Discussiones Mathematicae
General Algebra and Applications 43 (2023) 135–140
https://doi.org/10.7151/dmgaa.1421

SUPER STRONGLY CLEAN GROUP RINGS

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Abstract

In this paper, we study super strongly clean group rings for different classes of rings and groups. Mainly, we prove the following results:

(1) Let R be a ring with $2 \in J(R)$ and G be a locally finite 2-group. Then the group ring RG is super strongly clean if and only if R is super strongly clean.

(2) If R is a local ring with $p \in J(R)$ and G is a locally finite p-group, then the group ring RG is super strongly clean.

(3) If R is an abelian exchange ring with $2 \in J(R)$ and G is a locally finite 2-group, then the group ring RG is super strongly clean.

Keywords: super strongly clean ring, clean ring, group ring, locally finite p-group.

2020 Mathematics Subject Classification: 16U99, 16S34.

1. Introduction

All rings are assumed to be associative with non-zero identity and all groups are assumed to be non-trivial. The notion $J(R)$ denotes the Jacobson radical, $U(R)$ denotes the group of units, $Z(R)$ denotes the center of R, $Id(R)$ denotes the set of idempotents of R, C_n denotes the cyclic group of order n and C_∞ denotes the infinite cyclic group. If R is a ring and G is a group then RG denotes the group ring of G over R. In [9], Nicholson coined the concept of a clean ring. A ring R is called clean if every element of R is the sum of an idempotent and a unit. Further, Nicholson and Zhou [11] introduced the class of rings in which elements are uniquely the sum of an idempotent and a unit. They called such a ring a uniquely clean ring. Recently, Qu and Wei [13] introduced the concept of super strongly clean ring and defined as follows:

Definition 1.1. An element r is called super strongly clean if r is clean and $eu = ue$, whenever $r = e + u$ for any $e \in Id(R)$ and $u \in U(R)$. The ring R is called super strongly clean if every element of R is super strongly clean.

From [13], it is clear that the class of a super strongly clean rings lies strictly between the classes of uniquely clean rings and a strongly clean rings.

In [4, Example 1], Han and Nicholson answered the question of Park (whether the group ring RG is clean when R is clean and G is finite group such that $|G|$ is a unit in R) negatively. Moreover, they proved some results of clean group rings. They also asked whether RG is clean when R is a commutative regular ring. After that, Chen *et al.* [1] proposed a question "If R is a ring and G is a group, when is the group ring RG clean?" They proved that if R is a ring and G is a locally finite group, then RG is uniquely clean if and only if R is uniquely clean and G is a 2-group. Further, a number of authors contributed to the progress of cleanliness of the group rings; refer to [1, 4, 5, 14, 15]. Motivated by the above development of cleanliness of group rings and the question of Chen *et al.* [1], we study super strongly clean group rings for different classes of rings and groups.

In this article, mainly, we prove the following results:

(1) Let R be a ring with $2 \in J(R)$ and G be a locally finite 2-group. Then, the group ring RG is super strongly clean if and only if R is super strongly clean.

(2) If R is a local ring with $p \in J(R)$ and G is a locally finite p-group, then the group ring RG is super strongly clean.

(3) If R is an abelian exchange ring with $2 \in J(R)$ and G is a locally finite 2-group, then the group ring RG is super strongly clean.

We also provide some examples of group rings which are not super strongly clean rings.

2. Super Strongly Clean Group Rings

The group G is called a torsion group if every element of G is of finite order. If p is prime, $g \in G$ is called a p-torsion element if the order of g is p^k for some $k \geq 0$.

The group G is called a p-group if every element of G is p-torsion. A group G is called locally finite if every finitely generated subgroup of G is finite. The class of locally finite groups includes finite groups and torsion abelian groups. The homomorphism $\omega : RG \to R$ given by

$$\omega\left(\sum_{g\in G} a_g g\right) = \sum_{g\in G} a_g$$

is called the augmentation mapping of RG, and its kernel, denoted by $\omega(RG)$ is called the augmentation ideal of RG. It is known that

$$\omega(RG) = \left\{\sum_{g\in G} a_g(g-1) | g \in G, g \neq 1, a_g \in R\right\}.$$

Theorem 2.1 [13, Theorem 2.2]**.** *R is a super strongly clean if and only if R is Abel and clean.*

Lemma 2.2 [1, Lemma 1]**.** *Let R be a ring in which every idempotent is central and assume that $2 \in J(R)$. If G is a locally finite 2-group, then every idempotent in RG is in R.*

Here, we prove the main results of this paper.

Theorem 2.3. *Let R be a ring with $2 \in J(R)$ and G be a locally finite 2-group. Then the group ring RG is super strongly clean if and only if R is super strongly clean.*

Proof. Suppose RG is super strongly clean. Then RG is Abel and clean. Thus, R is clean being an image of clean ring RG. Let $a \in R$ and $e^2 = e \in Id(R)$. Then, $a \in RG$ and $e \in Id(RG)$. Since $Id(RG) \subseteq Z(RG)$, $ea(1-e) = 0$. It follows that R is Abel. Hence R is super strongly clean.

Conversely, suppose R is super strongly clean. Then from [15, Theorem 4], RG is clean so it suffices to show that it is Abel. Let $a \in RG$ and $e^2 = e \in Id(RG)$. We want to show that $ae = ea$ from which we conclude that RG has central idempotents. We have $\omega(a)$ and $\omega(e) \in R$. But by Lemma 2.2, $e = \omega(e)$ in R. Thus $\omega(e)\omega(a)\omega(1-e) = 0$ since R is super strongly clean. It follows that $ea(1-e) = 0$. Hence, RG is super strongly clean. ∎

Theorem 2.4. *If R is a local ring with $p \in J(R)$ and G is a locally finite p-group, then the group ring RG is super strongly clean.*

Proof. Since R is a local ring with $p \in J(R)$ and G is a locally finite p-group, RG is local from [10, Theorem (2)]. Thus RG is clean and $Id(RG)$ has only trivial idempotents. Therefore $Id(RG) \subseteq Z(RG)$. Then RG is an Abel clean ring. Hence, RG is a super strongly clean ring. ∎

Theorem 2.5. *If R is an abelian exchange ring with $2 \in J(R)$ and G is a locally finite 2-group, then the group ring RG is super strongly clean.*

Proof. Since R is abelian exchange, RG is clean from [6, Theorem 5]. Then from [1, Lemma 11], we have $Id(RG) \subseteq Z(R)$ because G is locally finite 2-group and $2 \in J(R)$. It follows that RG is Abel. Hence, RG is super strongly clean. ∎

In the following examples we show that the group ring is not super strongly clean, even if R is commutative.

Example 2.6. If R is commutative, then RC_∞ is not super strongly clean.

Proof. Since R is commutative, RC_∞ is not clean from [2, Example 3.3]. It follows that RC_∞ is not super strongly clean. ∎

Example 2.7. If $n > 2$, then there exists a prime p such that $\mathbb{Z}_pC_n$ is not super strongly clean.

Proof. From [2, Example 3.6], for $n > 2$ there exists a prime p such that $\mathbb{Z}_pC_n$ is not clean. While $\mathbb{Z}_pC_n$ is Abel because $\mathbb{Z}_pC_n$ is commutative. Thus, $\mathbb{Z}_pC_n$ is not super strongly clean. ∎

With the help of the following example, we show that for every super strongly clean ring R, the group ring RG need not be a super strongly clean ring.

Example 2.8. For any prime p, the group ring $\mathbb{Z}_pS_3$ is not super strongly clean.

Proof. *Case* 1. If $p = 2$. Suppose $\mathbb{Z}_2S_3$ is super strongly clean. Then from Theorem 2.1, $\mathbb{Z}_2S_3$ is clean and Abel. Now, we verify the following two claims.

Claim 1. *$\mathbb{Z}_2S_3$ is clean. Since $\mathbb{Z}_2$ is abelian and exchange, $\mathbb{Z}_2S_3$ is clean from* [6, Theorem 4]. *Therefore, the Claim* 1 *is true.*

Claim 2. *$\mathbb{Z}_2S_3$ is not Abel. If it were, then $Id(\mathbb{Z}_2S_3) \subseteq Z(\mathbb{Z}_2S_3)$, from which it follows that for any idempotent $x^2 = x \in \mathbb{Z}_2S_3$ and $y \in \mathbb{Z}_2S_3$, $xy = yx$. Let $g, h \in S_3$ satisfy $|g| = 2$ and $|h| = 3$, and set $x = gh^2 + g + h \in \mathbb{Z}_pS_3$. Then, we have*

$$x^2 = \left(gh^2 + g + h\right)^2 = (gh^2)^2 + g^2 + h^2 + gh^2g + ggh^2 + gh^2h + hgh^2 + gh + hg$$

$$= h + g + hg = h + g + gh^2 = x.$$

Thus $x^2 = x \in Id(\mathbb{Z}_2S_3)$. Now, for $y = h \in \mathbb{Z}_2S_3$, we get,

$$xy = \left(gh^2 + g + h\right) h = g + gh + h^2$$

$$yx = h\left(gh^2 + g + h\right) = gh + gh^2 + h^2.$$

It follows that $xy \neq yx$ which is a contradiction. Therefore, $Id(\mathbb{Z}_2S_3) \nsubseteq Z(\mathbb{Z}_2S_3)$. Thus, claim 2 is not true. Hence, $\mathbb{Z}_2S_3$ is not super strongly clean.

Case 2. If $p \geq 3$. Suppose $\mathbb{Z}_pS_3$ is super strongly clean. Then from Theorem 2.1, $\mathbb{Z}_pS_3$ is clean and Abel. Now, we verify the following two claims:

Claim 1. *$\mathbb{Z}_pS_3$ is clean. Since $\mathbb{Z}_p$ is a field and S_3 is a torsion group, $\mathbb{Z}_pS_3$ is clean from* [8, Corollary 2.10]. *Therefore, the Claim* 1 *is true.*

Claim 2. *$\mathbb{Z}_pS_3$ is not Abel for all $p \geq 3$. If it were, then $Id(\mathbb{Z}_pS_3) \subseteq Z(\mathbb{Z}_pS_3)$, from which it follows that for any idempotent $x^2 = x \in \mathbb{Z}_pS_3$ and $y \in \mathbb{Z}_pS_3$, $xy = yx$. Let $g, h \in S_3$ satisfy $|g| = 2$ and $|h| = 3$, and set $x = a + bgh^2 \in \mathbb{Z}_pS_3$, where $a, b \in \mathbb{Z}_p$. Then $x^2 = x$ if $a^2 + b^2 = a(modp)$ and $2ab = b(modq)$. In particular, for $a = b$, $x^2 = x$ if $2a^2 = a(modp)$. It follows that $x^2 = x$ if $a = \frac{p+1}{2} < p$. Therefore, for all $p \geq 3$, there exists an element $x = \left(\frac{p+1}{2}\right) + \left(\frac{p+1}{2}\right) gh \in \mathbb{Z}_pS_3$ such that $x^2 = x$. Now, for $y = g \in \mathbb{Z}_pS_3$, we have,*

$$xy = \left(\left(\frac{p+1}{2}\right) + \left(\frac{p+1}{2}\right) gh\right) g = \left(\frac{p+1}{2}\right) g + \left(\frac{p+1}{2}\right) h^2$$

$$yx = g\left(\left(\frac{p+1}{2}\right) + \left(\frac{p+1}{2}\right) gh\right) = \left(\frac{p+1}{2}\right) g + \left(\frac{p+1}{2}\right) h.$$

Then, $xy \neq yx$ for an element $x \in Id(\mathbb{Z}_pS_3)$ and some $y \in \mathbb{Z}_pS_3$. Therefore, the Claim 2 is not true. Hence, $\mathbb{Z}_pS_3$ is not super strongly clean. ■

Now, from the above example, we can observe that the study of super strongly clean group rings for different classes of rings and groups is not superfluous.

Acknowledgment

The authors are thankful to the referee for his comments and suggestions which improved the presentation of the paper.

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Received 29 July 2021
First Revised 30 September 2021
Second Revised 27 October 2021
Accepted 7 November 2021