

## $\sigma$ -FILTERS OF COMMUTATIVE *BE*-ALGEBRAS

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### Abstract

The concept of  $\sigma$ -filters is introduced in commutative *BE*-algebras and some properties of these classes of filters are studied. Some equivalent conditions are derived for every filter of a commutative *BE*-algebra to become a  $\sigma$ -filter. Some necessary and sufficient conditions are given for every regular filter of a commutative *BE*-algebra to become a  $\sigma$ -filter. A set of equivalent conditions is given for the class of all  $\sigma$ -filters of a commutative *BE*-algebra to become a sublattice to the lattice of all filters.

**Keywords:** commutative *BE*-algebra, dual annihilator filter, prime filter,  $\sigma$ -filter, regular filter, O-filter.

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### INTRODUCTION

The notion of *BE*-algebras was introduced and extensively studied by Kim and Kim in [5]. These classes of *BE*-algebras were introduced as a generalization of the class of *BCK*-algebras of Iseki and Tanaka [4]. Some properties of filters of *BE*-algebras were studied by Ahn and Kim in [1] and by Meng in [6]. In [12], Walendziak discussed some significant properties of commutative *BE*-algebras. He also investigated the relationship between *BE*-algebras, implicative algebras and *J*-algebras. In [6], Meng introduced the notion of prime filters in *BCK*-algebras, and then gave a description of the filter generated by a set, and obtained some of fundamental properties of prime filters. In [4], some properties of prime ideals are investigated in *BCK*-algebras. In [8], the author studied some properties of prime filters in *BE*-algebras. In this paper, the author extensively studied the algebraic as well as the topological properties of prime filters of commutative

$BE$ -algebras. In [9], the authors introduced the notion of dual annihilators of commutative  $BE$ -algebra and studied extensively the properties of these dual annihilators. In 2020, the authors introduced the notions of regular filters [10] and  $O$ -filters [11] in commutative  $BE$ -algebras and the interconnection between those two special classes of filters is studied.

In this paper, the concept of  $\sigma$ -filters is introduced in commutative  $BE$ -algebras and their properties are studied analogous to that in a distributive lattice [3]. A set of equivalent conditions is given for every filter of a commutative  $BE$ -algebra to become a  $\sigma$ -filter. It is observed that every  $\sigma$ -filter of a commutative  $BE$ -algebra is a regular filter but not the converse in general. However, some equivalent conditions are proved for every regular filter of a commutative  $BE$ -algebra to become a  $\sigma$ -filter. It is also observed that every  $O$ -filter of a commutative  $BE$ -algebra is a  $\sigma$ -filter but not the converse in general. Some necessary and sufficient conditions are given for every  $\sigma$ -filter of a commutative  $BE$ -algebra to become an  $O$ -filter. Some equivalent conditions are given to prove that the class of all  $\sigma$ -filters of a commutative  $BE$ -algebra to become a sublattice to the lattice of all filters of a commutative  $BE$ -algebra.

## 1. PRELIMINARIES

In this section, we present certain definitions and results which are taken mostly from the papers [1, 5, 9, 10], and [11] for the ready reference of the reader.

**Definition 1.1** [5]. An algebra  $(X, *, 1)$  of type  $(2, 0)$  is called a  $BE$ -algebra if it satisfies the following properties:

- (1)  $x * x = 1$ ,
- (2)  $x * 1 = 1$ ,
- (3)  $1 * x = x$ ,
- (4)  $x * (y * z) = y * (x * z)$  for all  $x, y, z \in X$ .

A  $BE$ -algebra  $X$  is called self-distributive if  $x * (y * z) = (x * y) * (x * z)$  for all  $x, y, z \in X$ . A  $BE$ -algebra  $X$  is called transitive if  $y * z \leq (x * y) * (x * z)$  for all  $x, y, z \in X$ . A  $BE$ -algebra  $X$  is called commutative if  $(x * y) * y = (y * x) * x$  for all  $x, y \in X$ . Every commutative  $BE$ -algebra is transitive. For any  $x, y \in X$ , define  $x \vee y = (y * x) * x$ . If  $X$  is commutative then  $(X, \vee)$  is a semilattice [12]. We introduce a relation  $\leq$  on a  $BE$ -algebra  $X$  by  $x \leq y$  if and only if  $x * y = 1$  for all  $x, y \in X$ . Clearly  $\leq$  is reflexive. If  $X$  is commutative, then  $\leq$  is transitive, anti-symmetric and hence a partial order on  $X$ .

**Theorem 1.2** [5]. Let  $X$  be a transitive  $BE$ -algebra and  $x, y, z \in X$ . Then

- (1)  $1 \leq x$  implies  $x = 1$ ,

- (2)  $y \leq z$  implies  $x * y \leq x * z$  and  $z * x \leq y * x$ .

**Definition 1.3** [1]. A non-empty subset  $F$  of a  $BE$ -algebra  $X$  is called a filter of  $X$  if, for all  $x, y \in X$ , it satisfies the following properties:

- (1)  $1 \in F$ ,  
 (2)  $x \in F$  and  $x * y \in F$  imply that  $y \in F$ .

For any non-empty subset  $A$  of a transitive  $BE$ -algebra  $X$ , the set  $\langle A \rangle = \{x \in X \mid a_1 * (a_2 * (\dots * (a_n * x) \dots)) = 1 \text{ for some } a_1, a_2, \dots, a_n \in A\}$  is the smallest filter containing  $A$ . For any  $a \in X$ ,  $\langle a \rangle = \{x \in X \mid a^n * x = 1 \text{ for some } n \in \mathbb{N}\}$ , where  $a^n * x = a * (a * (\dots * (a * x) \dots))$  with the repetition of  $a$  is  $n$  times, is called the principal filter generated  $a$ . Let  $F$  be a filter of a transitive  $BE$ -algebra and  $a \in X$ , then  $\langle F \cup \{a\} \rangle = \{x \in X \mid a^n * x = 1 \text{ for some } n \in \mathbb{N}\}$ . A proper filter  $P$  of a  $BE$ -algebra is called prime [8] if  $F \cap G \subseteq P$  implies  $F \subseteq P$  or  $G \subseteq P$  for any two proper filters  $F, G$  of  $X$ . A proper filter  $P$  of a  $BE$ -algebra is called prime [8] if  $\langle x \rangle \cap \langle y \rangle \subseteq P$  implies  $x \in P$  or  $y \in P$  for any  $x, y \in X$ . A proper filter  $M$  of a transitive  $BE$ -algebra  $X$  is called maximal if there exist no proper filters  $Q$  such that  $M \subset Q$ . "Every maximal filter of a commutative  $BE$ -algebra is prime".

**Theorem 1.4** [8]. Let  $F$  and  $G$  be two filters of a transitive  $BE$ -algebra  $X$ . Then

$$F \vee G = \{x \in X \mid a * (b * x) = 1 \text{ for some } a \in F, b \in G\}$$

is the supremum of  $F$  and  $G$ . Hence the set  $\mathcal{F}(X)$  of all filters of  $X$  is a lattice.

**Lemma 1.5** [9]. Let  $X$  be a commutative  $BE$ -algebra. Then for any  $x, y, a \in X$

- (1)  $y * z \leq (z * x) * (y * x)$ ,  
 (2)  $(x * y) \vee a \leq (x \vee a) * (y \vee a)$ .

For any non-empty subset  $A$  of a  $BE$ -algebra  $X$ , the *dual annihilator* [9] of  $A$  is defined as  $A^+ = \{x \in X \mid x \vee a = 1 \text{ for all } a \in A\}$ . In a commutative  $BE$ -algebra  $X$ , the set  $A^+$  forms a filter of  $X$  such that  $A \cap A^+ = \{1\}$ . In case of  $A = \{a\}$ , we have  $(a)^+ = \{x \in X \mid a \vee x = 1\}$ . For  $a \in X$ , the set  $(a)^+$  is called the *dual annulet* of  $a$ . Clearly  $X^+ = \{1\}$  and  $\{1\}^+ = X$ .

**Proposition 1.6** [9]. Let  $X$  be a commutative  $BE$ -algebra and  $\emptyset \neq A, B \subseteq X$ . Then

- (1) if  $A \subseteq B$ , then  $B^+ \subseteq A^+$ ,  
 (2)  $A \subseteq A^{++}$ ,  
 (3)  $A^+ = A^{+++}$ .

**Proposition 1.7** [9]. Let  $F$  and  $G$  be two filters of a commutative  $BE$ -algebra  $X$ . Then

- (1)  $F \cap G = \{1\}$  if and only if  $F \subseteq G^+$ ,
- (2)  $(F \vee G)^+ = F^+ \cap G^+$ ,
- (3)  $(F \cap G)^{++} = F^{++} \cap G^{++}$ .

**Proposition 1.8** [9]. *Let  $X$  be a commutative  $BE$ -algebra and  $a, b \in X$ . Then we have*

- (1)  $\langle a \rangle \subseteq (a)^{++}$ ,
- (2)  $a \leq b$  implies  $(a)^+ \subseteq (b)^+$ ,
- (3)  $a \in (b)^{++}$  implies  $(b)^+ \subseteq (a)^+$ .

A filter  $F$  of a commutative  $BE$ -algebra  $X$  is called a *dual annihilator filter* [9] if  $F = F^{++}$ . A filter  $F$  of a commutative  $BE$ -algebra  $X$  is called a *regular filter* [10] if  $(x)^{++} \subseteq F$  whenever  $x \in F$ . A filter  $F$  of a commutative  $BE$ -algebra  $X$  is called an *O-filter* [11] if  $F = O(S)$  for some  $\vee$ -closed subset  $S$  of  $X$ , where  $O(S) = \{x \in X \mid x \vee s = 1 \text{ for some } s \in S\}$ . Every O-filter of a commutative  $BE$ -algebra is a regular filter.

## 2. $\sigma$ -FILTERS OF $BE$ -ALGEBRAS

In this section, the concept of  $\sigma$ -filters is introduced in commutative  $BE$ -algebras. Some properties of  $\sigma$ -filters are proved. A set of equivalent conditions is given for every prime filter of a commutative  $BE$ -algebra to become a  $\sigma$ -filter. Interconnections among  $\sigma$ -filters, regular filters, O-filters of commutative  $BE$ -algebras are established.

**Lemma 2.1.** *Let  $X$  be a commutative  $BE$ -algebra. For any  $x, y \in X$ , we have*

- (1)  $(x)^+ \cap (x * y)^+ \subseteq (y)^+$ ,
- (2)  $(x \vee y)^{++} = (x)^{++} \cap (y)^{++}$ ,
- (3)  $(x)^+ \cap (y)^+ = \{1\}$  if and only if  $(x)^+ \subseteq (y)^{++}$ ,
- (4)  $x \in (y)^+$  if and only if  $(x)^{++} \subseteq (y)^+$ .

**Proof.** (1) Let  $a \in (x)^+ \cap (x * y)^+$ . Then  $x \vee a = 1$  and  $(x * y) \vee a = 1$ . Hence

$$\begin{aligned}
 1 &= (x * y) \vee a \\
 &\leq (x \vee a) * (y \vee a) && \text{by Lemma 1.5(2)} \\
 &= 1 * (y \vee a) \\
 &= y \vee a
 \end{aligned}$$

which means  $y \vee a = 1$ . Hence  $a \in (y)^+$ . Therefore  $(x)^+ \cap (x * y) \subseteq (y)^+$ .

(2) Let  $x, y \in X$ . Since  $x, y \leq x \vee y$ , we get  $(x)^+, (y)^+ \subseteq (x \vee y)^+$ . Hence  $(x \vee y)^{++} \subseteq (x)^{++}, (y)^{++}$ . Thus  $(x \vee y)^{++} \subseteq (x)^{++} \cap (y)^{++}$ . Conversely, let  $a \in (x)^{++} \cap (y)^{++}$ . Suppose  $b \in (x \vee y)^+$  be an arbitrary element. Since  $b \in (x \vee y)^+$ , we get

$$\begin{aligned} b \vee (x \vee y) = 1 &\Rightarrow b \vee x \in (y)^+ \\ &\Rightarrow a \vee b \vee x = 1 \quad \text{since } a \in (y)^{++} \\ &\Rightarrow a \vee b \in (x)^+ \\ &\Rightarrow a \vee (a \vee b) = 1 \quad \text{since } a \in (x)^{++} \\ &\Rightarrow a \vee b = 1 \quad \text{for all } b \in (x \vee y)^+ \end{aligned}$$

which means that  $a \in (x \vee y)^{++}$ . Therefore  $(x)^{++} \cap (y)^{++} \subseteq (x \vee y)^{++}$ .

(3) Let  $x, y \in X$ . Assume that  $(x)^+ \cap (y)^+ = \{1\}$ . Let  $a \in (x)^+$ . Let  $b \in (y)^+$  be any element. Then, we get that  $a \vee b \in (x)^+ \cap (y)^+ = \{1\}$ . Hence  $a \in (b)^+$  for all  $b \in (y)^+$ . Therefore  $a \in (y)^{++}$ , which gives that  $(x)^+ \subseteq (y)^{++}$ . Conversely, suppose that  $(x)^+ \subseteq (y)^{++}$ . Then  $(x)^+ \cap (y)^+ \subseteq (y)^{++} \cap (y)^+ = \{1\}$ . Therefore  $(x)^+ \cap (y)^+ = \{1\}$ .

(4) Let  $x, y \in X$ . Suppose  $x \in (y)^+$ . Then  $x \vee y = 1$ . Hence  $(x)^{++} \cap (y)^{++} = (x \vee y)^{++} = (1)^{++} = \{1\}$ . Thus by (3), we get  $(x)^{++} \subseteq (y)^{+++} = (y)^+$ . Converse is clear. ■

**Definition 2.2.** For any prime filter  $P$  of a commutative  $BE$ -algebra  $X$ , define  $O(P) = \{x \in X \mid (x)^+ \not\subseteq P\}$ .

**Proposition 2.3.** For any prime filter  $P$  of a commutative  $BE$ -algebra  $X$ , the set  $O(P)$  is a filter of  $X$  such that  $O(P) \subseteq P$ .

**Proof.** Clearly  $1 \in O(P)$ . Suppose  $x, x * y \in O(P)$ . Then  $(x)^+ \not\subseteq P$  and  $(x * y)^+ \not\subseteq P$ . Since  $P$  is prime, we get  $(x)^+ \cap (x * y)^+ \not\subseteq P$ . By Lemma 2.1(1), we get  $(y)^+ \not\subseteq P$ . Hence  $y \in O(P)$ . Therefore  $O(P)$  is a filter of  $X$ . Again, let  $x \in O(P)$ . Then  $(x)^+ \not\subseteq P$ . Then there exists  $y \in (x)^+$  such that  $y \notin P$ . Since  $y \in (x)^+$ , we get  $x \vee y = 1$ . Hence  $(x)^{++} \cap (y)^{++} = (x \vee y)^{++} = \{1\}^{++} = \{1\} \subseteq P$ . Since  $P$  is prime, we get  $(x)^{++} \subseteq P$  or  $(y)^{++} \subseteq P$ . Suppose  $(y)^{++} \subseteq P$ . Since  $y \in (y)^{++}$ , we get  $y \in P$  which is a contradiction. Hence  $(x)^{++} \subseteq P$ , which means  $x \in P$ . Therefore  $O(P) \subseteq P$ . ■

**Definition 2.4.** Let  $X$  be a commutative  $BE$ -algebra. For any filter  $F$  of  $X$ , define

$$\sigma(F) = \{x \in X \mid (x)^+ \vee F = X\}.$$

Clearly  $\sigma(X) = X$ . For  $F = \{1\}$ , obviously we get  $\sigma(\{1\}) = \{1\}$ .

**Lemma 2.5.** *For any filter  $F$  of a commutative BE-algebra  $X$ ,  $\sigma(F)$  is a filter of  $X$ .*

**Proof.** Clearly  $1 \in \sigma(F)$ . Let  $x, x * y \in \sigma(F)$ . Then  $(x)^+ \vee F = X$  and  $(x * y)^+ \vee F = X$ . Hence

$$\begin{aligned} X &= X \cap X \\ &= \{(x)^+ \vee F\} \cap \{(x * y)^+ \vee F\} \\ &= \{(x)^+ \cap (x * y)^+\} \vee F \\ &\subseteq (y)^+ \vee F. \end{aligned}$$

which gives  $(y)^+ \vee F = X$ . Hence  $y \in \sigma(F)$ . Therefore  $\sigma(F)$  is a filter of  $X$ . ■

In the following result, some elementary properties of  $\sigma(F)$  are derived.

**Lemma 2.6.** *For any two filters  $F, G$  of a commutative BE-algebra  $X$ , we have*

- (1)  $\sigma(F) \subseteq F$ ,
- (2)  $F \subseteq G$  implies  $\sigma(F) \subseteq \sigma(G)$ ,
- (3)  $\sigma(F \cap G) = \sigma(F) \cap \sigma(G)$ ,
- (4)  $\sigma(F) \vee \sigma(G) \subseteq \sigma(F \vee G)$ .

**Proof.** (1) Let  $x \in \sigma(F)$ . Then  $(x)^+ \vee F = X$ . Hence  $a * (b * x) = 1$  for some  $a \in (x)^+$  and  $b \in F$ . Since  $a \in (x)^+$ , we get  $(a * x) * x = a \vee x = 1$ . Since  $X$  is commutative, we get  $1 = a * (b * x) = b * (a * x) \leq ((a * x) * x) * (b * x) = 1 * (b * x) = b * x$ . Hence  $b * x = 1$ , which gives  $b \leq x$ . Since  $b \in F$  and  $F$  is a filter, it concludes that  $x \in F$ . Therefore  $\sigma(F) \subseteq F$ .

(2) Suppose  $F \subseteq G$ . Let  $x \in \sigma(F)$ . Then  $X = (x)^+ \vee F \subseteq (x)^+ \vee G$ . Therefore  $x \in \sigma(G)$ .

(3) Clearly  $\sigma(F \cap G) \subseteq \sigma(F) \cap \sigma(G)$ . Conversely, let  $x \in \sigma(F) \cap \sigma(G)$ . Then  $(x)^+ \vee F = (x)^+ \vee G = X$ . Now  $(x)^+ \vee (F \cap G) = \{(x)^+ \vee F\} \cap \{(x)^+ \vee G\} = X \cap X = X$ . Hence  $x \in \sigma(F \cap G)$ . Thus  $\sigma(F) \cap \sigma(G) \subseteq \sigma(F \cap G)$ . Therefore  $\sigma(F \cap G) = \sigma(F) \cap \sigma(G)$ .

(4) By (2), it is obvious. ■

**Proposition 2.7.** *Let  $P$  be a proper filter of a commutative BE-algebra  $X$ . Then*

- (1) if  $P$  is prime, then  $\sigma(P) \subseteq O(P)$ ,
- (2) if  $P$  is maximal, then  $\sigma(P) = O(P)$ .

**Proof.** (1) Let  $x \in \sigma(P)$ . Then  $(x)^+ \vee P = X$ . Suppose that  $(x)^+ \subseteq P$ . Then we get  $P = X$ , which is a contradiction. Hence  $(x)^+ \not\subseteq P$ . Thus  $x \in O(P)$ . Therefore  $\sigma(P) \subseteq O(P)$ .

(2) Since every maximal filter is prime, we get  $\sigma(P) \subseteq O(P)$ . Conversely, let  $x \in O(P)$ . Then  $a \vee x = 1$  for some  $a \notin P$ . Thus there exists  $a \in (x)^+$  and  $a \notin P$ . Hence  $(x)^+ \not\subseteq P$ . Since  $P$  is maximal, we get  $(x)^+ \vee P = X$ . Thus  $x \in \sigma(P)$ . Therefore  $\sigma(P) = O(P)$ . ■

**Definition 2.8.** A filter  $F$  of a  $BE$ -algebra  $X$  is called a  $\sigma$ -filter if  $F = \sigma(F)$ .

Clearly the improper filters  $\{1\}$  and  $X$  are trivial  $\sigma$ -filters of  $X$ . In the following, we observe a non-trivial example for  $\sigma$ -filters of a  $BE$ -algebra.

**Example 2.9.** Let  $X = \{a, b, c, d, 1\}$  be a set. Define a binary operation  $*$  on  $X$  as

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	1	d
b	1	c	1	c	d
c	1	b	b	1	d
d	1	a	b	c	1

$\vee$	1	a	b	c	d
1	1	1	1	1	1
a	1	a	b	c	1
b	1	b	b	1	1
c	1	c	1	c	1
d	1	1	1	c	1

Clearly  $(X, *, \vee, 1)$  is a commutative  $BE$ -algebra. Consider the filter  $F = \{1, a, b, c\}$ . It can be easily verified that  $(a)^+ = \{1, d\}$ ,  $(b)^+ = \{1, c, d\}$ ,  $(c)^+ = \{1, b, d\}$  and  $(d)^+ = \{1, a, b, c\}$ . Clearly  $(1)^+ \vee F = X$ . Observe that  $(a)^+ \vee F = (b)^+ \vee F = (c)^+ \vee F = X$ . Thus  $\sigma(F) = \{1, a, b, c\} = F$ . Therefore  $F$  is a  $\sigma$ -filter of  $X$ .

It is observed that a proper  $\sigma$ -filter of a commutative  $BE$ -algebra contains no dual dense elements (an element  $x$  of a commutative  $BE$ -algebra is called *dual dense* if  $(x)^+ = \{1\}$ ) and the converse is not true. For this, consider the following example.

**Example 2.10.** Let  $X = \{1, a, b, c, d\}$  be a set. Define a binary operation  $*$  on  $X$  as

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	a	c	d
b	1	1	1	c	d
c	1	a	b	1	d
d	1	a	b	c	1

$*$	1	a	b	c	d
1	1	1	1	1	1
a	1	a	a	1	1
b	1	a	b	1	1
c	1	1	1	c	1
d	1	1	1	1	d

Clearly  $(X, *, \vee, 1)$  is a commutative  $BE$ -algebra. Now  $(a)^+ = \{1, c, d\}$ ;  $(b)^+ = \{1, c, d\}$ ;  $(c)^+ = \{1, a, b, d\}$  and  $(d)^+ = \{1, a, b, d\}$ . Consider the filter  $F = \{1, d\}$  of  $X$  which is not containing dual dense elements. Hence  $(a)^+ \vee F = (b)^+ \vee F = \{1, c, d\}$ ,  $(c)^+ \vee F = F$  and  $(d)^+ \vee F = F$ . Thus  $\sigma(F) = \{1\}$ . Therefore  $F$  is not a  $\sigma$ -filter of  $X$ .

**Theorem 2.11.** *Following assertions are equivalent in a commutative BE-algebra  $X$ :*

- (1) *every filter is a  $\sigma$ -filter;*
- (2) *every prime filter is a  $\sigma$ -filter;*
- (3) *for every prime filter  $P$ ,  $O(P) = P$ .*

**Proof.** (1)  $\Rightarrow$  (2): It is clear.

(2)  $\Rightarrow$  (3): Assume that every prime filter is a  $\sigma$ -filter. Let  $P$  be a prime filter of  $X$ . Since  $P$  is proper, there exists  $c \in X$  such that  $c \notin P$ . Since by (2),  $P$  is a  $\sigma$ -filter of  $X$ , we have  $\sigma(P) = P$ . Clearly  $O(P) \subseteq P$ . Conversely, let  $x \in P = \sigma(P)$ . Then  $(x)^+ \vee P = X$ . Since  $c \in X$ , we get  $c \in (x)^+ \vee P$ . Then  $a * (b * c) = 1$  for some  $a \in (x)^+$  and  $b \in P$ . Hence  $a \leq b * c$ . Suppose  $a \in P$ . Then  $b * c \in P$ . Since  $b \in P$ , we get  $c \in P$ , which is a contradiction. Thus  $a \notin P$ . Hence  $a \vee x = 1$  for some  $a \notin P$ . Therefore  $x \in O(P)$ , which gives that  $P = O(P)$ .

(3)  $\Rightarrow$  (1): Assume that  $O(P) = P$  for every prime filter of  $X$ . Let  $F$  be an arbitrary filter of  $X$ . By Lemma 2.6(1),  $\sigma(F) \subseteq F$ . Conversely, let  $x \in F$ . Suppose  $(x)^+ \vee F \neq X$ . Then there exists a maximal filter  $P$  such that  $(x)^+ \vee F \subseteq P$ . Since every maximal filter is prime, we get that  $P$  is prime. Hence  $(x)^+ \subseteq P$  and  $F \subseteq P$ . Since  $(x)^+ \subseteq P$ , we get that  $x \notin O(P) = P$ . Since  $x \in F$ , we get  $x \in P$  which is a contradiction. Hence  $(x)^+ \vee F = X$ . Therefore  $F$  is a  $\sigma$ -filter of  $X$ . ■

In [10], the class of all regular filters of a commutative BE-algebra  $X$  is characterized in terms of dual annihilators. In the following theorem, it is proved that the class of all regular filters of  $X$  contains properly the class of all  $\sigma$ -filters of  $X$ .

**Proposition 2.12.** *Every  $\sigma$ -filter of a commutative BE-algebra is a regular filter.*

**Proof.** Let  $F$  be a  $\sigma$ -filter of a commutative BE-algebra  $X$ . Then  $\sigma(F) = F$ . Let  $x \in F$ . Then  $(x)^+ \vee F = X$ . Now, let  $t \in (x)^{++}$ . Then, by Proposition 1.8(3),  $(x)^+ \subseteq (t)^+$ . Hence  $X = (x)^+ \vee F \subseteq (t)^+ \vee F$ . Thus  $t \in \sigma(F) = F$ . Thus  $(x)^{++} \subseteq F$ . Therefore  $F$  is a regular filter of  $X$ . ■

The converse of the above proposition is not true, i.e., every regular filter of a commutative BE-algebra need not be a  $\sigma$ -filter. Indeed, consider Example 2.9. Here,  $F = \{1, d\}$  is clearly a regular filter, because  $(d)^{++} \subseteq F$ . But  $F$  is not a  $\sigma$ -filter of  $X$ , because of  $(d)^+ \vee F \neq X$ . However, some equivalent conditions are given for every regular filter of a commutative BE-algebra to become a  $\sigma$ -filter.

**Theorem 2.13.** *Following assertions are equivalent in a commutative BE-algebra  $X$ :*



- (1) every regular filter is a  $\sigma$ -filter;
- (2) every dual annihilator filter is a  $\sigma$ -filter;
- (3) for each  $x \in X$ ,  $(x)^{++}$  is a  $\sigma$ -filter;
- (4) for each  $x \in X$ ,  $(x)^+ \vee (x)^{++} = X$ .

**Proof.** (1)  $\Rightarrow$  (2): Since every dual annihilator filter is a regular filter, it is clear.

(2)  $\Rightarrow$  (3): Since each  $(x)^{++}$  is a dual annihilator filter, it is clear.

(3)  $\Rightarrow$  (4): Assume the statement (3). Let  $x \in X$ . Since  $(x)^{++}$  is a  $\sigma$ -filter of  $X$ , we get  $(x)^{++} = \sigma((x)^{++})$ . Clearly  $x \in (x)^{++} = \sigma((x)^{++})$ . Hence  $(x)^+ \vee (x)^{++} = X$ .

(4)  $\Rightarrow$  (1): Assume that  $(x)^+ \vee (x)^{++} = X$  for each  $x \in X$ . Let  $F$  be a regular filter of  $X$ . Clearly  $\sigma(F) \subseteq F$ . Conversely, let  $x \in F$ . Since  $F$  is a regular filter, we get  $(x)^{++} \subseteq F$ . Hence  $X = (x)^+ \vee (x)^{++} \subseteq (x)^+ \vee F$ . Thus  $x \in \sigma(F)$ . Therefore  $F$  is a  $\sigma$ -filter of  $X$ . ■

Recall that a filter  $F$  of a commutative  $BE$ -algebra  $X$  is called an O-filter if  $F = O(S)$  for some  $\vee$ -closed subset  $S$  of  $X$ . In [11], authors studied the properties of O-filters and proved that every O-filter of a self-distributive and commutative  $BE$ -algebra is the intersection of all minimal prime filters containing it. In the following result, it is proved that the class of all  $\sigma$ -filters of a commutative  $BE$ -algebra  $X$  is properly contained in the class of all O-filters of  $X$ .

**Theorem 2.14.** *Suppose  $X$  is a commutative  $BE$ -algebra with a dual-dense element (i.e.,  $(x)^+ = \{1\}$ ). Then every  $\sigma$ -filter of  $X$  is an O-filter.*

**Proof.** Let  $F$  be a  $\sigma$ -filter of  $X$ . Then  $\sigma(F) = F$ . Consider the set  $S = \{ x \in X \mid (x)^{++} \vee F = X \}$ . It can be easily verified, by using Lemma 2.1(2), that  $S$  is a  $\vee$ -closed subset of  $X$ . We now show that  $F = O(S)$ . Let  $x \in O(S)$ . Then  $x \vee y = 1$  for some  $y \in S$ . Now

$$\begin{aligned}
 x \vee y = 1 &\Rightarrow y \in (x)^+ \\
 &\Rightarrow (y)^{++} \subseteq (x)^+ && \text{by Lemma 2.1(4)} \\
 &\Rightarrow X = (y)^{++} \vee F \subseteq (x)^+ \vee F && \text{since } y \in S \\
 &\Rightarrow x \in \sigma(F) = F && \text{since } F \text{ is a } \sigma\text{-filter}
 \end{aligned}$$

which concludes that  $O(S) \subseteq F$ . Conversely, let  $x \in F = \sigma(F)$  and  $d$  a dual-dense element of  $X$ . Then  $(x)^+ \vee \sigma(F) = X$ . Therefore  $d \in (x)^+ \vee \sigma(F)$ . Hence  $a * (b * d) = 1$  for some  $a \in (x)^+$  and  $b \in \sigma(F)$ . Thus  $a \vee x = 1$  and  $(b)^+ \vee F = X$ . Now

$$\begin{aligned}
 a * (b * d) = 1 &\Rightarrow a \leq b * d \\
 &\Rightarrow (a)^+ \subseteq (b * d)^+
 \end{aligned}$$

$$\begin{aligned}
&\Rightarrow (a)^+ \cap (b)^+ \subseteq (b)^+ \cap (b * d)^+ \\
&\Rightarrow (a)^+ \cap (b)^+ \subseteq (d)^+ = \{1\} \quad \text{by Lemma 2.1(1)} \\
&\Rightarrow (b)^+ \subseteq (a)^{++} \quad \text{by Lemma 2.1(3)} \\
&\Rightarrow X = (b)^+ \vee F \subseteq (a)^{++} \vee F \quad \text{since } b \in \sigma(F) \\
&\Rightarrow a \in S \text{ and } a \vee x = 1 \\
&\Rightarrow x \in O(S)
\end{aligned}$$

which gives  $F = \sigma(F) \subseteq O(S)$ . Hence  $F = O(S)$ . Therefore  $F$  is an  $O$ -filter of  $X$ . ■

The converse of the above theorem is not true, i.e., every  $O$ -filter of a commutative  $BE$ -algebra need not be a  $\sigma$ -filter. For, consider the following example.

**Example 2.15.** Let  $X = \{1, a, b, c\}$  be a set. Define a binary operation  $*$  on  $X$  as

$*$	$1$	$a$	$b$	$c$
$1$	$1$	$a$	$b$	$c$
$a$	$1$	$1$	$a$	$c$
$b$	$1$	$1$	$1$	$c$
$c$	$1$	$a$	$b$	$1$

$\vee$	$1$	$a$	$b$	$c$
$1$	$1$	$1$	$1$	$1$
$a$	$1$	$a$	$a$	$1$
$b$	$1$	$a$	$b$	$1$
$c$	$1$	$1$	$1$	$c$

It can be routinely verified that  $(X, *, \vee, 1)$  is a commutative  $BE$ -algebra. Observe that  $(a)^+ = (b)^+ = \{1, c\}$ , and  $(c)^+ = \{1, a, b\}$ . Consider the filter  $F = \{1, c\}$  of  $X$ . Clearly  $S = \{a, b\}$  is a  $\vee$ -closed subset of  $X$ . It is easy to observe that  $F = O(S)$ . Hence  $F$  is an  $O$ -filter of  $X$ . Now  $\sigma(F) = \{1\} \subset F$ . Therefore  $F$  is not a  $\sigma$ -filter of  $X$ .

**Lemma 2.16.** *In a commutative  $BE$ -algebra, every dual annulet is an  $O$ -filter.*

**Proof.** Let  $X$  be a commutative  $BE$ -algebra and  $a \in X$ . Consider  $[a] = \{x \in X \mid x \leq a\}$ . Let  $x, y \in [a]$ . Then  $x \leq a$  and  $y \leq a$ . Since  $X$  is commutative, it is partially ordered. Hence  $x \vee y \leq a$ , which gives that  $x \vee y \in [a]$ . Therefore  $[a]$  is a  $\vee$ -closed subset of  $X$ . We now show that  $(a)^+ = O([a])$ . Let  $x \in (a)^+$ . Then  $a \vee x = 1$  and  $a \in [a]$ . Hence  $x \in O([a])$ , which gives that  $(a)^+ \subseteq O([a])$ . Conversely, let  $x \in O([a])$ . Then  $x \vee y = 1$  for some  $y \in [a]$ . Since  $y \in [a]$ , we get  $y \leq a$ . Hence  $1 = x \vee y \leq x \vee a$ . Thus  $x \in (a)^+$ . Hence  $O([a]) \subseteq (a)^+$ . Therefore  $(a)^+$  is an  $O$ -filter of  $X$ . ■

**Theorem 2.17.** *Following assertions are equivalent in a commutative  $BE$ -algebra  $X$ :*

- (1) *every  $O$ -filter is a  $\sigma$ -filter;*
- (2) *each dual annulet is a  $\sigma$ -filter;*

(3) for any  $x, y \in X$ ,  $x \vee y = 1$  implies  $(x)^+ \vee (y)^+ = X$ .

**Proof.** (1)  $\Rightarrow$  (2): Since each dual annulet is an  $O$ -filter, it is clear.

(2)  $\Rightarrow$  (3): Assume that each dual annulet is a  $\sigma$ -filter of  $X$ . Let  $x, y \in X$  be such that  $x \vee y = 1$ . Hence  $x \in (y)^+$ . By (2), we get that  $(y)^+$  is a  $\sigma$ -filter of  $X$ . Hence  $x \in (y)^+ = \sigma((y)^+)$ . Thus we get  $(x)^+ \vee (y)^+ = X$ . Therefore, condition (3) is proved.

(3)  $\Rightarrow$  (1): Assume that condition (3) holds. Let  $F$  be an  $O$ -filter of  $X$ . Then  $F = O(S)$  for some  $\vee$ -closed subset  $S$  of  $X$ . Clearly  $\sigma(F) \subseteq F$ . We claim that  $O(S) \subseteq \sigma(F)$ . Now

$$\begin{aligned} x \in O(S) &\Rightarrow x \vee y = 1 \text{ for some } y \in S \\ &\Rightarrow (x)^+ \vee (y)^+ = X && \text{by (3)} \\ &\Rightarrow X = (x)^+ \vee (y)^+ \subseteq (x)^+ \vee O(S) && \text{since } y \in S \\ &\Rightarrow x \in \sigma(O(S)) = \sigma(F) \end{aligned}$$

Hence  $O(S) \subseteq \sigma(F)$ , which gives  $F = O(S) = \sigma(F)$ . Therefore  $F$  is a  $\sigma$ -filter of  $X$ . ■

**Theorem 2.18.** Let  $P$  be a prime filter of a commutative  $BE$ -algebra  $X$  such that  $P = O(P)$ . If  $X$  satisfies any one assertions of the above theorem, then  $P$  is a  $\sigma$ -filter.

**Proof.** Assume that  $X$  satisfies condition (3) of the above theorem. Let  $P$  be a prime filter of  $X$  such that  $P = O(P)$ . By Proposition 2.7(1), we have  $\sigma(P) \subseteq O(P) = P$ . Conversely, let  $x \in O(P)$ . Then there exists  $y \notin P$  such that  $x \vee y = 1$ . Since  $y \notin O(P)$ , we get  $(y)^+ \subseteq P$ . By (3) of the above theorem, we get that  $(x)^+ \vee (y)^+ = X$ . Hence  $X = (x)^+ \vee (y)^+ \subseteq (x)^+ \vee P$ . Thus  $x \in \sigma(P)$ . Hence  $P$  is a  $\sigma$ -filter of  $X$ . ■

Let us denote by  $\mu$  the set of all maximal filters of a  $BE$ -algebra  $X$ . For any filter  $F$  of a  $BE$ -algebra  $X$ , we also denote  $\mu(F) = \{M \in \mu \mid F \subseteq M\}$ . Since every maximal filter of a commutative  $BE$ -algebra is prime, by Proposition 2.3, we conclude that  $O(M)$  is a filter such that  $O(M) \subseteq M$  for every  $M \in \mu$ . Then we have the following result.

**Theorem 2.19.** For any filter  $F$  of a commutative  $BE$ -algebra  $X$ ,  $\sigma(F) = \bigcap_{M \in \mu(F)} O(M)$ .

**Proof.** Let  $x \in \sigma(F)$  and  $F \subseteq M$  where  $M \in \mu$ . Then  $X = (x)^+ \vee F \subseteq (x)^+ \vee M$ . Suppose  $(x)^+ \subseteq M$ , then  $M = X$ , which is a contradiction. Hence  $(x)^+ \not\subseteq M$ . Thus  $x \in O(M)$  for all  $M \in \mu(F)$ . Therefore  $\sigma(F) \subseteq \bigcap_{M \in \mu(F)} O(M)$ . Conversely, let  $x \in \bigcap_{M \in \mu(F)} O(M)$ . Then  $x \in O(M)$  for all  $M \in \mu(F)$ . Suppose

$(x)^+ \vee F \neq X$ . Then there exists a maximal filter  $M_0$  such that  $(x)^+ \vee F \subseteq M_0$ . Hence  $(x)^+ \subseteq M_0$  and  $F \subseteq M_0$ . Since  $F \subseteq M$ , by hypothesis, we get  $x \in O(M_0)$ . Hence  $(x)^+ \not\subseteq M_0$ , which is a contradiction. Therefore  $(x)^+ \vee F = X$ . Thus  $x \in \sigma(F)$ . Hence  $\bigcap_{M \in \mu(F)} O(M) \subseteq \sigma(F)$ . ■

From the above theorem, it can be easily observed that  $\sigma(F) \subseteq O(M)$  for every  $M \in \mu(F)$ . Now, in the following, a set of equivalent conditions is given for the class of all  $\sigma$ -filters of a commutative  $BE$ -algebra to become a sublattice to the lattice  $\mathcal{F}(X)$  of all filters of the commutative  $BE$ -algebra  $X$ .

**Theorem 2.20.** *The following assertions are equivalent in a commutative  $BE$ -algebra  $X$ :*

- (1) *for any  $M \in \mu$ ,  $O(M)$  is maximal;*
- (2) *for any  $F, G \in \mathcal{F}(X)$ ,  $F \vee G = X$  implies  $\sigma(F) \vee \sigma(G) = X$ ;*
- (3) *for any  $F, G \in \mathcal{F}(X)$ ,  $\sigma(F) \vee \sigma(G) = \sigma(F \vee G)$ ;*
- (4) *for any two distinct maximal filters  $M$  and  $N$ ,  $O(M) \vee O(N) = X$ ;*
- (5) *for any  $M \in \mu$ ,  $M$  is the unique member of  $\mu$  such that  $O(M) \subseteq M$ .*

**Proof.** (1)  $\Rightarrow$  (2): Assume the condition (1). Then clearly  $O(M) = M$  for all  $M \in \mu$ . Let  $F, G \in \mathcal{F}(X)$  be such that  $F \vee G = X$ . Suppose  $\sigma(F) \vee \sigma(G) \neq X$ . Then there exists a maximal filter  $M$  such that  $\sigma(F) \vee \sigma(G) \subseteq M$ . Hence  $\sigma(F) \subseteq M$  and  $\sigma(G) \subseteq M$ . Now

$$\begin{aligned} \sigma(F) \subseteq M &\Rightarrow \bigcap_{M_i \in \mu(F)} O(M_i) \subseteq M \\ &\Rightarrow O(M_i) \subseteq M \text{ for some } M_i \in \mu(F) \text{ (since } M \text{ is prime)} \\ &\Rightarrow M_i \subseteq M \quad \text{by condition (1)} \\ &\Rightarrow F \subseteq M \quad \text{since } F \subseteq M_i. \end{aligned}$$

Similarly, we can obtain that  $G \subseteq M$ . Hence  $X = F \vee G \subseteq M$ , which is a contradiction to the maximality of  $M$ . Therefore  $\sigma(F) \vee \sigma(G) = X$ .

(2)  $\Rightarrow$  (3): Assume the condition (2). Let  $F, G \in \mathcal{F}(X)$ . Clearly  $\sigma(F) \vee \sigma(G) \subseteq \sigma(F \vee G)$ . Let  $x \in \sigma(F \vee G)$ . Then  $\{(x)^+ \vee F\} \vee \{(x)^+ \vee G\} = (x)^+ \vee F \vee G = X$ . Hence by condition (2), we get  $\sigma((x)^+ \vee F) \vee \sigma((x)^+ \vee G) = X$ . Thus  $x \in \sigma((x)^+ \vee F) \vee \sigma((x)^+ \vee G)$ . Hence  $r * (s * x) = 1$  for some  $r \in \sigma((x)^+ \vee F)$  and  $s \in \sigma((x)^+ \vee G)$ . Now

$$\begin{aligned} r \in \sigma((x)^+ \vee F) &\Rightarrow (r)^+ \vee \{(x)^+ \vee F\} = X \\ &\Rightarrow X = \{(r)^+ \vee (x)^+\} \vee F \subseteq (r \vee x)^+ \vee F \\ &\Rightarrow (r \vee x)^+ \vee F = X \\ &\Rightarrow r \vee x \in \sigma(F). \end{aligned}$$

Similarly, we can get  $s \vee x \in \sigma(G)$ . Now, we have the following consequence:

$$\begin{aligned}
 r * (s * x) = 1 &\Rightarrow r \leq s * x \\
 &\Rightarrow r \vee x \leq (s * x) \vee x \leq (s \vee x) * (x \vee x) \\
 &\Rightarrow r \vee x \leq (s \vee x) * (x \vee x) \\
 &\Rightarrow r \vee x \leq (s \vee x) * x \\
 &\Rightarrow (r \vee x) * ((s \vee x) * x) = 1
 \end{aligned}$$

where  $r \vee x \in \sigma(F)$  and  $s \vee x \in \sigma(G)$ . Hence  $x \in \sigma(F) \vee \sigma(G)$ . Thus  $\sigma(F \vee G) \subseteq \sigma(F) \vee \sigma(G)$ . Therefore  $\sigma(F) \vee \sigma(G) = \sigma(F \vee G)$ .

(3)  $\Rightarrow$  (4): Assume the condition (3). Let  $M, N$  be two distinct maximal filters of  $X$ . Choose  $x \in M - N$  and  $y \in N - M$ . Since  $x \notin N$ , we get  $N \vee \langle x \rangle = X$ . Since  $y \notin M$ , we get  $M \vee \langle y \rangle = X$ . Now

$$\begin{aligned}
 X &= \sigma(X) \\
 &= \sigma(X \vee X) \\
 &= \sigma(\{N \vee \langle x \rangle\} \vee \{M \vee \langle y \rangle\}) \\
 &= \sigma(\{M \vee \langle x \rangle\} \vee \{N \vee \langle y \rangle\}) \\
 &= \sigma(M \vee N) && \text{since } x \in M \text{ and } y \in N \\
 &= \sigma(M) \vee \sigma(N) && \text{by condition (3)} \\
 &\subseteq O(M) \vee O(N) && \text{by Proposition 2.7(1).}
 \end{aligned}$$

Therefore  $O(M) \vee O(N) = X$ .

(4)  $\Rightarrow$  (5): Assume condition (4). Let  $M \in \mu$ . Suppose  $N \in \mu$  such that  $N \neq M$  and  $O(N) \subseteq M$ . Since  $O(M) \subseteq M$ , by hypothesis, we get  $X = O(M) \vee O(N) = M$ , which is a contradiction. Hence  $M$  is the unique maximal filter such that  $O(M) \subseteq M$ .

(5)  $\Rightarrow$  (1): Let  $M \in \mu$ . Suppose  $O(M)$  is not maximal. Let  $M_0$  be a maximal filter of  $X$  such that  $O(M) \subseteq M_0$ . We have always  $O(M_0) \subseteq M_0$ , which is a contradiction. ■

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### REFERENCES

- [1] S.S. Ahn, Y.H. Kim and J.M. Ko, *Filters in commutative BE-algebras*, Commun. Korean. Math. Soc. **27** (2) (2012) 233–242.  
<https://doi.org/10.4134/CKMS.2012.27.2.233>

- [2] A. Borumand Saeid, A. Rezaei and R.A. Borzooei, *Some types of filters in BE-algebras*, Math. Comput. Sci. **7** (2013) 341–352.  
<https://doi.org/10.1007/s11786-013-0157-6>
- [3] W.H. Cornish, *O-ideals, congruences, sheaf representation of distributive lattices*, Rev. Roum. Math. Pures et Appl. **22** (8) (1977) 1059–1067.
- [4] K. Iseki and S. Tanaka, *An introduction to the theory of BCK-algebras*, Math. Japon. **23** (1) (1979) 1–6.
- [5] H.S. Kim and Y.H. Kim, *On BE-algebras*, Sci. Math. Jpn. **66** (1) (2007) 113–116.
- [6] B.L. Meng, *On filters in BE-algebras*, Sci. Math. Japon. (2010) 105–111.  
<https://doi.org/10.32219/isms.71.2-201>
- [7] S. Rasouli, *Generalized co-annihilators in residuated lattices*, Annals of the University of Craiova, Mathematics and Computer Science Series **45** (2) (2019) 190–207.
- [8] M. Sambasiva Rao, *Prime filters of commutative BE-algebras*, J. Appl. Math. Inf. **33** (5–6) (2015) 579–591.  
<https://doi.org/10.14317/jami.2015.579>
- [9] V.V. Kumar and M.S. Rao, *Dual annihilator filters of commutative BE-algebras*, Asian-European J. Math. **10** (1) (2017) 1750013(11 pages).  
<https://doi.org/10.1142/s1793557117500139>
- [10] V.V. Kumar and M.S. Rao and S.K. Vali, *Regular filters of commutative BE-algebras*, TWMS J. Appl. & Engg. Math. **11** (4) (2021) 1023–1035.
- [11] V.V. Kumar and M.S. Rao, and S.K. Vali, *Quasi-complemented BE-algebras*, Discuss. Math. Gen. Algebra Appl. **41** (2021) 265–281.  
<https://doi.org/10.2307/1996101>
- [12] A. Walendziak, *On commutative BE-algebras*, Sci. Math. Japon. (2008) 585–588.  
<https://doi.org/10.32219/isms.69.2-281>

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