Discussiones Mathematicae
General Algebra and Applications 43 (2023) 327–337
https://doi.org/10.7151/dmgaa.1422

ADDITIVE MAPPINGS SATISFYING ALGEBRAIC IDENTITIES IN SEMIPRIME RINGS

Abu Zaid Ansari

Department of Mathematics
Faculty of Science, Islamic University of Madinah, K.S.A

e-mail: ansari.abuzaid@gmail.com
ansari.abuzaid@iu.edu.sa

Abstract

Let R be a k-torsion free semiprime and $F, d : R \to R$ be two additive mappings which satisfy the algebraic identity $F(x^{2n}) = F(x^n)\alpha(x^n) + \beta(x^n)d(x^n)$ for all $x \in R$, where α and β are automorphisms on R. Then F is a generalized (α, β)-derivation with associated (α, β)-derivation d on R, where $k \in \{2, n, 2n-1\}$. On the other hand, it is proved that f is a generalized Jordan left (α, β)-derivation associated with Jordan left (α, β)-derivation δ on R if they satisfy the algebraic identity $f(x^{2n}) = \alpha(x^n)f(x^n) + \beta(x^n)\delta(x^n)$ for all $x \in R$ together with some restrictions on R.

Keywords: semiprime rings, generalized (α, β)-derivation, generalized left (α, β)-derivation and additive mappings.

2020 Mathematics Subject Classification: 16N60, 16B99, 16W25.

1. Introduction

Throughout the present paper R will denote an associative ring with identity. $Z(R)$ denotes the center of R, $Q_l(R_C)$ is left Martindale ring of quotients and $\mathcal{C}$ its extended centroid. A ring R is termed as n-torsion free if $nx = 0$, implies $x = 0$, $\forall\, x \in R$, where $n > 1$ is an integer. The commutator $xy - yx$ will be represented as usual by $[x, y]$. Recall that a ring R is known as prime if $aRb = \{0\}$ implies either $a = 0$ or $b = 0$, and is called semiprime if $aRa = \{0\}$ implies $a = 0$. An additive mapping d from R to itself is known as a derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$ and is said to be a Jordan derivation in case $d(x^2) = d(x)x + xd(x)$ is fulfilled for all $x \in R$. Clearly, every derivation is a Jordan derivation, but

[1]Author is thankful to the referee for his/her valuable suggestions.

the converse is not always true. Herstein's classical conclusion [8, Theorem 3.3] argues that a Jordan derivation on a prime ring with a characteristic other than two is a derivation. This conclusion has been generalized by Cusack [6] to the 2-torsion free semiprime ring. An additive mapping F from R to itself is known as a generalized derivation if there exists a derivation d from R to itself such that $F(xy) = F(x)y + xd(y)$ for each pair $x, y \in R$. An additive mapping $F : R \to R$ is known as a generalized Jordan derivation if there exists a Jordan derivation $d : R \to R$ such that $F(x^2) = F(x)x + xd(x)$ for all $x \in R$. It is easy to verify that every generalized derivation is generalized Jordan derivation but the converse is not true in general. Suppose that α and β are two endomorphisms defined on R. An additive mapping d from R to itself is termed as an (α, β)-derivation (respectively Jordan (α, β)-derivation) if $d(xy) = d(x)\alpha(y) + \beta(x)d(y)$ (respectively $d(x^2) = d(x)\alpha(x) + \beta(x)d(x)$) fulfills for all $x, y \in R$. Every (α, β)-derivation is a Jordan (α, β)-derivation, although the converse is not always true. On 2-torsion free semiprime ring, both are same (for details see [9]). An additive mapping F from R to itself is known as a generalized (α, β)-derivation (respectively generalized Jordan (α, β)-derivation) if there exists an (α, β)-deviation (respectively Jordan (α, β)-deviation) d from R to itself such that $F(xy) = F(x)\alpha(y) + \beta(x)d(y)$ (respectively $F(x^2) = F(x)\alpha(x) + \beta(x)d(x)$) for all $x, y \in R$. Every generalized (α, β)-derivation is generalized Jordan (α, β)-derivation but in general, the converse does not hold. If R is 2-torsion free semiprime ring it is valid (see [2]). Now, if F is a generalized (α, β)-derivation (respectively generalized Jordan (α, β)-derivation) associated with an (α, β)-derivation (respectively Jordan (α, β)-derivation) d on R, then the identity $F(x^{2n}) = F(x^n)\alpha(x^n) + \beta(x^n)d(x^n)$ holds for all $x \in R$ but what about the converse? In this article, we have studied the converse of the this statement. Specifically, we planned under what condition on R, F is a generalized (α, β)-derivation associated with an (α, β)-derivation d if it satisfies the algebraic identity $F(x^{2n}) = F(x^n)\alpha(x^n) + \beta(x^n)d(x^n)$ for all $x \in R$. In the present paper, author uses the tools of Vander Monde determinant to relax the torsion condition on R in respect of the complimentary work carried by the authors in [4]. The suitable arguments and substantial modification are made to establish the proof of Theorem 2.1.

Next, an additive mapping $\delta : R \to R$ is said to be a left derivation (respectively Jordan left derivation) if $\delta(xy) = x\delta(y) + y\delta(x)$ (respectively $\delta(x^2) = 2x\delta(x)$) holds for all $x, y \in R$. An additive mapping $\delta : R \to R$ is said to be a right derivation (respectively Jordan right derivation) if $\delta(xy) = \delta(x)y + \delta(y)x$ (respectively $\delta(x^2) = 2\delta(x)x$) holds for all $x, y \in R$. If δ is both left as well as right derivation, then it is a derivation. Clearly, every left (respectively right) derivation on a ring R is a Jordan left (respectively Jordan right) derivation but the converse need not be true in general (Details are present in [12]). Following [5], an additive mapping f from R to itself is known as a generalized left derivation (respectively generalized Jordan left derivation) if there exists a Jordan left deviation δ from R to itself such that $f(xy) = xf(y) + y\delta(x)$ (respectively $f(x^2) = xf(x) + x\delta(x)$) for all $x, y \in R$. Following Zalar [13], an additive mapping $T : R \to R$ is termed as a left (respectively right) centralizer of R if $T(xy) = T(x)y$ (respectively $T(xy) = xT(y)$) for all $x, y \in R$. Particularly, T will be a Jordan left (respectively Jordan right) centralizer of R if $x = y$. It is obvious that f is a generalized left derivation if and only if f has the form $f = \delta + T$, where δ is a left derivation and T is the right centralizer of R. Generalized left derivations encompass the notion of left derivation. A generalized left derivation with $\delta = 0$ incorporates the idea of right centralizer. The sum of two generalized left derivations will also be a generalized left derivation. If δ is any left derivation of R, then for any fixed element $a \in R$, every mapping of the form $f(x) = xa + \delta(x)$ will be a generalized left derivation. An additive mapping $\delta : R \to R$ is called a left (α, β)-derivation (respectively Jordan left (α, β)-derivation) if $\delta(xy) = \alpha(x)\delta(y) + \beta(y)\delta(x)$ (respectively $\delta(x^2) = \alpha(x)\delta(x) + \beta(x)\delta(x)$) for all $x, y \in R$. An additive mapping $f : R \to R$ is termed as a generalized left (α, β)-derivation (respectively generalized Jordan left (α, β)-derivation) if there is a Jordan left (α, β)-deviation $\delta : R \to R$ such that $f(xy) = \alpha(x)f(y) + \beta(y)\delta(x)$ (respectively $f(x^2) = \alpha(x)f(x) + \beta(x)\delta(x)$) for all $x, y \in R$. An additive mapping $T : R \to R$ is known as a left (respectively right) α-centralizer of R if $T(xy) = T(x)\alpha(y)$ (respectively $T(xy) = \alpha(x)T(y)$) for all $x, y \in R$. An additive mapping $T : R \to R$ is called Jordan left (respectively Jordan right) α-centralizer of R if $T(x^2) = T(x)\alpha(x)$ (respectively $T(x^2) = \alpha(x)T(x)$) for all $x \in R$. Obviously, every left α-centralizer is a Jordan left α-centralizer on R. Same conclusion holds for right α-centralizer on R. It is obvious that f is a generalized left (α, β)-derivation if and only if $f = \delta + T$, where δ and T are a left (α, β)-derivation and a right α-centralizer of R, respectively. Generalized left (α, β)-derivations encompass the idea of left (α, β)-derivation. A generalized left (α, β)-derivation with $\delta = 0$ incorporates the concept of right α-centralizer. Furthermore, the sum of two generalized left (α, β)-derivations will be a generalized left (α, β)-derivation. If δ is any left (α, β)-derivation of R and a is any fixed element in R, then every mapping of the form $f(x) = \alpha(x)a + \delta(x)$ will be a generalized left (α, β)-derivation on R. Suppose that $a \in R$ is a fixed element, then for any generalized left (α, β)-derivation f, the mapping g from R to itself such that $g(x) = f(x) + \alpha(x)a$ or $g(x) = f(x) - \alpha(x)a$ will also be a generalized left (α, β)-derivation on R. If f is a generalized Jordan left (α, β)-derivation with associated Jordan left (α, β)-derivation δ on R, then $f(x^{2n}) = \alpha(x^n)f(x^n) + \beta(x^n)\delta(x^n)$ holds for all $x \in R$ but the converse is not true in general. In this paper, we study the converse of this statement. More precisely, f is a generalized Jordan left (α, β)-derivation associated with Jordan left (α, β)-derivation δ on R if $f(x^{2n}) = \alpha(x^n)f(x^n) + \beta(x^n)\delta(x^n)$ holds for all

$x \in R$ with some restrictions on R.

Let us start with the following theorem.

2. Main theorems

Theorem 2.1. *Let $n \geq 1$ be any fixed integer and R be a k-torsion free semiprime ring. Suppose that $F, d : R \to R$ are two additive mappings which satisfy the algebraic identity $F(x^{2n}) = F(x^n)\alpha(x^n) + \beta(x^n)d(x^n)$ for all $x \in R$, where α and β are automorphisms on R. Then F is a generalized (α, β)-derivation with associated (α, β)-derivation d on R, where $k \in \{2, n, 2n-1\}$.*

Proof. We have given that

$$F(x^{2n}) = F(x^n)\alpha(x^n) + \beta(x^n)d(x^n) \text{ for all } x \in R. \tag{2.1}$$

If we replace x by $x + py$ in the above equation, then we find

$$\begin{aligned}
&F\Big(x^{2n} + \tbinom{2n}{1}x^{2n-1}py + \tbinom{2n}{2}x^{2n-2}p^2y^2 + \cdots + p^{2n}y^{2n}\Big) \\
&= F\Big(x^n + \tbinom{n}{1}x^{n-1}py + \tbinom{n}{2}x^{n-2}p^2y^2 + \cdots + p^ny^n\Big)\Big(\alpha(x^n) + \tbinom{n}{1}\alpha(x^{n-1}py) \\
&+ \tbinom{n}{2}\alpha(x^{n-2}p^2y^2) + \cdots + p^n\alpha(y^n)\Big) + \Big(\beta(x^n) + \tbinom{n}{1}\beta(x^{n-1}py) + \tbinom{n}{2}\beta(x^{n-2}p^2y^2) \\
&+ \cdots + p^n\beta(y^n)\Big)d\Big(x^n + \tbinom{n}{1}x^{n-1}py + \tbinom{n}{2}x^{n-2}p^2y^2 + \cdots + p^ny^n\Big)
\end{aligned}$$

for all $x, y \in R$, i.e.,

$$\begin{aligned}
&p\Big[\tbinom{2n}{1}F(x^{2n-1}y) - \tbinom{n}{1}F(x^n)\alpha(x^{n-1}y) - \tbinom{n}{1}F(x^{n-1}y)\alpha(x^n) - \tbinom{n}{1}\beta(x^n)d(x^{n-1}y) \\
&- \tbinom{n}{1}\beta(x^{n-1}y)d(x^n)\Big] + p^2\Big[\tbinom{2n}{2}F(x^{2n-2}y^2) - \tbinom{n}{2}F(x^n)\alpha(x^{n-2}y^2) \\
&- \tbinom{n}{1}\tbinom{n}{1}F(x^{n-1}y)\alpha(x^{n-1}y) - \tbinom{n}{2}F(x^{n-2})\alpha(y^2x^n) - \tbinom{n}{2}d(x^{n-2}y^2) \\
&- \tbinom{n}{1}\tbinom{n}{1}\beta(x^{n-1}y)d(x^{n-1}y) - \tbinom{n}{2}\beta(x^{n-2}y^2)d(x^n)\Big] + \cdots \\
&+ p^{2n}\Big[F(x^{2n}) - F(x^n)\alpha(x^n) - \beta(x^n)d(x^n)\Big] = 0 \text{ for all } x, y \in R.
\end{aligned}$$

Rewrite the above expression by using (2.1) as

$$pf_1(x, y) + p^2f_2(x, y) + \cdots + p^{2n-1}f_{2n-1}(x, y) = 0,$$

where $f_i(x, y)$ stand for the coefficients of p^i's for all $i = 1, 2, \ldots, 2n-1$. If we replace p by $1, 2, \ldots, 2n-1$, then we find a system of $2n-1$ homogeneous equations. It gives us a Vander Monde matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2^2 & \cdots & 2^{2n-1} \\ \cdots & & & \\ \cdots & & & \\ 2n-1 & (2n-1)^2 & \cdots & (2n-1)^{2n-1} \end{bmatrix}.$$

Which yields that $f_i(x,y)=0$ for all $x,y\in R$ and for all $i=1,2,\ldots,2n-1$. In particular, we have

$$\begin{aligned} f_1(x,y) \;=\; & \tbinom{2n}{1}F(x^{2n-1}y)-\tbinom{n}{1}F(x^n)\alpha(x^{n-1}y)-\tbinom{n}{1}F(x^{n-1}y)\alpha(x^n) \\ - \; & \tbinom{n}{1}\beta(x^n)d(x^{n-1}y)-\tbinom{n}{1}\beta(x^{n-1}y)d(x^n)=0 \text{ for all } x,y\in R. \end{aligned}$$

Let us put $x=e$ and making use of $d(e)=0$ and $\alpha(e)=\beta(e)=e$ to appear $2nF(y)=nF(e)\alpha(y)+nF(y)+nd(y)$. Since R is n-torsion free, we have

$$F(y)=F(e)\alpha(y)+d(y) \text{ for all } y\in R. \tag{2.2}$$

Next observe that

$$\begin{aligned} f_2(x,y) \;=\; & \tbinom{2n}{2}F(x^{2n-2}y^2)-\tbinom{n}{2}F(x^n)\alpha(x^{n-2}y^2)-\tbinom{n}{1}\tbinom{n}{1}F(x^{n-1}y)\alpha(x^{n-1}y) \\ - \; & \tbinom{n}{2}F(x^{n-2})\alpha(y^2x^n)-\tbinom{n}{2}d(x^{n-2}y^2)-\tbinom{n}{1}\tbinom{n}{1}\beta(x^{n-1}y)d(x^{n-1}y) \\ - \; & \tbinom{n}{2}\beta(x^{n-2}y^2)d(x^n)=0 \text{ for all } x,y\in R. \end{aligned}$$

Rewrite the above expression by substituting e for x to obtain

$$\begin{aligned} \tbinom{2n}{2}F(y^2) \;=\; & \tbinom{n}{2}F(e)\alpha(y^2)+\tbinom{n}{1}\tbinom{n}{1}F(y)\alpha(y)+\tbinom{n}{2}F(y^2) \\ + \; & \tbinom{n}{2}d(y^2)+\tbinom{n}{1}\tbinom{n}{1}\beta(y)d(y) \text{ for all } y\in R. \end{aligned}$$

This implies that

$$\begin{aligned} \tfrac{2n(2n-1)}{2}F(y^2) \;=\; & \tfrac{n(n-1)}{2}F(e)\alpha(y^2)+n^2F(y)\alpha(y)+\tfrac{n(n-1)}{2}F(y^2) \\ + \; & \tfrac{n(n-1)}{2}d(y^2)+n^2\beta(y)d(y). \end{aligned}$$

Since R is n-torsion free, we get

$$\begin{aligned} 2(2n-1)F(y^2) \;=\; & (n-1)F(e)\alpha(y^2)+2nF(y)\alpha(y)+n(n-1)F(y^2) \\ + \; & (n-1)d(y^2)+2n\beta(y)d(y). \end{aligned}$$

A simple manipulation give us

$$(3n-1)F(y^2)=(n-1)F(e)\alpha(y^2)+2nF(y)\alpha(y)+(n-1)d(y^2)+2n\beta(y)d(y).$$

An application of (2.2) yields that

$$\begin{aligned}(3n-1)\Big[F(e)\alpha(y^2)+d(y^2)\Big] &= (n-1)F(e)\alpha(y^2)+2n\Big[F(e)\alpha(y)+d(y)\Big]\alpha(y)\\ &+\ (n-1)d(y^2)+2n\beta(y)d(y).\end{aligned}$$

On simplifying above expression, we obtain

$$2nd(y^2)=2nd(y)\alpha(y)+2n\beta(y)d(y) \text{ for all } y\in R.$$

$2n$-torsion freeness of R allow us to write last expression as $d(y^2)=d(y)\alpha(y)+\beta(y)d(y)$. That is d is a Jordan (α,β)-derivation. Since R is a 2-torsion free semiprime ring, then use [9] to get that d is an (α,β)-derivation on R. Consider (2.2) once again, so that

$$\begin{aligned}F(y^2) &= F(e)\alpha(y^2)+d(y^2)\\ &= \big[F(e)\alpha(y)+d(y)\big]\alpha(y)+\beta(y)d(y)\\ &= F(y)\alpha(y)+\beta(y)d(y).\end{aligned}$$

Hence F is generalized Jordan (α,β)-derivation on R associated with the (α,β)-derivation d. Using main theorem from [2], we get the required result. ■

There are immediate consequences of the above theorem.

Corollary 2.1. *Let $n\geq 1$ be any fixed integer and R be any k-torsion free semiprime ring. If $F:R\to R$ is an additive mapping satisfying $F(x^{2n})=F(x^n)\alpha(x^n)$ for all $x\in R$, where α is an automorphisms on R. Then, F is an α-centralizer on R, where $k\in\{2,n,2n-1\}$.*

Proof. Taking $d=0$ in Theorem 2.1, we get the required result.

Corollary 2.2. *Let $n\geq 1$ be any fixed integer and R be a k-torsion free semiprime ring. Suppose that $d:R\to R$ is an additive mapping which satisfies the algebraic identity $d(x^{2n})=d(x^n)\alpha(x^n)+\beta(x^n)d(x^n)$ for all $x\in R$, where α and β are automorphisms on R. Then d is an (α,β)-derivation on R, where $k\in\{2,n,2n-1\}$.*

Proof. Considering d as F and using same steps as we did in Theorem 2.1, we get the required result.

Corollary 2.3. *Let $n\geq 1$ be any fixed integer and R be any k-torsion free semiprime ring. If $F:R\to R$ is an additive mapping satisfying $F(x^{2n})=F(x^n)x^n$ for all $x\in R$. Then, F is a centralizer on R, where $k\in\{2,n,2n-1\}$.*

Proof. Taking $\alpha=I_{identity}$ in Corollary 2.1, we get the required result.

Corollary 2.4. *Let $n \geq 1$ be any fixed integer and R be a k-torsion free semiprime ring. Suppose that $d : R \to R$ is an additive mapping which satisfies the identity $d(x^{2n}) = d(x^n)x^n + x^n d(x^n)$ for all $x \in R$. Then d is a derivation on R, where $k \in \{2, n, 2n-1\}$.*

Proof. Considering $\alpha = \beta = I_{identity}$ in Corollary 2.2, we get the required result.

Come to the next main theorem of the paper.

Theorem 2.2. *Let $n \geq 1$ be any fixed integer and R be k-torsion free ring. If $f, \delta : R \longrightarrow R$ are two additive mappings which satisfy the algebraic identity $f(x^{2n}) = \alpha(x^n)f(x^n) + \beta(x^n)\delta(x^n)$ for all $x \in R$, where α and β are automorphisms on R, then f is generalized Jordan left (α, β)-derivation associated with Jordan left (α, β)-derivation δ on R, where $k \in \{2, n, 2n-1\}$.*

Proof. We have given that

$$f(x^{2n}) = \alpha(x^n)f(x^n) + \beta(x^n)\delta(x^n) \text{ for all } x \in R. \tag{2.3}$$

Replacing x by $x + qy$, we get

$$\begin{aligned}
&q\Big[\tbinom{2n}{1}F(x^{2n-1}y) - \tbinom{n}{1}\alpha(x^{n-1}y)F(x^n) - \tbinom{n}{1}\alpha(x^n)F(x^{n-1}y) - \tbinom{n}{1}\beta(x^n)\delta(x^{n-1}y)\\
&- \tbinom{n}{1}\beta(x^{n-1}y)\delta(x^n)\Big] + q^2\Big[\tbinom{2n}{2}F(x^{2n-2}y^2) - \tbinom{n}{2}\alpha(x^{n-2}y^2)F(x^n)\\
&- \tbinom{n}{1}\tbinom{n}{1}\alpha(x^{n-1}y)F(x^{n-1}y) - \tbinom{n}{2}\alpha(y^2x^n)F(x^{n-2}) - \tbinom{n}{2}\delta(x^{n-2}y^2)\\
&- \tbinom{n}{1}\tbinom{n}{1}\beta(x^{n-1}y)\delta(x^{n-1}y) - \tbinom{n}{2}\beta(x^{n-2}y^2)\delta(x^n)\Big] + \cdots\\
&+ q^{2n}\Big[F(x^{2n}) - \alpha(x^n)F(x^n) - \beta(x^n)\delta(x^n)\Big] = 0 \text{ for all } x, y \in R.
\end{aligned}$$

Rewrite the above expression by using (2.3) as $qP_1(x,y) + q^2P_2(x,y) + \cdots + q^{2n-1}P_{2n-1}(x,y) = 0$, where $P_i(x,y)$ stand for the coefficients of q^i's for all $i = 1, 2, \ldots, 2n-1$. If we replace q by $1, 2, \ldots, 2n-1$, then we find a system of $2n-1$ homogeneous equations. It gives us a Vander Monde matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2^2 & \cdots & 2^{2n-1} \\ \cdots & & & \\ \cdots & & & \\ 2n-1 & (2n-1)^2 & \cdots & (2n-1)^{2n-1} \end{bmatrix}.$$

Which yields that $P_i(x,y) = 0$ for all $x, y \in R$ and for all $i = 1, 2, .., 2n-1$. In particular, We have

$$\begin{aligned}
P_1(x,y) \;=\; &\tbinom{2n}{1}F(x^{2n-1}y) - \tbinom{n}{1}\alpha(x^{n-1}y)F(x^n) - \tbinom{n}{1}\alpha(x^n)F(x^{n-1}y)\\
&- \tbinom{n}{1}\beta(x^n)\delta(x^{n-1}y) - \tbinom{n}{1}\beta(x^{n-1}y)\delta(x^n) = 0 \text{ for all } x, y \in R.
\end{aligned}$$

Putting $x = e$ and making use of $d(e) = 0$ and n-torsion freeness of R, we have

$$f(y) = \alpha(y)f(e) + \delta(y) \text{ for all } y \in R. \tag{2.4}$$

Next,

$$\begin{aligned} P_2(x,y) \;=\; & \tbinom{2n}{2}F(x^{2n-2}y^2) - \tbinom{n}{2}\alpha(x^{n-2}y^2)F(x^n) - \tbinom{n}{1}\tbinom{n}{1}\alpha(x^{n-1}y)F(x^{n-1}y) \\ - \; & \tbinom{n}{2}\alpha(y^2x^n)F(x^{n-2}) - \tbinom{n}{2}\delta(x^{n-2}y^2) - \tbinom{n}{1}\tbinom{n}{1}\beta(x^{n-1}y)\delta(x^{n-1}y) \\ - \; & \tbinom{n}{2}\beta(x^{n-2}y^2)\delta(x^n) = 0 \text{ for all } x, y \in R. \end{aligned}$$

Rewrite the above expression by substituting e for x to obtain

$$\begin{aligned} \tbinom{2n}{2}F(y^2) \;=\; & \tbinom{n}{2}\alpha(y^2)F(e) + \tbinom{n}{1}\tbinom{n}{1}\alpha(y)F(y) + \tbinom{n}{2}F(y^2) \\ + \; & \tbinom{n}{2}\delta(y^2) + \tbinom{n}{1}\tbinom{n}{1}\beta(y)\delta(y). \end{aligned}$$

That is,

$$\begin{aligned} \tfrac{2n(2n-1)}{2}F(y^2) \;=\; & \tfrac{n(n-1)}{2}\alpha(y^2)F(e) + n^2\alpha(y)F(y) + \tfrac{n(n-1)}{2}F(y^2) \\ + \; & \tfrac{n(n-1)}{2}d(y^2) + n^2\beta(y)d(y). \end{aligned}$$

After simple manipulation, we arrive at

$$(3n^2 - n)f(y^2) = n(n-1)\alpha(y^2)f(e) + 2n^2\alpha(y)f(y) + n(n-1)\delta(y^2) + 2n^2\delta d(y).$$

Since R is n torsion free, we get

$$(3n-1)f(y^2) = (n-1)\alpha(y^2)f(e) + 2n\alpha(y)f(y) + (n-1)\delta(y^2) + 2n\beta(y)\delta(y).$$

Use (2.4) to get the following

$$\begin{aligned} (3n-1)\Big[\alpha(y^2)f(e) + \delta(y^2)\Big] &= (n-1)\alpha(y^2)f(e) \\ + 2n\Big[\alpha(y)f(e) + \delta(y)\Big]\alpha(y) &+ (n-1)\delta(y^2) + 2n\beta(y)\delta(y). \end{aligned}$$

Simplify the above expression and making use of $2n$-torsion freeness of R, we have

$$\delta(y^2) = \alpha(y)\delta(y) + \beta(y)\delta(y) \text{ for all } y \in R.$$

Hence δ is a Jordan left (α, β)-derivation on R. Now, from (2.4), we get

$$\begin{aligned} f(y^2) &= \alpha(y^2)f(e) + \delta(y^2) \\ &= \alpha(y)\big[\alpha(y)f(e) + \delta(y)\big] + \beta(y)\delta(y) \\ &= \alpha(y)f(y) + \beta(y)\delta(y) \end{aligned}$$

Hence F is generalized Jordan left (α, β)-derivation on R associated with Jordan left (α, β)-derivation δ on R. ■

The following corollary is a consequence of the above theorem by assuming $\alpha = \beta = I_{identity}$.

Corollary 2.5 ([7], Theorem 2.4). *Let $n \geq 1$ be any fixed integer and R be k-torsion free semiprime ring. If $f, \delta : R \longrightarrow R$ are two additive mappings which satisfy the algebraic identity $f(x^{2n}) = x^n f(x^n) + x^n \delta(x^n)$ for all $x \in R$, then f is generalized left derivation associated with left derivation δ on R, where $k \in \{2, n, 2n-1\}$.*

Corollary 2.6 ([7], Theorem 2.5). *Let $n \geq 1$ be any fixed integer and R be a k-torsion free semiprime ring. If $f, \delta : R \to R$ are additive mappings satisfying $f(x^{2n}) = x^n f(x^n) + x^n \delta(x^n)$ for all $x \in R$, where $k \in \{2, n, 2n-1\}$. Then*

(1) *$[\delta(x), y] = 0$ for all $x, y \in R$, where δ is a derivation on R,*

(2) *δ maps R into $Z(R)$,*

(3) *δ is zero or R is commutative,*

(4) *For some $q \in Q_l(R_C)$, $f(x) = xq$ for all $x \in R$,*

(5) *f is a generalized derivation on R.*

The following example is in the favour of our theorems.

Example 2.1. Define a ring $R = \left\{ \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \mid m_1, m_2 \in 2\mathbb{Z}_8 \right\}$, $\mathbb{Z}_8$ has its usual meaning. Define mappings $F, d, f, \delta, \alpha, \beta : R \to R$ by $F\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & m_2 \end{pmatrix}$, $d\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} = \begin{pmatrix} m_1 & 0 \\ 0 & 0 \end{pmatrix}$, $f\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & m_2 \end{pmatrix}$, $\delta\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} = \begin{pmatrix} m_1 & 0 \\ 0 & 0 \end{pmatrix}$, $\alpha\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & m_2 \end{pmatrix}$ and $\beta\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} = \begin{pmatrix} m_1 & 0 \\ 0 & 0 \end{pmatrix}$. It is clear that F is not a generalized (α, β)-derivation and f is not a generalized Jordan left (α, β)-derivation on R but F, d, f, δ satisfy the algebraic conditions $F(x^6) = \alpha(x^2)F(x^4) + \beta(x^2)D(x^4)$ and $f(x^6) = f(x^2)\alpha(x^4) + \beta(x^4)\delta(x^2)$ for all $x \in R$. Which shows that semipriness and torsion restriction on R are essential conditions in Theorem 2.1 and Theorem 2.2.

Acknowledgement

The researcher of the paper extends his sincere gratitude to the Deanship of Scientific Research at the Islamic University of Madinah for the support provided to the Post-Publishing Program 2.

References

[1] S. Ali, *On generalized left derivations in rings and Banach algebras*, Aequat. Math. **81** (2011) 209–226.
https://doi.org/10.1007/s00010-011-0070-5

[2] S. Ali and C. Haetinger, *Jordan α-centralizer in rings and some applications*, Bol. Soc. Paran. Mat. **26** (2008) (1–2) 71–80.
https://doi.org/10.5269/bspm.v26i1-2.7405

[3] A.Z. Ansari, *On identities with additive mappings in rings*, Iranian J. Math. Sci. Inform. **15** (1) (2020) 125–133.
http://ijmsi.ir/article-1-1051-en.html

[4] A.Z. Ansari and F. Shujat, *Additive mappings covering generalized (α_1, α_2)-derivations in semiprime rings*, Gulf J. Math. **11** (2) (2021) 19–26.
https://gjom.org/index.php/gjom/article/view/495

[5] M. Ashraf and S. Ali, *On generalized Jordan left derivations in rings*, Bull. Korean Math. Soc. **45** (2) (2008) 253–261.
https://doi.org/10.4134/BKMS.2008.45.2.253

[6] J.M. Cusack, *Jordan derivations in rings*, Proc. Amer. Math. Soc. **53** (2) (1975) 321–324.
https://doi.org/10.1090/S0002-9939-1975-0399182-5

[7] F. Shujat and A.Z. Ansari, *Additive mappings satisfying certain identities of semiprime rings*, Preprint in Bulletin of Korean Mathematical Society.

[8] I.N. Herstein, *Derivations in prime rings*, Proc. Amer. Math. Soc. **8** (1957) 1104–1110.
https://doi.org/10.1090/S0002-9939-1957-0095864-2

[9] C. Lanski, *Generalized derivations and n-th power maps in rings*, Comm. Algebra **35** (11) (2007) 3660–3672.
https://doi.org/10.1080/00927870701511426

[10] E.C. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc. (1957) 1093–1100.
https://doi.org/10.1090/S0002-9939-1957-0095863-0

[11] J. Vukman, *On left Jordan derivations on rings and Banach algebras*, Aequationes Math. **75** (2008) 260–266.
https://doi.org/10.1007/s00010-007-2872-z

[12] S.M.A. Zaidi, M. Ashraf and S. Ali, *On Jordan ideals and left (θ, θ)-derivation in prime rings*, Int. J. Math. Math. Sci. **37** (2004) 1957–1965.
https://doi.org/10.1155/S0161171204309075

[13] B. Zalar, *On centralizers of semiprime rings*, Comment. Math. Univ. Carol. **32** (1991) 609–614.
http://eudml.org/doc/247321

[14] J. Zhu and C. Xiong, *Generalized derivations on rings and mappings of P-preserving kernel into range on Von Neumann algebras*, Acta Math. Sinica **41** (1998) 795–800.
https://doi.org/10.12386/A19980133

Received 29 August 2021
Revised 23 May 2022
Accepted 23 May 2022