefore ρ is transitive and $\rho = \rho \circ \rho = E_k^2$, contradicting the fact that ρ is a central relation. If $\operatorname{Pol} \sigma = \operatorname{Pol} \gamma$, then $\gamma \notin \{\emptyset, \Delta_{E_k}, E_k^2, \rho\}$. Hence γ is a nontrivial equivalence relation and we obtain a contradiction. ■

Lemma 5.34. *If the assumptions of Proposition* 5.19 *are satisfied,* $\sigma_h = \sigma'_h = E_k^h$ *and* σ *is not transitive, then* σ *is of type IX.*

Proof. We claim that σ is not symmetric. In fact, if σ is symmetric, then σ is reflexive and symmetric; from Lemma 5.30, σ is a central relation which is a contradiction. Hence σ is not symmetric. Set $\gamma = \sigma \cap \sigma^{-1}$. As σ is not symmetric, we have $\gamma \subsetneq \sigma$.

(i) If $\operatorname{Pol} \sigma = \operatorname{Pol} \gamma$, then γ is reflexive, symmetric and $\operatorname{Pol} \gamma$ is meet-irreducible and maximal below $\operatorname{Pol} \rho$. If γ is transitive, then γ is a nontrivial equivalence relation; contradiction. Hence $\gamma \subsetneq \gamma \circ \gamma$. Let $(a, b) \in (\gamma \circ \gamma) \setminus \gamma$. For $(u, v) \in \gamma$ such that $u \neq v$, the above operation f_0 preserves $\gamma \circ \gamma$ and does not preserve γ. Hence $\operatorname{Pol} \gamma \subsetneq \operatorname{Pol}(\gamma \circ \gamma)$. It follows that (1) $\operatorname{Pol}(\gamma \circ \gamma) = O_{E_k}$ or (2) $\operatorname{Pol}(\gamma \circ \gamma) = \operatorname{Pol} \rho$.

Suppose that (1) is satisfied, then $\gamma \circ \gamma = E_k^2$, therefore γ satisfies the assumptions of Lemma 5.30. We conclude that γ is a central relation; contradiction. Suppose that (2) is satisfied, then $\gamma \circ \gamma = \rho$ (see Lemma 5.18). We claim that $\rho \nsubseteq \sigma$. Assume that $\rho \subseteq \sigma$. As $\gamma \circ \gamma = \rho$ and γ is reflexive, we have $\gamma \subseteq \rho$, this inclusion is strict because $\operatorname{Pol} \gamma = \operatorname{Pol} \sigma \subsetneq \operatorname{Pol} \rho$. Let $(a, b) \in \rho \setminus \gamma$, then $(b, a) \in \rho \subseteq \sigma$; so $(a, b) \in \sigma \cap \sigma^{-1} = \gamma$, contradicting the fact that $(a, b) \notin \gamma$; therefore $\rho \nsubseteq \sigma$. Let $(a, b) \in \rho \setminus \sigma$ and $(x, y) \in \sigma \setminus \gamma$; since $\gamma \circ \gamma = \rho$ there exists w such that $(a, w), (w, b) \in \gamma$. Consider the unary operation f defined by $f(x) = a, f(y) = b$ and for $t \in E_k \setminus \{a, b\}$, $f(t) = w$. It is easy to see that f preserves γ and does not preserve σ. Thus $\operatorname{Pol} \sigma \subsetneq \operatorname{Pol} \gamma \subsetneq \operatorname{Pol} \rho$ contradicting the assumption $\operatorname{Pol} \gamma = \operatorname{Pol} \sigma$.

(ii) If $\operatorname{Pol} \sigma \neq \operatorname{Pol} \gamma$, then $\gamma \in \{\Delta_{E_k}, \rho\}$. Suppose that $\gamma = \Delta_{E_k}$, then σ is reflexive and antisymmetric. If $\operatorname{Pol} \sigma \subsetneq \operatorname{Pol}(tr(\sigma))$, then by assumptions on σ we have $tr(\sigma) \in \{\rho, E_k^2\}$. However, by Lemma 5.30, $\top$ and $\perp$ are such that $(\perp, x), (x, \top) \in \sigma$ for any $x \in E_k$. Hence $(\perp, \top) \in tr(\sigma)$, but $(\top, \perp) \notin tr(\sigma)$ (due to σ antisymmetric), contradiction. Hence $\operatorname{Pol} \sigma = \operatorname{Pol} tr(\sigma)$. It follows that $tr(\sigma)$ fulfills the assumptions of Lemma 5.33; thus we obtain a contradiction. Hence $\gamma = \rho$ and σ is of type IX. ■

We close the Case (9) with subcase $\sigma_h \neq E_k^h$ or $\sigma'_h \neq E_k^h$. Recall that

$$\sigma_l = \{(a_1, \dots, a_l) \in E_k^l : \exists\, u \in E_k, (a_1, u), \dots, (a_l, u) \in \sigma\}$$

$$\text{and } \sigma'_l = \{(a_1, \ldots, a_l) \in E_k^l : \exists\, u \in E_k, (u, a_1,), \ldots, (u, a_l) \in \sigma\}$$

for $2 \leq l \leq k$. We have shown that σ is reflexive or irreflexive.

Lemma 5.35. *If the assumptions of Proposition* 5.19 *are satisfied,* $\sigma_2 = E_k^2 = \sigma'_2$ *and* $h \geq 3$*, then the following implications hold.*

(i) *If* $\sigma_h \neq E_k^h$*, then* $\sigma_h = \rho$.

(ii) *If* $\sigma'_h \neq E_k^h$*, then* $\sigma'_h = \rho$.

Proof. (i) Assume that $\sigma_2 = \sigma'_2 = E_k^2$ and $\sigma_h \neq E_k^h$. Let n be the least integer N such that $\sigma_N \neq E_k^N$, then $2 < n \leq h$. Since $\sigma_{n-1} = E_k^{n-1}$, σ_n is totally reflexive. Let $(a, b) \in E_k^2 \setminus \sigma$ and $(u, v) \in \sigma$ such that $u \neq v$. Then, the above unary operation f_0 preserves σ_n and does not preserve σ; therefore $\operatorname{Pol}\sigma \subsetneq \operatorname{Pol}\sigma_n$. Assume that $n < h$. If $\sigma_n = \iota_k^n$, then $\operatorname{Pol}\sigma_n \subsetneq \operatorname{Pol}\iota_k^k \neq \operatorname{Pol}\rho$, contradiction (due to $\operatorname{Pol}\sigma$ is meet-irreducible below $\operatorname{Pol}\rho$ and $\operatorname{Pol}\iota_k^k$ a maximal clone). Hence $\sigma_n \neq \iota_k^n$. Let $(a_1, \ldots, a_n) \in E_k^n \setminus \sigma_n$, $(u_1, \ldots, u_n) \in \sigma_n \setminus \iota_k^n$. Consider the unary operation g_1 defined on E_k by $g_1(x) = a_i$ if $x = u_i$, for some $1 \leq i \leq n$, and $g_1(x) = a_1$ elsewhere. Since $n < h$, $\operatorname{Im}(g_1) = \{a_1, \ldots, a_n\}$ and ρ is totally reflexive, g_1 preserves ρ. But, $(u_1, \ldots, u_n) \in \sigma_n$ and $(g_1(u_1), \ldots, g_1(u_n)) = (a_1, \ldots, a_n) \notin \sigma_n$, so $g_1 \notin \operatorname{Pol}\sigma_n$; thus $\operatorname{Pol}\rho \nsubseteq \operatorname{Pol}\sigma_n$. Therefore $\operatorname{Pol}\sigma \subsetneq \operatorname{Pol}\sigma_n \neq O_{E_k}$. Indeed, since $\operatorname{Pol}\sigma$ is meet-irreducible and maximal below $\operatorname{Pol}\rho$, we must have $\operatorname{Pol}\rho \subseteq \operatorname{Pol}\sigma_n$ which is a contradiction. It follows that $n = h$ and $\operatorname{Pol}\sigma \subsetneq \operatorname{Pol}\sigma_h$; since $\operatorname{Pol}\sigma$ is meet-irreducible and maximal below $\operatorname{Pol}\rho$ we get $\operatorname{Pol}\sigma_h = O_{E_k}$ or $\operatorname{Pol}\sigma_h = \operatorname{Pol}\rho$. If $\operatorname{Pol}\sigma_h = O_{E_k}$, then σ_h is a diagonal relation and as σ_h is totally reflexive, we deduce that $\sigma_h = E_k^h$, contradiction with $\sigma_h \neq E_k^h$. Thus $\operatorname{Pol}\sigma_h = \operatorname{Pol}\rho$, using Lemma 5.18 we get that $\sigma_h = \rho$.

The proof of (ii) is similar to that of (i). ■

Now, consider the set $\Gamma = \{B \subseteq E_k : B^h \subseteq \rho\}$. Let $m = \max\{\operatorname{Card}(B) : B \in \Gamma\}$; we have $h \leq m$. For all $h \leq l \leq m$, set

$$\rho_l = \big\{(a_1, \ldots, a_l) \in E_k^l : \{a_1, \ldots, a_l\}^h \subseteq \rho\big\}.$$

It is easy to check that for all $h \leq l \leq m$, $\sigma_l \subseteq \rho_l$ and $\sigma'_l \subseteq \rho_l$ where $\sigma_h = \sigma'_h = \rho$.

Lemma 5.36. *Under the assumptions of Proposition* 5.19, *the following statements hold.*

(i) *If* $\sigma_h \neq E_k^h$*, then for every maximal* ρ*-chain* B *there exists* $\top_B \in E_k$ *such that for all* $a \in B$, $(a, \top_B) \in \sigma$.

(ii) *If* $\sigma'_h \neq E_k^h$*, then for every maximal* ρ*-chain* B *there exists* $\bot_B \in E_k$ *such that for all* $a \in B$, $(\bot_B, a) \in \sigma$.

Proof. (i) Assume that $\sigma_h \neq E_k^h$, then from (i) of Lemma 5.35, we have $\sigma_h = \rho$. We will show that $\operatorname{Pol}\sigma_m \subseteq \operatorname{Pol}\rho$. Let $f \in \operatorname{Pol}\sigma_m$ be an n-ary operation and let $\boldsymbol{x}_i = (a_{1,i}, \ldots, a_{h,i}) \in \rho, 1 \leq i \leq n$; set $\boldsymbol{x}'_i = (a_{1,i}, \ldots, a_{h,i}, \underbrace{a_{h,i} \ldots, a_{h,i}}_{m-h \text{ times}}), 1 \leq i \leq n$. For $1 \leq i \leq m$, $\boldsymbol{x}'_i \in \sigma_m$, (due to $\sigma_h = \rho$). Hence

$$f(\boldsymbol{x}'_1, \ldots, \boldsymbol{x}'_n) = (f(a_{1,1}, \ldots, a_{1,n}), f(a_{2,1}, \ldots, a_{2,n}), \ldots, f(a_{h,1}, \ldots, a_{h,n})) \in \sigma_m.$$

Therefore $f(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) = (f(a_{1,1}, \ldots, a_{1,n}), \ldots, f(a_{h,1}, \ldots, a_{h,n})) \in \rho$. Thus $f \in \operatorname{Pol}\rho$. Therfore $\operatorname{Pol}\sigma_m \subseteq \operatorname{Pol}\rho$. Since $\sigma_m \in [\{\sigma\}]$, we get $\operatorname{Pol}\sigma \subseteq \operatorname{Pol}\sigma_m$. So $\operatorname{Pol}\sigma \subseteq \operatorname{Pol}\sigma_m \subseteq \operatorname{Pol}\rho$. Assume that $\sigma_m \subsetneq \rho_m$. Let $(c_1, \ldots, c_m) \in \rho_m \setminus \sigma_m$ and $(a_1, \ldots, a_h) \in \rho \setminus \iota_k^h$; we set

$$W = \{(i_1, \ldots, i_h) : 1 \leq i_1 < \ldots < i_h \leq m\}$$

denoted for reason of simple notation by

$$W = \{(i_1^j, \ldots, i_h^j) : 1 \leq j \leq q\}.$$

For all $1 \leq j \leq q$, set $\boldsymbol{y}_j = (x_{j,1}, \ldots, x_{j,m})$ and for all $1 \leq i \leq m$, set $\boldsymbol{x}_i = (x_{1,i}, \ldots, x_{q,i})$ with $x_{j,i} = a_l$ if $i = i_l^j$, for some $1 \leq l \leq h$, and $x_{j,i} = a_1$ otherwise. Then the q-ary operation f defined by $f(\boldsymbol{x}) = c_i$ if $\boldsymbol{x} = \boldsymbol{x}_i$ for some $1 \leq i \leq m$ and $f(\boldsymbol{x}) = c_1$ otherwise, is well defined (because $|\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m\}| = m$), preserves ρ (because $\operatorname{Im}(f) = \{c_1, \ldots, c_m\}$ and $\{c_1, \ldots, c_m\}$ is a ρ-chain) and does not preserve σ_m, because $\{\boldsymbol{y}_j : 1 \leq j \leq q\} \subseteq \sigma_m$ and $f(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_q) = (f(\boldsymbol{x}_1), \ldots, f(\boldsymbol{x}_m)) = (c_1, \ldots, c_m) \notin \sigma_m$. Therefore $\operatorname{Pol}\sigma_m \subsetneq \operatorname{Pol}\rho$. The above unary operation f_0 preserves σ_m and not σ. Thus $\operatorname{Pol}\sigma \subsetneq \operatorname{Pol}\sigma_m \subsetneq \operatorname{Pol}\rho$, contradiction with the fact that $\operatorname{Pol}\sigma$ is maximal below $\operatorname{Pol}\rho$. Thus $\sigma_m = \rho_m$. Let $B = \{a_1, \ldots, a_n\}$ be a maximal ρ-chain, then $(a_1, \ldots, a_n, \underbrace{a_n, \ldots, a_n}_{m-n \text{ times}}) \in \rho_m = \sigma_m$; so there exists $\top_B \in E_k$ such that for all $1 \leq i \leq n, (a_i, \top_B) \in \sigma$.

The proof of (ii) is obained similarly. ■

Lemma 5.37. *If the assumptions of Proposition* 5.19 *are satisfied and* $\sigma_h \neq E_k^h$ *or* $\sigma'_h \neq E_k^h$, *then* σ *is reflexive.*

Proof. Assume that $\sigma_h \neq E_k^h$. Let B be a maximal ρ-chain. From Lemma 5.36, there exists $\top_B \in E_k$ such that $(x, \top_B) \in \sigma$ for every $x \in B$. Let $c \in C_\rho$; for all $0 \leq i \leq t-1$, $(c, \top_{A_i}) \in \sigma$. We have $h \geq 3$ (due to $\sigma_h \neq E_k^h$ and $\sigma_2 = E_k^2$). From Lemma 5.35, $T = \{\top_{A_0}, \top_{A_1}, \ldots, \top_{A_{t-1}}\}$ is a ρ-chain. So there exists a maximal ρ-chain D such that $T \subseteq D$. Without loss of generality we suppose that $D = A_0$. Therefore $(\top_{A_0}, \top_{A_0}) \in \sigma$. Hence $\top_{A_0} \in \eta = \{x \in E_k : (x, x) \in \sigma\}$ and by Lemma 5.21 we get that $\eta = E_k$ and σ is reflexive. The case $\sigma'_h \neq E_k^h$ can be solved using a similar argument as above. ■

We have shown that if $\mathrm{Pol}\,\sigma$ is a meet-irreducible submaximal clone of $\mathrm{Pol}\,\rho$ and $\sigma_h \neq E_k^h$ or $\sigma'_h \neq E_k^h$, then σ is reflexive. From now on we suppose that σ is reflexive. Since σ is not a diagonal relation, we have $\Delta_{E_k} \subsetneq \sigma \subsetneq E_k^2$. We consider again the binary relation $\gamma = \sigma \cap \sigma^{-1}$; thus $\Delta_{E_k} \subseteq \gamma \subseteq \sigma$. We distinguish the following three cases: (i) $\Delta_{E_k} = \gamma$, (ii) $\Delta_{E_k} \subsetneq \gamma \subsetneq \sigma$ and (iii) $\gamma = \sigma$.
First, we study the subcase (ii).

Lemma 5.38. *If the assumptions of Proposition* 5.19 *are satisfied,* $\sigma_h \neq E_k^h$ *or* $\sigma'_h \neq E_k^h$ *and* $\sigma_2 = \sigma'_2 = E_k^2$*, then the subcase* (ii) *is impossible.*

Proof. Suppose that $\sigma_h \neq E_k^h, \sigma_2 = \sigma'_2 = E_k^2$ and $\Delta_{E_k} \subsetneq \gamma \subsetneq \sigma$ hold, then we have $h \geq 3$. From Lemma 5.35, we have $\sigma_h = \rho$. In addition $\mathrm{Pol}\,\sigma \subseteq \mathrm{Pol}\,\gamma$ and $\gamma \subseteq \gamma \circ \gamma \subseteq E_k^2$; this yields the following possible two subcases:

(a) $\gamma = \gamma \circ \gamma$ and (b) $\gamma \subsetneq \gamma \circ \gamma$.

Assume that subcase (a) holds, then γ is a nontrivial equivalence relation and $\mathrm{Pol}(\sigma) \subsetneq \mathrm{Pol}(\gamma)$, contradiction.

Assume that subcase (b) holds. Recall that γ is the symmetric part of σ and $\gamma \circ \gamma$ is symmetric. We claim that $\gamma \circ \gamma \nsubseteq \sigma$. Assume that $\gamma \subsetneq \gamma \circ \gamma \subseteq \sigma$; let $(a, b) \in \gamma \circ \gamma \setminus \gamma$, then by symmetry of $\gamma \circ \gamma$, we get $(a, b), (b, a) \in \sigma$; so $(a, b) \in \sigma \cap \sigma^{-1} = \gamma$, contradiction. It follows that $\gamma \circ \gamma \nsubseteq \sigma$. Let $(a, b) \in (\gamma \circ \gamma) \setminus \sigma, (e, d) \in \sigma \setminus \gamma$; then there exists $u \in E_k$ such that $(a, u), (u, b) \in \gamma$ $(**)$. Let f_2 be the unary operation defined on E_k by $f_2(e) = a, f_2(d) = b$ and $f_2(x) = u$ elsewhere. Since $(e, d) \notin \gamma$, using $(**)$, the reflexivity and the symmetry of γ we obtain that f_2 preserves γ; but $(e, d) \in \sigma$ and $(f_2(e), f_2(d)) = (a, b) \notin \sigma$; so f_2 does not preserve σ. Hence $\mathrm{Pol}(\sigma) \subsetneq \mathrm{Pol}(\gamma)$. Since $\Delta_{E_k} \subsetneq \gamma \subsetneq E_k^2$, there is $(a, b) \in E_k^2 \setminus \gamma$. Let $(e, d) \in \gamma \setminus \Delta_{E_k}$. The function $g : E_k \to E_k$ defined by $g(e) = a$ and $g(x) = b$ otherwise preserves ρ (due to $h \geq 3$ and ρ totally reflexive) and does not preserve γ (due to $(e, d) \in \gamma$ and $(g(e), g(d)) = (a, b) \notin \gamma$). Hence $\mathrm{Pol}\,\rho \nsubseteq \mathrm{Pol}\,\gamma$. Hence $\mathrm{Pol}\,\sigma \subsetneq \mathrm{Pol}\,\gamma \nsubseteq \mathrm{Pol}\,\rho$ contradicting the meet-irreducibility of $\mathrm{Pol}\,\sigma$ below $\mathrm{Pol}\,\rho$. ∎

From Lemma 5.38, σ is symmetric or antisymmetric. The next lemma shows that σ is symmetric.

Lemma 5.39. *If the assumptions of Proposition* 5.19 *are satisfied,* $\sigma_2 = \sigma'_2 = E_k^2$ *and* $\sigma_h \neq E_k^h$ *or* $\sigma'_h \neq E_k^h$*, then* σ *is symmetric.*

Proof. Assume that $\sigma_2 = \sigma'_2 = E_k^2$, $\sigma_h \neq E_k^h$ and σ is not symmetric. As $\sigma_2 = E_k^2$ and $\sigma_h \neq E_k^h$, we have $h \geq 3$. From Lemma 5.38, σ is antisymmetric. From Lemma 5.35, $\sigma_h = \rho$ and σ is reflexive. Furthermore by Lemma 5.36, for any maximal ρ-chain A, there exists $\top_A \in E_k$ such that for all $x \in A$, $(x, \top_A) \in \sigma$. Let $\{A_i : 0 \leq i \leq t - 1\}$ be the set of all maximal ρ-chains on E_k and set $T = \{\top_{A_0}, \top_{A_1}, \ldots, \top_{A_{t-1}}\}$. We recall that $C_\rho \subseteq A_i, 0 \leq i \leq t - 1$. We distinguish the following two subcases: (i) $\sigma'_h = E_k^h$, (ii) $\sigma'_h \neq E_k^h$.

Assume that (i) holds. Then from Lemma 5.30, there exists $\perp \in E_k$ such that for any $x \in E_k$, $(\perp, x) \in \sigma$. Assume that σ is not transitive, then $\sigma \subsetneq tr(\sigma)$. Let $(a,b) \in tr(\sigma) \setminus \sigma$ and $(u,v) \in \sigma \setminus \Delta_{E_k}$ and consider the unary operation l defined on E_k by $l(x) = a$ if $(x,u) \in \sigma$ and $l(x) = b$ otherwise, then l preserves $tr(\sigma)$ and does not preserve σ; thus $\operatorname{Pol}\sigma \subsetneq \operatorname{Pol} tr(\sigma)$. Assume that $tr(\sigma) \neq E_k^2$, let $(a,b) \in E_k^2 \setminus tr(\sigma)$ and $(u,v) \in tr(\sigma)$ with $u \neq v$. The unary operation g defined on E_k by $g(x) = a$ is $x = u$ and $g(x) = b$ elsewhere preserves ρ (due to $h \geq 3$ and ρ is totally reflexive) and does not preserve $tr(\sigma)$ (due to $(u,v) \in tr(\sigma)$ and $(g(u), g(v)) = (a,b) \notin tr(\sigma)$), hence $\operatorname{Pol}\rho \nsubseteq \operatorname{Pol} tr(\sigma)$, contradicting the meet-irreducibility of $\operatorname{Pol}\sigma$ below $\operatorname{Pol}\rho$. Let $b \in E_k$ such that $\perp \neq b$; then by the antisymmetry of σ we get $(b, \perp) \notin \sigma$. Since $(b, \perp) \in E_k^2 = tr(\sigma)$, there exist $u_1, \ldots, u_n \in E_k$ such that $(b, u_n), (u_n, u_{n-1}), \ldots, (u_2, u_1), (u_1, \perp) \in \sigma$. It follows that $(u_1, \perp), (\perp, u_1) \in \sigma$. The antisymmetry of σ yields $\perp = u_1$. By induction we obtain $u_n = \perp$. Thus $(b, \perp) \in \sigma$, contradiction.

Assume that $\sigma = tr(\sigma)$, then σ is transitive and σ is a partial order. Let B be a maximal ρ-chain, then there exists $\top_B \in B$ such that for any $x \in B$, $(x, \top_B) \in \sigma$. Let $a \in E_k \setminus B$, then $\{\top_B, a\}$ is a ρ-chain (because $h \geq 3$ and ρ is totally reflexive), therefore there exists $u \in E_k$ such that $(\top_B, u), (a, u) \in \sigma$. Since σ is transitive, then for any $x \in B, (x,u) \in \sigma$; so $B \cup \{u\}$ is a ρ-chain (due to $\sigma_h = \rho$). Hence $u \in B$ and $(u, \top_B), (\top_B, u) \in \sigma$. Thus $\top_B = u$ and $(a, \top_B) \in \sigma$. As the choice of a was arbitrary we deduce that $\top_B$ is the greatest element of σ; this observation together with $(\perp, x) \in \sigma$ for any $x \in E_k$, yield that σ is a bounded partial order, contradiction. Thus the subcase $\sigma'_h = E_k^h$ is impossible.

Assume (ii): $\sigma'_h \neq E_k^h$ holds, then we get by Lemma 5.35 that $\sigma'_h = \rho$. We distinguish three subcases. (a) $\sigma \subsetneq tr(\sigma) \subsetneq E_k^2$, (b) $\sigma = tr(\sigma)$, (c) $tr(\sigma) = E_k^2$.

Since $tr(\sigma) \in [\{\sigma\}]$, we have $\operatorname{Pol}\sigma \subseteq \operatorname{Pol} tr(\sigma)$. If (a) holds, then using the above unary operation l, we obtain $\operatorname{Pol}\sigma \subsetneq \operatorname{Pol} tr(\sigma)$. Hence $\operatorname{Pol}\sigma \subsetneq \operatorname{Pol} tr(\sigma) \neq \operatorname{Pol}\rho$ (due to Lemma 5.18, $\Delta_{E_k} \neq tr(\sigma) \neq E_k^2$ and ρ is not transitive), contradicting the meet-irreducibility of $\operatorname{Pol}\sigma$ below $\operatorname{Pol}\rho$.

Assume that (b) holds(i.e σ is transitive). Since $\rho = \sigma_h = \sigma'_h$ and σ reflexive, from Lemma 5.36, for any maximal ρ-chain B, there exist $\top_B, \perp_B \in B$ such that for any $x \in B, (x, \top_B), (\perp_B, x) \in \sigma$. Let B be a maximal ρ-chain, and $\top_B$ and $\perp_B$ as above. A similar argument use in the subcase (i) with σ transitive shows that $\top_B$ is the greatest element of σ and dually $\perp_B$ is the least element of σ; therefore σ is a bounded partial order, contradiction.

Assume that (c) holds. Let $\{A_i : 0 \leq i \leq t-1\}$ be the set of maximal ρ-chains $(t \geq 2)$. Since $\sigma_h \neq E_k^h$ and $\sigma'_h \neq E_k^h$, applying Lemma 5.35 we obtain $\sigma_h = \sigma'_h = \rho$, from Lemma 5.36, for any maximal ρ-chain B, there exist $\top_B, \perp_B \in B$ such that for any $x \in B$, $(x, \top_B), (\perp_B, x) \in \sigma$. In addition, from (1) of Proposition 3.4, $C_\rho \subseteq A_i, 0 \leq i \leq t-1$. Therefore $\{\top_{A_i} : 0 \leq i \leq t-1\}$ is a ρ-chain. So there exists a maximal ρ-chain D such that $\{\top_{A_i} : 0 \leq i \leq t-1\} \subseteq D$. As D

is a maximal ρ-chain, $\perp_D, \top_D \in D$ and $(\top_D, \perp_D) \in E_k^2 = tr(\sigma)$. So there exist $u_1, \ldots, u_n \in E_k$ such that $(\top_D, u_1), (u_1, u_2), \ldots, (u_{n-1}, u_n), (u_n, \perp_D) \in \sigma$. Since $\{u_1, \top_D\}$ is also a ρ-chain (due to $h \geq 3$), there exists a maximal ρ-chain B such that $\{u_1, \top_D\} \subseteq B$ (due to every ρ-chain is contained in a maximal ρ-chain). We have $\top_B \in D$ (due to $\top_B \in \{\top_{A_i}, 0 \leq i \leq t-1\} \subseteq D$ and D is a maximal ρ-chain), so $(\top_B, \top_D) \in \sigma$; in addition $(\top_D, \top_B) \in \sigma$ (due to $\top_D \in B$); therefore $(\top_B, \top_D), (\top_D, \top_B) \in \sigma$ and $\top_B = \top_D$ (due to σ is anti-symmetric). We have also $u_1 \in B$, so $(u_1, \top_B) = (u_1, \top_D) \in \sigma$, therefore $u_1 = \top_D = \top_B$ (due to $(\top_D, u_1) \in \sigma$ and σ anti-symmetric). By induction we obtain $u_i = \top_D, 1 \leq i \leq n$. Hence $\top_D = \perp_D$ and $\top_{A_0} = \ldots = \top_{A_{t-1}}$; so E_k is a ρ-chain (due to $\sigma_h = \rho$ and $(x, \top_D) \in \sigma$ for all $x \in E_k$); contradiction. Therefore (c) is impossible.

Hence the case σ antisymmetric is impossible. Thus σ is symmetric. ■

We have shown that σ is reflexive and symmetric. Recall that ρ has t maximal ρ-chains $A_0, A_1, \ldots, A_{t-1}$. Let $i \in \{0, 1, \ldots, t-1\}$ and $m = |A_i|$. Set

$$\gamma = \big\{(a_1, \ldots, a_k) \in E_k^k : \exists\, u \in E_k, (a_1, u), \ldots, (a_m, u) \in \sigma \text{ and } \{a_1, \ldots, a_k\}^{h-1} \times \{u\} \subseteq \rho\big\}$$

and $\beta = \{(a_1, \ldots, a_k) \in E_k^k : \{a_1, \ldots, a_m\}^h \subseteq \rho\}$.

Lemma 5.40. *If the assumptions of Proposition* 5.19 *are satisfied,* $\sigma_2 = \sigma_2' = E_k^2$ *and* $(\sigma_h \neq E_k^h$ *or* $\sigma_h' \neq E_k^h)$*, then* $\beta = \gamma$.

Proof. Assume that $\gamma \neq \beta$ and $\sigma_h \neq E_k^h$. From Lemma 5.35, we get $\sigma_h = \rho$; using Lemma 5.39, we get that σ is symmetric. Therefore $\sigma_h = \sigma_h' = \rho$. Hence $\gamma \subsetneq \beta$. Let $(v_1, \ldots, v_k) \in \beta \setminus \gamma$. It is easy to check that $\operatorname{Pol} \sigma \subseteq \operatorname{Pol} \gamma$ (using $\sigma_h = \rho$). Now we show that $\operatorname{Pol} \gamma \subseteq \operatorname{Pol} \rho$. Let $f \in \operatorname{Pol} \gamma$ be an n-ary operation. Let $\boldsymbol{a}_i = (a_{1,i}, \ldots, a_{h,i}) \in \rho, 1 \leq i \leq n$, set

$$\boldsymbol{a}_i' = (a_{1,i}, \ldots, a_{h,i}, \underbrace{a_{h,i}, \ldots, a_{h,i}}_{k-h \text{ times}}), 1 \leq i \leq n.$$

Using $\sigma_h = \rho$ one can check that $\boldsymbol{a}_i' \in \gamma$, $1 \leq i \leq n$ $(*_6)$. Using $\sigma_h = \rho$ and $(*_6)$ one can check that $(f(a_{1,1}, \ldots, a_{1,n}), \ldots, f(a_{h,1}, \ldots, a_{h,n})) \in \rho$; therfore $\operatorname{Pol} \gamma \subseteq \operatorname{Pol} \rho$. Hence $\operatorname{Pol} \sigma \subseteq \operatorname{Pol} \gamma \subseteq \operatorname{Pol} \rho$. We show that these inclusions are proper. Let $(a, b) \in E_k^2 \setminus \sigma$ and $(u, v) \in \sigma \setminus \Delta_{E_k}$. The unary operation g defined on E_k by $g(x) = a$ if $x = u$ and $g(x) = b$ otherwise preserve γ due to $\rho = \sigma_h, E_k^2 = \sigma_2, 3 \leq h \leq m$, σ reflexive and symmetric, $\operatorname{Im} g = \{a, b\}$ and ρ totally reflexive; but does not preserve σ due to $(u, v) \in \sigma$ and $(g(u), g(v)) = (a, b) \notin \sigma$. Therfore $\operatorname{Pol} \sigma \subsetneq \operatorname{Pol} \gamma$. To finish we show that $\operatorname{Pol} \gamma \subsetneq \operatorname{Pol} \rho$. From Lemma 5.36, for all $0 \leq i \leq t-1$, there exists $u_{A_i} \in E_k$ such that $(x, u_{A_i}) \in \sigma$ for every $x \in A_i$. Since $C_\rho \subseteq A_i, 0 \leq i \leq t-1$, $\{u_{A_0}, \ldots, u_{A_{t-1}}\}$ is contained in a maximal

ρ-chain D. We suppose that $D = A_0$. Therefore $(u_{A_0}, u_{A_1}), \ldots, (u_{A_0}, u_{A_{t-1}}) \in \sigma$. Let $a_1, \ldots, a_{h-1} \in E_k$; $\{a_1, \ldots, a_{h-1}\}$ is contained in a maximal ρ-chain A_i for some $0 \leq i \leq t-1$. Hence $(a_1, \ldots, a_{h-1}, u_{A_0}) \in \sigma_h = \rho$. Therefore $u_{A_0} \in C_\rho$. We choose $a_1, a_2, \ldots, a_{h-1} \in A_0$ and $a_h \notin A_0$ such that $(a_1, \ldots, a_h) \notin \rho$ (due to $A_0 \neq E_k$). Set

$$W = \{(i_1, \ldots, i_h) \in \{1, \ldots, k\}^h : 1 \leq i_1 < \ldots < i_h \leq k\},$$

denoted simply by $W = \{(i_1^j, \ldots, i_h^j) : 1 \leq j \leq q\}$ where $q = |W|$. For all $1 \leq j \leq q$, set $y_j = (x_{1,j}, x_{2,j}, \ldots, x_{k,j}) \in E_k^k$ with

$$x_{l,j} = \begin{cases} a_n & \text{if } l = i_n^j \text{ and } (n \neq h \text{ or } l > m) \text{ for some } 1 \leq n \leq h, \\ u_{A_0} & \text{otherwise;} \end{cases}$$

for $1 \leq l \leq q$. Furthermore, for $1 \leq i \leq k$ we set $\boldsymbol{x}_i = (x_{i,1}, x_{i,2}, \ldots, x_{i,q})$. From construction of $\boldsymbol{x}_i$, for all $1 \leq i_1 < i_2 < \ldots < i_h \leq k$, $(\boldsymbol{x}_{i_1}, \boldsymbol{x}_{i_2}, \ldots, \boldsymbol{x}_{i_h}) \in \rho$ if and only if $i_h \leq m$. We define the q-ary operation f on E_k by $f(\boldsymbol{x}) = v_i$ if $\boldsymbol{x} = \boldsymbol{x}_i$ for some $1 \leq i \leq k$ and $f(\boldsymbol{x}) = u_{A_0}$ otherwise. We have $\{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_q\} \subseteq \gamma$ (due to $u_{A_0} \in C_\rho$ and $(a_1, u_{A_0}), \ldots, (a_{h-1}, u_{A_0}) \in \sigma$) and $f(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_q) = (f(\boldsymbol{x}_1), \ldots, f(\boldsymbol{x}_k)) = (v_1, \ldots, v_k) \notin \gamma$. It is easy to check that $f \in \operatorname{Pol}\rho$. Thus $\operatorname{Pol}\sigma \subsetneq \operatorname{Pol}\gamma \subsetneq \operatorname{Pol}\rho$, contradicting the maximality of $\operatorname{Pol}\sigma$ in $\operatorname{Pol}\rho$. Thus $\gamma = \beta$. ∎

Lemma 5.41. *If the assumptions of Proposition 5.19 are satified, $\sigma_2 = \sigma_2' = E_k^2$ and ($\sigma_h \neq E_k^h$ or $\sigma_h' \neq E_k^h$), then σ is of type VIII.*

Proof. We have shown above that σ is reflexive and symmetric. Furthermore $\sigma \circ \sigma = E_k^2$. Let $i \in \{0, 1, \ldots, t-1\}$ and $m = |A_i|$. From Lemma 5.40, $\gamma = \beta$. We suppose that $A_i = \{a_1, \ldots, a_m\}$. Let $a_{m+1}, \ldots, a_k \in E_k$ such that $E_k = \{a_1, \ldots, a_k\}$. We have $(a_1, a_2, \ldots, a_k) \in \beta = \gamma$; therefore there exists $u_{A_i} \in E_k$ such that for all $1 \leq j \leq m$ $(a_j, u_{A_i}) \in \sigma$ and for all $1 \leq i_1 < i_2 < \ldots < i_{h-1} \leq k, (a_{i_1}, a_{i_2}, \ldots, a_{i_{h-1}}, u_{A_i}) \in \rho$. Hence $u_{A_i} \in C_\rho$ and σ fulfills condition VIII of Theorem 3.2. ∎

Now we are ready to prove Proposition 5.19.

Proof. (Proof of Proposition 5.19) Combining Lemmas 5.20–5.41, we obtain the result. ∎

Proof. (Proof of Theorem 3.2) Combining Propositions 4.1, 4.3, 4.4, 4.8, Corollary 4.5, Propositions 5.1, 5.9, 5.16, 5.19 and Corollary 5.17 we have the result. ∎

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References

[1] K.A. Baker and A.F. Pixley, *Polynomial interpolation and the Chinese remainder theorem for algebraic systems*, Math. Z. **143** (1975) 165–174.

[2] L. Diekouam, Y.L.J. Tenkeu, E.R.A. Temgoua and M. Tonga, *Meet-reducible submaximal clones determined by nontrivial equivalence relations*, J. Algebra, Number Theory; Adv. Appl. **17** (2017) 29–93.
http://dx.doi.org/jantaa_7100121803

[3] D. Lau, *Submaximale Klassen von P_3*, J. Inf. Process. Cybern. EIK **18** (4–5) (1982) 227–243.

[4] D. Lau, Function Algebras on finite sets. A basic course on many-valued logic and clone theory (Springer-Verlag, New York, Heidelberg, Berlin, 2006).

[5] E.L. Post, *Introduction to a general theory of elementary propositions*, Amer. J. Math. **43** (1921) 163–185.

[6] E.L. Post, The Two-Valued Iterative Systems of Mathematical Logic, Ann. Math. Studies **5** (Princeton University Press, 1941).

[7] I.G. Rosenberg, *La structure des fonctions de plusieurs variables sur un ensemble fini*, C.R. Acad. Sci. Paris, Ser. A–B **260** (1965) 3817–3819.

[8] I.G. Rosenberg and Á. Szendrei, *Submaximal clones with a prime order automorphism*, Acta Sci. Math. **49** (1985) 29–48.

[9] R. Pöschel and L.A. Kalužnin, Funktionen-Und Relationenalgebren (VEB Deutscher Verlag der Wissenschaften, Berlin, 1979).

[10] E.R.A. Temgoua and I.G. Rosenberg, *Binary central relations and submaximal clones determined by nontrivial equivalence relations*, Algebra Univ. **67** (2012) 299–311.
https://doi.org/10.1007/s00012-012-0183-2

[11] E.R.A. Temgoua, *Meet-irreducible submaximal clones determined by nontrivial equivalence relations*, Algebra Univ. **70** (2013) 175–196.
https://doi.org/10.1007/s00012-013-0244-1

[12] Y.L.J. Jeufack, L. Diekouam and E.R.A. Temgoua, *Meet-reducible submaximal clones determined by two central relations*, Asian Eur. J. Math. **12** (2018) 26 pages.
https://doi.org/10.1142/1793557119500645

[13] Á. Szendrei, Clones in Universal Algebra. Séminaire de Mathématiques Supérieures **99** (Les presses de l'université de Montréal, 1986).

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