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ON THE FINITE GOLDIE DIMENSION OF SUM OF TWO IDEALS OF AN R-GROUP

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Abstract

We consider an R-group G, where R is a zero symmetric right nearring. We obtain the Ω-dimension of sum of two ideals of G, as a natural generalization of sum of two subspaces of a finite dimensional vector space; indeed, difficulty due to non-linearity in G. However, in this paper we overcome the situation under a suitable assumption. More precisely, we prove that for a proper ideal Ω of G with Ω-finite Goldie dimension (Ω-FGD), if K_1, K_2 are ideals of G wherein $K_1 \cap K_2$ is an Ω-complement, then $dim_\Omega(K_1 + K_2) = dim_\Omega(K_1) + dim_\Omega(K_2) - dim_\Omega(K_1 \cap K_2)$. In the sequel, we prove several properties.

Keywords: Nearring, essential ideal, uniform ideal, finite dimension.

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1. Introduction

It is customary to generalize the theory of vector spaces over a field to that of modules over rings in a natural way. The dimension of the vector space over a field is a maximal set of linearly independent vectors or a minimal set of vectors which spans the space. The former statement when generalized to modules over associative rings or to module over nearrings (known as R-groups) become the concept of Goldie dimension (or uniform dimension). A module is uniform if every non-zero submodule is essential. Uniform submodules play a significant role to establish various finite dimension conditions in modules over associative rings. However, certain results may not be possible to generalize to modules over rings unless we impose some assumption(s). In particular, in the theory of finite dimensional vector spaces, for a subspace W of a vector space V, $dim(V/W) = dim(V) \setminus dim(W)$ is well known. This condition may not hold in general, when we consider a module over rings. For simplicity, $K = 6\mathbb{Z}$ is a uniform submodule of $\mathbb{Z}$-module over $\mathbb{Z}$, and $dim(\mathbb{Z}/6\mathbb{Z}) = dim(\mathbb{Z}_2 \times \mathbb{Z}_3) = 2 \neq 0 = dim(\mathbb{Z}) \setminus dim(6\mathbb{Z})$. Also, a similar situation can arise when we generalize the dimension of sum of two subspaces of a vector space. This motivates us to obtain the dimension of sum of two ideals of an R-group with relative finite Goldie dimension (where R is a nearring) under the assumption that their intersection is a complement ideal. Goldie [10] characterized several equivalent conditions for a module to have finite uniform dimension. In Bhavanari [6], uniform dimension was generalized to R-groups and obtained characterization for an R-group to have finite Goldie dimension. Goldie dimension aspects in R-groups were extensively studied in [4, 7, 15, 18].

Let Ω be a proper ideal of G. Our aim in this paper is to prove, if G has Ω-FGD and $\mathcal{K} = K_1 \cap K_2$ is an Ω-complement, then $dim_\Omega(K_1 + K_2) = dim_\Omega(K_1) + dim_\Omega(K_2) - dim_\Omega(\mathcal{K})$, where K_1 and K_2 are ideals of G.

2. Preliminaries

A (right) nearring $(R, +, \cdot)$ (Pilz [16]) is an algebraic system, where R is an additive group (need not be abelian) and a multiplicative semigroup, satisfying only one distributive axiom: $(r_1 + r_2)r_3 = r_1r_3 + r_2r_3$ for all $r_1, r_2, r_3 \in R$. If $(R, +, \cdot)$ is a right nearring, then $0a = 0$ and $(-a)b = -ab$, for all $a,\ b \in R$, but in general $a0 \neq 0$ for some $a \in R$. We call R is zero symmetric if $r0 = 0$ for all $r \in R$, denoted by $R = R_0$. An additive group $(G, +)$ is called an R-group (or module over a nearring R), denoted by $_RG$ or simply by G if there exists a mapping $R \times G \to G$ (image $(r, g) \to rg$), satisfying (1) $(r + s)g = rg + sg$; (2) $(rs)g = r(sg)$ for all $g \in G$ and $r,\ s \in R$. It is evident that every nearring is an R-group (over itself). Also, if R is a ring, then each (left) module over R is an

R-group. Throughout, G denotes an R-group where R is a zero-symmetric right nearring. A normal subgroup H of G is called an ideal if $r(g+h)-rg \in H$ for all $r \in R$, $h \in H$, $g \in G$. Since R is zero symmetric, for any ideals A and B of G, $A+B$ is an ideal of G ([16], Corollary 2.3), and we use $A \oplus B$ to denote the direct sum of A and B.

An ideal H of G is essential ([18]), if for any ideal K of G, $H \cap K = (0)$ implies $K = (0)$. If every ideal $(0) \neq H$ of G is essential then G is uniform.

For standard definitions and notations in nearrings, we refer to [8, 16].

3. Relative finite dimension in R-group

We start this section with the definitions of Ω-uniform ideal, Ω-direct sum and Ω-FGD

Definition 3.1. (1) An ideal H of G is said to be relative essential, if there exists a proper ideal Ω of G such that

(a) $H \nsubseteq \Omega$,

(b) for any ideal K of G, $H \cap K \subseteq \Omega$ implies $K \subseteq \Omega$.

We denote it by $H \leq^e_\Omega G$ (or H is Ω-essential in G).

(2) We denote $H_1 \leq^e_\Omega H_2$ when H_2 considered as an R-group. In case, $\Omega = (0)$, this is referred as G-essential, by [6]. An ideal I of G is relative uniform, if every ideal J of G, $J \subseteq I$, then $J \leq^e_\Omega G$.

(3) Let Ω be a proper ideals of G and let $\{I_i\}_{i \in I}$ be a family of ideals of Ω. We say that $\{I_i\}_{i \in I}$ is Ω-direct if $I_i \cap \left(\sum_{j \neq i} I_j\right) \subseteq \Omega$.

(4) G has Ω-FGD (or G has finite Goldie dimension with respect to a proper ideal Ω of G), if G does not contain an infinite number of ideals $H_i \nsubseteq \Omega$, whose sum is Ω-direct.

Lemma 3.1. *If G has Ω-FGD, then every ideal $H \nsubseteq \Omega$ of G, contains an Ω-uniform ideal.*

Proof. Suppose that G has Ω-FGD. In a contrary way, suppose H contains no Ω-uniform ideal. Then H is not Ω-uniform. So there exist ideals H_1 and H'_1 of G contained in H, and $H_1, H'_1 \nsubseteq \Omega$ such that $H_1 \cap H'_1 \subseteq \Omega$, $H_1 + H'_1 \subseteq H$. Then by supposition H'_1 is not Ω-uniform, which implies that there exist ideals H_2, H'_2 contained in H'_1 and $H_2, H'_2 \nsubseteq \Omega$ such that $H_2 \cap H'_2 \subseteq \Omega, H_2 + H'_2 \subseteq H'_1$. If we continue, then we get $\{H_i\}_1^\infty$, $\{H'_i\}_1^\infty$ of two infinite sequences of ideals of G, not contained in Ω such that $H_i \cap H'_i \subseteq \Omega$ and $H_i + H'_i \subseteq H'_{i-1}$, for $i \geq 2$. Thus, the sum $\sum_{i=1}^\infty H_i$ is infinite Ω-direct, a contradiction that G has Ω-FGD. ∎

Proposition 3.1. *Let I, J be ideals of G and Ω proper ideal of G. If $I \subseteq J$ and J is Ω-uniform, then I is also Ω-uniform.*

Proof. Easy verification. ■

Lemma 3.2. *Let H_i, $1 \leq i \leq 3$, be ideals of G, and Ω a proper ideal of G. Then*

(1) *$H_1 \leq_\Omega^e H_3$, $H_2 \leq_\Omega^e H_3$ and $H_1 \cap H_2 \nsubseteq \Omega$ implies that $H_1 \cap H_2 \leq_\Omega^e H_3$.*

(2) *Let $H_1 \subseteq H_2 \subseteq H_3$. Then $H_1 \leq_\Omega^e H_3$ if and only if $H_1 \leq_\Omega^e H_2$ and $H_2 \leq_\Omega^e H_3$.*

Proof. (1) Suppose that $H_1 \leq_\Omega^e H_3$ and $H_2 \leq_\Omega^e H_3$ and $(H_1 \cap H_2) \nsubseteq \Omega$. Let $(H_1 \cap H_2) \cap K \subseteq \Omega$, where K is an ideal of G, $K \subseteq H_3$. This implies $H_1 \cap (H_2 \cap K) \subseteq \Omega$. Since $H_2 \cap K$ is an ideal of G, $K \subseteq H_3$ and $H_1 \leq_\Omega^e H_3$, we get $H_2 \cap K \subseteq \Omega$. Again, since $H_2 \leq_\Omega^e H_3$, we have $K \subseteq T$.

(2) Suppose $H_1 \leq_\Omega^e H_3$. Let K be an ideal of G such that $H_1 \cap K \subseteq \Omega$ and $K \subseteq H_2$. Since $K \subseteq H_2 \subseteq H_3$ and $H_1 \leq_\Omega^e H_3$, we have that $K \subseteq \Omega$. Next, let K be an ideal of G such that $H_2 \cap K \subseteq \Omega$ and $K \subseteq H_3$. Now $H_1 \cap K \subseteq H_2 \cap K \subseteq \Omega$ and since $H_1 \leq_\Omega^e H_3$, we have $K \subseteq \Omega$. Conversely, suppose $H_1 \leq_\Omega^e H_2$ and $H_2 \leq_\Omega^e H_3$. To prove $H_1 \leq_\Omega^e H_3$, let K be an ideal of G such that $H_1 \cap K \subseteq \Omega$ and $K \subseteq H_3$. We have $H_1 \cap (H_2 \cap K) \subseteq H_1 \cap K \subseteq \Omega$. Since $H_2 \cap K$ is ideal of G, $H_2 \cap K \subseteq H_2$, and $H_1 \leq_\Omega^e H_2$, we have $H_2 \cap K \subseteq \Omega$. Also, as $H_2 \leq_\Omega^e H_3$, we get $K \subseteq \Omega$. ■

Following the Notation 3.4.6 of [8], $\langle A \rangle$ denotes the ideal generated by A, for a given subset A of G and for $a \in G$, $\langle a \rangle$ denotes $\langle \{a\} \rangle$.

Notation 3.1. Let $u \in G$. Then $\langle u \rangle = \bigcup_{i=1}^{\infty} A_{i+1}$, where $A_{i+1} = A_i^* \cup A_i^0 \cup A_i^+$ with $A_0 = \{u\}$, and

$$\begin{aligned} A_i^* &= \{s + y - s : s \in G, y \in A_i\}, \\ A_i^+ &= \{r(s + a) - rs : r \in R, s \in G, a \in A_i\}, \\ A_i^0 &= \{a - b : a, b \in A_i\} \cup \{a + b : a, b \in A_i\}. \end{aligned}$$

Lemma 3.3. *Let K, L and Ω (proper) be ideals of G such that $K \cap L = \Omega$. If $a \in K$, $b \in L$, then for any $a_1 \in \langle a \rangle$, there exists $b_1 \in \langle b \rangle$ such that $a_1 + b_1 \in \langle a + b \rangle + \Omega$.*

Proof. Write $\mathrm{X} = \{a\}$, $\mathrm{Y} = \{b\}$ and $\mathrm{Z} = \{a + b\}$. Let $\langle a \rangle = \bigcup_{k=1}^{\infty} \mathrm{X}_k$, $\langle b \rangle = \bigcup_{k=1}^{\infty} Y_k$, $\langle a + b \rangle = \bigcup_{k=1}^{\infty} \mathrm{Z}_k$. Let S$(k)$ be the statement: $a_1 \in \mathrm{X}_k$, implies there exists $b_1 \in \mathrm{Y}_k$ such that $a_1 + b_1 \in \mathrm{Z}_k + \Omega$. Then S(1) is trivially true. To verify S(2), let $a_1 \in \mathrm{X}_2 = \mathrm{X}_1^* \cup \mathrm{X}_1^+ \cup \mathrm{X}_1^o$, as in Notation 3.1. Now $a_1 = g + x_1 - g$ or $a_1 = r(g + x_1) - rg$ or $a_1 = x_1 - x_2$, where $r \in R$, $g \in G$ and $x_1, x_2 \in \mathrm{X}_1$. If $a_1 = g + x_1 - g$, then write $b_1 = g + y_1 - g \in \mathrm{Y}_1$, so that $a_1 + b_1 = g + x_1 - g + g + y_1 - g = g + (x_1 + y_1) - g \in \mathrm{Z}_1^* + \Omega$. If

$a_1 = r(g+x_1)-rg \in \mathrm{X}^+$, then write $b_1 = r(g+x_1+y_1)-r(g+x_1) \in \mathrm{Y}_1$. Clearly, $a_1+b_1-a_1-b_1 \in \langle a\rangle \cap \langle b\rangle \subseteq \Omega$, implies $a_1+b_1 = t+b_1+a_1$, for some $t \in \Omega$. Now, $a_1+b_1 = t+r(g+x_1+y_1)-r(g+x_1)+r(g+x_1)-rg = t+r(g+x_1+y_1)-rg = \Omega+\mathrm{Z}_1^+ = \mathrm{Z}_1^+ +\Omega$, as Ω is normal. If $a_1 = x_1-x_2 \in \mathrm{X}_1^o$, write $b_1 = y_1-y_2 \in \mathrm{Y}_1^o$, where $y_1, y_2 \in \mathrm{Y}_1$. Since $-x_1+y_1+x_2-y_1 \in \mathrm{X}_1 \cap \mathrm{Y}_1 \subseteq K \cap L \subseteq \Omega$, we have $-x_2+y_1 = \omega+y_1-x_2$, for some $\omega \in \Omega$. Then, $a_1+b_1 = (x_1-x_2)+(y_1-y_2) = x_1+\omega+y_1-x_2-y_2 = \omega_1+x_1+y_1+\omega_2-(x_2+y_2)$, where $\omega_1, \omega_2 \in \Omega$. This implies $a_1+b_1 = (\omega_1+\omega_2)+(x_1+y_1)-(x_2+y_2) \in \Omega+\mathrm{Z}_1^o = \mathrm{Z}_1^o+\Omega$, as Ω is normal. Therefore, from all the above three cases, it follows that $a_1+b_1 \in (\mathrm{Z}_1^*+\Omega)\cup(\mathrm{Z}_1^+ +\Omega)\cup(\mathrm{Z}_1^o+\Omega) \subseteq (\mathrm{Z}_1^* \cup \mathrm{Z}_1^+ \cup \mathrm{Z}_1^o)+\Omega = \mathrm{Z}_2+\Omega$. Hence S(2) is true. Suppose the induction hypothesis. That is, $\mathrm{S}(k-1)$ is true, for some k, with $k-1 \geq 2$. Let $a_1 \in \mathrm{X}_k$. Then $a_1 = g+x_1-g$ or $a_1 = r(g+x_1)-rg$ or $a_1 = x_1-x_2$, where $r \in R$, $g \in G$ and $x_1, x_2 \in \mathrm{X}_{k-1}$. Then there exist $y_1, y_2 \in \mathrm{Y}_{k-1}$ such that $x_1+y_1, x_2+y_2 \in \mathrm{Z}_{k-1}$. If $a_1 = g+x_1-g$, then we take $b_1 = g+y_1-g$. If $a_1 = r(g+x_1)-rg$, then take $b_1 = r(g+x_1+y_1)-r(g+x_1)$. If $a_1 = x_1-x_2$, then we take $b_1 = y_1-y_2$. In either case, we have $b_1 \in \mathrm{Y}_k$ and $a_1+b_1 \in (\mathrm{Z}_{k-1}^* \cup \mathrm{Z}_{k-1}^+ \cup \mathrm{Z}_{k-1}^o)+\Omega = \mathrm{Z}_k+\Omega$. Thus $\mathrm{S}(k)$ is true. ■

Lemma 3.4. *Let $\Omega \subset L_i \subseteq K_i$, for $i = 1,2$ be ideals of G such that $K_1 \cap K_2 = \Omega$. Then $L_i \leq_\Omega^e K_i$, for $i = 1,2$ if and only if $L_1+L_2 \leq_\Omega^e K_1+K_2$.*

Proof. Assume that $L_i \leq_\Omega^e K_i$, for $i = 1,2$. Write $A_1 = L_1+K_2$ and $A_2 = K_1+L_2$. Clearly, since $L_1 \nsubseteq \Omega$ and $L_2 \nsubseteq \Omega$ and $K_2 \nsubseteq \Omega$, we have $A_1 = L_1+K_2 \nsubseteq \Omega$. Now to show $A_1 \leq_\Omega^e K_1+K_2$. Let $a \in K_1+K_2$ such that $a \notin \Omega$, then $a = k_1+k_2$, for some $k_1 \in K_1, k_2 \in K_2$ and $\langle a\rangle \nsubseteq \Omega$. If $k_1 \in \Omega$, then $a = k_1+k_2 \in \Omega+K_2 \subseteq L_1+K_2 = A_1$. Therefore, $\langle a\rangle \subseteq A_1$, implies $\langle a\rangle \cap A = \langle a\rangle \nsubseteq \Omega$. If $k_1 \notin \Omega$, then since $L_1 \leq_\Omega^e K_1$, and $\langle k_1\rangle \subseteq K_1$, we get $L \cap \langle k_1\rangle \nsubseteq \Omega$. Then there exists $x_1 \in L_1 \cap \langle k_1\rangle$ such that $x_1 \notin \Omega$. Since $x_1 \in \langle k_1\rangle$, by Lemma 3.3, there exists $x_2 \in \langle k_2\rangle$ such that $x_1+x_2 \in \langle k_1+k_2\rangle+\Omega = \langle a\rangle+\Omega$. Clearly, since $x_1 \notin \Omega$, we have $x_1+x_2 \notin \Omega$. This shows that $x_1+x_2 \in \langle a\rangle$. Also, $x_1+x_2 \in L_1+K_2 = A_1$, implies $x_1+x_2 \in \langle a\rangle \cap A_1$, but $x_1+x_2 \notin \Omega$. Therefore, $\langle a\rangle \cap A_1 \nsubseteq \Omega$, shows that $A_1 \leq_\Omega^e K_1+K_2$. Similarly, $A_2 \leq_\Omega^e K_1+K_2$. Then by Lemma 3.2(1), we have $A_1 \cap A_2 \leq_\Omega^e K_1+K_2$. Now to show $L_1+L_2 = A_1 \cap A_2$, let $x \in A_1 \cap A_2$. Then $x = l_1+k_2 = k_1+l_2$, implies $l_1+k_2 = k_1+l_2$. Now $-k_1+l_1 = l_2-k_2 \in (K_1+L_1)\cap(K_2+L_2) = K_1 \cap K_2 = \Omega$. Therefore, $-k_1+l_1 = l_2-k_2 = \omega$ for some $\omega \in \Omega$. Now $k_2 = l_2-\omega \in L_2+\Omega = L_2$. Hence $x = l_1+k_2 \in L_1+L_2$, shows that $A_1 \cap A_2 \subseteq L_1+L_2$. Also, $L_1+L_2 \subseteq L_1+K_2 = A_1$ and $L_1+L_2 \subseteq K_1+L_2 = A_2$, implies $L_1+L_2 \subseteq A_1 \cap A_2$. Therefore, $L_1+L_2 = A_1 \cap A_2 \leq_\Omega^e K_1+K_2$. Conversely, suppose that $L_1+L_2 \leq_\Omega^e K_1+K_2$ and $L_1 \nleq_\Omega^e K_1$. Then there exists an ideal A of G such that $A \subseteq G_1$, $L_1 \cap A \subseteq \Omega$ but $A \nsubseteq \Omega$. Now we show that $(L_1+L_2)\cap A \subseteq \Omega$. Let $x \in (L_1+L_2)\cap A$. Then $x = l_1+l_2$, for some $x \in A$, $l_1 \in L_1$, $l_2 \in L_2$. Now $l_2 = -l_1+x \in (L_1+A)\cap L_2 \subseteq K_1 \cap K_2 = \Omega \subset L_1$. Therefore, $l_2 \in L_1$. Hence

$x = l_1 + l_2 \in L_1 \cap A \subseteq \Omega$, which shows that, $(L_1 + L_2) \cap A \subseteq \Omega$, a contradiction. Hence $L_1 \leq^e_\Omega K_1$. In a similar way, we will get $L_2 \leq^e_\Omega K_2$. ∎

Corollary 3.1. *Let $\Omega \subset H_i \subseteq G_i$ for $i = 1$ to n such that $\bigcap_{i=1}^n G_i = \Omega$. Then $\sum_{i=1}^n H_i \leq^e_\Omega \sum_{i=1}^n G_i$ if and only if $H_i \leq^e_\Omega G_i$, $1 \leq i \leq n$.*

Proof. By using Lemma 3.4 and induction on n. ∎

Proposition 3.2. *Let Ω be a proper ideal of G. If G has Ω-FGD, then there exist Ω-uniform ideals $\Omega \subset H_i$, $1 \leq i \leq n$, such that their sum is Ω-direct and Ω-essential in G (in this case, we denote as $H_1 \oplus \cdots \oplus H_n \leq^e_\Omega G$). The integer 'n' is independent of Ω-uniform ideals (the relative dimension of G with respect to Ω, and we write $dim_\Omega(G) = n$).*

Proof. Suppose G has Ω-uniform ideals $H_1, H_2, \ldots, H_n$ such that its sum is Ω-direct and $\sum_{i=1}^n H_i \leq^e_\Omega G$. Let $K_1, K_2, \ldots, K_m$ be ideals of G such that $K_i \nsubseteq \Omega$, and $\sum_{i=1}^m K_i$ is Ω-direct. Now to show $m \leq n$, first we show that if L is an ideal of G such that $L \cap H_i \nsubseteq \Omega$ for all i, then $L \leq^e_\Omega G$. Suppose $L \cap H_i \nsubseteq \Omega$. Since H_i is Ω-uniform, by definition, every ideal contained in H_i is Ω-essential. In particular, $L \cap H_i$ is an ideal contained in H_i and so $L \cap H_i \leq^e_\Omega H_i$. Now by Lemma 3.4, $\sum_{i=1}^n (L \cap H_i) \leq^e_\Omega \sum_{i=1}^n H_i$ and since $\sum_{i=1}^n H_i \leq^e_\Omega G$, by Lemma 3.2(2), we have $\sum_{i=1}^n (L \cap H_i) \leq^e_\Omega G$. Again by Lemma 3.2(2), since $\sum_{i=1}^n (L \cap H_i) \subseteq L \subseteq G$ and $\sum_{i=1}^n (L \cap H_i) \leq^e_\Omega G$, we get $L \leq^e_\Omega G$. Now if $\sum_{i=2}^m K_i \leq^e_\Omega G$, then since $\sum_{i=2}^m K_i$ is Ω-direct, we have $\sum_{i=2}^m K_i \cap K_1 \subseteq \Omega$, but $K_1 \nsubseteq \Omega$, a contradiction. Hence $\sum_{i=2}^m K_i \nleq^{se}_\Omega G$. So there exists an $j \in \{1, 2, \ldots, n\}$ such that $\sum_{i=2}^m K_i \cap H_j \subseteq \Omega$, and $H_j \nsubseteq \Omega$. Suppose $j = 1$. Then $\sum_{i=2}^m K_i \cap H_1 \subseteq \Omega$, which shows that $\sum_{i=2}^m K_i + H_1$ is Ω-direct. Again, since $H_1 + \sum_{i=3}^m K_i \nleq^e_\Omega G$, there exists $j \in \{2, \ldots, n\}$ such that $\sum_{i=3}^m K_i + H_j \subseteq \Omega$, and $H_j \nsubseteq \Omega$, say $j = 2$, which implies that $\sum_{i=3}^m K_i + H_1 + H_2$ is Ω-direct. Continuing this process, we get $m \leq n$. Hence G has Ω-*FGD*. ∎

Remark 3.1. If an ideal H of G is Ω-uniform, then $dim_\Omega(H) = 1$.

Example 3.1. Let $R = (\mathbb{Z}_{24}, +_{24}, \cdot_{24})$ and $G = R$. Consider the ideals $H_1 = \langle \overline{8} \rangle$, $H_2 = \langle \overline{12} \rangle$, $H_3 = \langle \overline{4} \rangle$, $H_4 = \langle \overline{6} \rangle$, $H_5 = \langle \overline{2} \rangle$, $H_6 = \langle \overline{3} \rangle$. Take $\Omega = H_4$. Then G has Ω-*FGD*, and Ω-*dim* 2, but G has no *FGD*.

Example 3.2. Consider $(R = (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3), +, \cdot)$ given in (*Sonata* [1] 12/2, 1) and $G = R$. The ideals of G are $H_1 = \{0, 3\}$, $H_2 = \{0, 6\}$, $H_3 = \{0, 9\}$, $H_4 = \{0, 3, 6, 9\}$, $H_5 = \{0, 2, 4\}$, $H_6 = \{0, 1, 2, 3, 4, 5\}$, $H_7 = \{0, 2, 4, 6, 8, 10\}$, $H_8 = \{0, 2, 4, 7, 9, 11\}$. Take $\Omega = H_5$. Then G has Ω-*FGD* with Ω-*dim* 2, but G has no *FGD*.

Example 3.3. Let $R = \left(\begin{pmatrix} 0 & \mathbb{Z}_{p^n} \\ 0 & 0 \end{pmatrix}, +_{p^n}, \cdot_{p^n} \right)$, where p is prime and $n \in \mathbb{Z}^+$. Here R non-commutative matrix ring and let $G = R$. Now G is considered as an R-group. The ideals of G are $H_i = \left\{ \begin{pmatrix} 0 & p^n\mathbb{Z}_{p^n} \\ 0 & 0 \end{pmatrix} : n \in \mathbb{Z}^+ \right\}$. Consider $H = H_{n-1}$ and $\Omega = H_1$. Then, $H_i \leq^e_\Omega H_{i+1}$, for all $i \geq 2$. Therefore, H has Ω-*FGD*.

Definition 3.2. Let H be an ideal of G. An ideal H' of G is called a relative complement of H if there exists a proper ideal Ω of G such that H' is maximal with respect to $H \cap H' \subseteq \Omega$. In this case, we call H' as Ω-complement of H.

If $\Omega = (0)$, then the Ω-complement corresponds to just the complement defined in [18].

Lemma 3.5. *Let A and Ω (proper) be ideals of G. If $B \subseteq \Omega$ is the maximal among the ideals of G with $A \cap B \subseteq \Omega$, then $A \oplus B \leq^e_\Omega G$ and B is an Ω-complement of A.*

Proof. It is sufficient to show the Ω-essentiality. Suppose D is an ideal of G such that $(A+B)\cap D \subseteq \Omega$. To show, $D \subseteq \Omega$, first we show that $A\cap(B+D) \subseteq \Omega$. For if $A\cap(B+D) \nsubseteq \Omega$, there exists $a \in A\cap(B+D)$, but $a \notin \Omega$. Then $a = b+d$, for some $a \in A$, $b \in B$ and $d \in D$, but $a \notin \Omega$. Then $d = a - b \in (A + B) \cap D \subseteq \Omega$. Also, $b \in B \subseteq \Omega$, implies $a = b + d \in \Omega$, a contradiction. Therefore, $A \cap (B + D) \subseteq \Omega$. Now by maximality of B, we have $B + D = B$, shows that $D \subseteq B \subseteq A + B$. Hence $D = (A + B) \cap D \subseteq \Omega$, proves $A \oplus B \leq^e_\Omega G$. ∎

Lemma 3.6. *Let Ω be a proper ideal of G. If H and K are ideals of G with $H \cap K \subseteq \Omega$, then $dim_\Omega(H + K)$=$dim_\Omega(H)$+$dim_\Omega(K)$.*

Proof. Suppose $dim_\Omega(H) = t$ and $dim_\Omega(K) = s$. Then, there exist Ω-uniform ideals $A_1, \dots, A_t$ of H such that $A_1 + A_2 + \cdots + A_t$ is Ω-direct and Ω-essential in H. Similarly, there exist Ω-uniform ideals $B_1, \dots, B_s$ of K such that $B_1 + B_2 + \cdots + B_s$ is Ω-direct and Ω-essential in K. Since $H \cap K \subseteq \Omega$, $A_1 + A_2 + \cdots + A_t + B_1 + B_2 + \cdots + B_s$ is Ω-direct in $H + K$. By Corollary 3.1, we have $A_1+A_2+\cdots+A_t+B_1+B_2+\cdots+B_s \leq^e_\Omega H+K$. Therefore, $dim_\Omega(H+K) = t+s$. Hence $dim_\Omega(H + K)$=$dim_\Omega(H) + dim_\Omega(K)$. ∎

Corollary 3.2. *If $A_1, \dots, A_n$ are ideals of G, then*

$$dim_\Omega(A_1 \oplus A_2 \oplus \cdots \oplus A_n) = dim_\Omega(A_1) + \cdots + dim_\Omega(A_n).$$

Proof. Follows from Lemma 3.6 and the mathematical induction. ∎

Theorem 3.1. *Let Ω be a proper ideal of G and $dim_\Omega(G) = n$. Then for any ideal H of G, $H \leq^e_\Omega G$ if and only if $dim_\Omega(H) = dim_\Omega(G)$.*

Proof. Suppose $H \leq_{\Omega}^{e} G$ and $dim_{\Omega}(H) = n$. Then there exist Ω-uniform ideals $A_1, \ldots, A_n$ of H such that $A_1 + A_2 + \cdots + A_n$ is Ω-direct and Ω-essential in H. Now, $A_1 \oplus A_2 \oplus \cdots \oplus A_n \leq_{\Omega}^{e} H$ and $H \leq_{\Omega}^{e} G$, implies by Lemma 3.2(2), $A_1 \oplus A_2 \oplus \cdots \oplus A_n \leq_{\Omega}^{e} G$. Therefore, $dim_{\Omega}(G) = n$, shows that $dim_{\Omega}(G) = dim_{\Omega}(H)$. Conversely, suppose $dim_{\Omega}(H) = dim_{\Omega}(G) = n$. Then there exist Ω-uniform ideals $A_1, \ldots, A_n$ of H such that $A_1 + A_2 + \cdots + A_n$ is Ω-direct and Ω-essential in H. If $A_1 \oplus A_2 \oplus \cdots \oplus A_n \not\leq_{\Omega}^{e} G$, we can get an Ω-unform ideal A_{n+1} of G such that $A_1 + A_2 + \cdots + A_n + A_{n+1}$ is Ω-direct. Then $dim_{\Omega}(G) \geq n+1$, a contradiction. So, $A_1 \oplus A_2 \oplus \cdots \oplus A_n \leq_{\Omega}^{e} G$. Since $A_1 \oplus A_2 \oplus \cdots \oplus A_n \subseteq H \subseteq G$, we get $H \leq_{\Omega}^{e} G$. ■

The proof of the following corollary is straightforward.

Corollary 3.3. *Let Ω be a proper ideal of G and $dim_{\Omega}(G) = n$. Then for any ideal H of G, $dim_{\Omega}(H) = dim_{\Omega}(G)$ if and only if H contains an Ω-direct sum of n Ω-uniform ideals.*

Lemma 3.7. *Let Ω be a proper ideal of G. If G has Ω-FGD and H an ideal of G with $dim_{\Omega}(H) < dim_{\Omega}(G)$. Then there exist Ω-uniform ideals $A_1, \ldots, A_k$ such that $H + A_1 + \cdots + A_k$ is Ω-direct and Ω-essential in G. Moreover, $k = dim_{\Omega}(G) - dim_{\Omega}(H)$.*

Proof. Since $dim_{\Omega}(H) < dim_{\Omega}(G)$, by Theorem 3.1, $H \not\leq_{\Omega}^{e} G$. Write

$$\mathcal{B} = \{K : H \cap K \subseteq \Omega, \text{ where } K \text{ is an ideal of } G\}.$$

By Zorn's lemma, there is an ideal H' which is maximal with respect to $H \cap H' \subseteq \Omega$. Then by Lemma 3.5, $H \oplus H' \leq_{\Omega}^{e} G$. Let $k = dim_{\Omega}(H')$. Now there exist Ω-uniform ideals $A_1, \ldots, A_k$ such that $A_1 \oplus \cdots \oplus A_k \leq_{\Omega}^{e} H'$. By Corollary 3.1, we have $H \oplus A_1 \oplus \cdots \oplus A_k \leq_{\Omega}^{e} H \oplus H'$.

By Lemma 3.2(2), we have

$$H \oplus A_1 \oplus \cdots \oplus A_k \leq_{\Omega}^{e} G.$$

Then by Corollary 3.2 and Theorem 3.1,

$$\begin{aligned} dim_{\Omega}(G) &= dim_{\Omega}(H \oplus A_1 \oplus \cdots \oplus A_k) \\ &= dim_{\Omega}(H) + dim_{\Omega}(A_1) + \cdots + dim_{\Omega}(A_k), \text{ by Corollary 3.2} \\ &= dim_{\Omega}(H) + k, \text{ since each } A_i \text{ is } \Omega\text{-uniform}, dim_{\Omega}(A_i) = 1. \end{aligned}$$

Therefore, $k = dim_{\Omega}(G) - dim_{\Omega}(H)$. ■

Theorem 3.2. *Suppose G has Ω-FGD and K_1, K_2 are ideals of G such that $\mathcal{K} = K_1 \cap K_2$ is an Ω-complement, contained in Ω. Then $dim_{\Omega}(K_1 + K_2) = dim_{\Omega}(K_1) + dim_{\Omega}(K_2) - dim_{\Omega}(\mathcal{K})$.*

Proof. Let A be an Ω-complement of $\mathcal{K}$ in K_1 and B an Ω-complement of $\mathcal{K}$ in K_2. Then by Lemma 3.5, $A \oplus \mathcal{K} \leq^e_\Omega K_1$ and $B \oplus \mathcal{K} \leq^e_\Omega K_2$. To show ${}^{(A+\mathcal{K})}/_{\mathcal{K}} \leq^e_{\Omega/\mathcal{K}} {}^{K_1}/_{\mathcal{K}}$, let ${}^{T}/_{\mathcal{K}}$ be an ideal of ${}^{G}/_{\mathcal{K}}$, contained in ${}^{K_1}/_{\mathcal{K}}$ such that ${}^{(A+\mathcal{K})}/_{\mathcal{K}} \cap {}^{T}/_{\mathcal{K}} \subseteq {}^{\Omega}/_{\mathcal{K}}$, where $\mathcal{K}$ is contained in T.

Then

$$^{((A+\mathcal{K})\cap T)}/_{\mathcal{K}} = {}^{(A+\mathcal{K})}/_{\mathcal{K}} \cap {}^{T}/_{\mathcal{K}} \subseteq {}^{\Omega}/_{\mathcal{K}},$$

implies $(A+\mathcal{K}) \cap T \subseteq \Omega$. Since $A \oplus \mathcal{K} \leq^e_\Omega K_1$, we get $T \subseteq \Omega$. Hence ${}^{T}/_{\mathcal{K}} \subseteq {}^{\Omega}/_{\mathcal{K}}$, shows that ${}^{(A+\mathcal{K})}/_{\mathcal{K}} \leq^e_{\Omega/\mathcal{K}} {}^{K_1}/_{\mathcal{K}}$. In a similar way, we get ${}^{(B+\mathcal{K})}/_{\mathcal{K}} \leq^e_{\Omega/\mathcal{K}} {}^{K_2}/_{\mathcal{K}}$.

Since

$$^{K_1}/_{\mathcal{K}} \cap {}^{K_2}/_{\mathcal{K}} = {}^{(K_1\cap K_2)}/_{\mathcal{K}} \subseteq {}^{\Omega}/_{\mathcal{K}},$$

we have

$$^{(A+B+\mathcal{K})}/_{\mathcal{K}} = {}^{(A+\mathcal{K})}/_{\mathcal{K}} + {}^{(B+\mathcal{K})}/_{\mathcal{K}} \leq^e_{\Omega/\mathcal{K}} {}^{K_1}/_{\mathcal{K}} + {}^{K_2}/_{\mathcal{K}} = {}^{(K_1+K_2)}/_{\mathcal{K}}.$$

First we show that $A+B+\mathcal{K} \leq^e_\Omega K_1 + K_2$. Let I be an ideal of G such that $(A+B+\mathcal{K}) \cap I \subseteq \Omega$. Let $x+\mathcal{K} \in ({}^{(A+B+\mathcal{K})}/_{\mathcal{K}}) \cap ({}^{(I+\mathcal{K})}/_{\mathcal{K}})$. Now $x+\mathcal{K} = a+b+\mathcal{K} = y+\mathcal{K}$ for some $a \in A, b \in B,$ and $y \in I$. Then $(a+b)-y = k$ for some $k \in \mathcal{K}$. So $y = (a+b)-k \in A+B+\mathcal{K}$, hence $y \in (A+B+\mathcal{K}) \cap I \subseteq \Omega$. Now $x+\mathcal{K} = y+\mathcal{K} \in {}^{\Omega}/_{\mathcal{K}}$. Therefore,

$$({}^{(A+B+\mathcal{K})}/_{\mathcal{K}}) \cap ({}^{(I+\mathcal{K})}/_{\mathcal{K}}) \subseteq {}^{\Omega}/_{\mathcal{K}}.$$

Since ${}^{(A+B+\mathcal{K})}/_{\mathcal{K}} \leq^e_{\Omega/\mathcal{K}} {}^{(K_1+K_2)}/_{\mathcal{K}}$, we have ${}^{(I+\mathcal{K})}/_{\mathcal{K}} \subseteq {}^{\Omega}/_{\mathcal{K}}$, implies that $I \subseteq I+\mathcal{K} \subseteq \Omega$. Therefore, $A+B+\mathcal{K} \leq^e_\Omega K_1+K_2$.

To prove $A+B+\mathcal{K}$ is Ω-direct, first we show that $(A+B)\cap\mathcal{K} \subseteq \Omega$. If not, there exist $a \in A, b \in B$ and $k \in \mathcal{K}$ such that $k = a+b \notin \Omega$, implies $a = k-b \in K_1 \cap K_2 = \mathcal{K}$. Hence $a \in A \cap \mathcal{K} \subseteq \Omega$. Also, $b = -a+k \in K_1 \cap \mathcal{K} \subseteq K_1 \cap K_2 = \mathcal{K}$, implies $b \in B \cap \mathcal{K} \subseteq \Omega$. Then, $k = a+b \in \Omega+\Omega \subseteq \Omega$, a contradiction. Therefore, $(A+B)\cap\mathcal{K} \subseteq \Omega$.

In a similar way we obtain $A \cap (B+\mathcal{K}) \subseteq \Omega$ and $B \cap (A+\mathcal{K}) \subseteq \Omega$, which shows that

$$(A \oplus B \oplus \mathcal{K}) \leq^e_\Omega K_1 + K_2.$$

Therefore,

$$\begin{aligned}
dim_\Omega(K_1+K_2) &= dim_\Omega(A \oplus B \oplus \mathcal{K}), \text{ (Theorem 3.1)}\\
&= dim_\Omega(A) + dim_\Omega(B) + dim_\Omega(\mathcal{K}), \quad \text{(Corollary 3.2)}\\
&= (dim_\Omega(K_1) - dim_\Omega(\mathcal{K})) + (dim_\Omega(K_2) - dim_\Omega(\mathcal{K})) + dim_\Omega(\mathcal{K})\\
&= dim_\Omega(K_1) + dim_\Omega(K_2) - dim_\Omega(\mathcal{K})
\end{aligned}$$

■

4. Conclusion

We have obtained Ω-dimension of sum of two ideals of an R-group, which is a generalization of sum of two subspaces of a finite dimensional vector space. This has been a significant attempt due to non-linearity in R-group. One can obtain spanning (or dualizing) dimensional aspects of R-groups. As an application, further one can extend the possible results on finite Goldie dimension in terms of join independent sets in a lattice, as defined by the authors in [17].

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