# $\sigma$-FILTERS OF COMMUTATIVE $\boldsymbol{B E}$-ALGEBRAS 

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#### Abstract

The concept of $\sigma$-filters is introduced in commutative $B E$-algebras and some properties of these classes of filters are studied. Some equivalent conditions are derived for every filter of a commutative $B E$-algebra to become a $\sigma$-filter. Some necessary and sufficient conditions are given for every regular filter of a commutative $B E$-algebra to become a $\sigma$-filter. A set of equivalent conditions is given for the class of all $\sigma$-filters of a commutative $B E$-algebra to become a sublattice to the lattice of all filters.


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## Introduction

The notion of $B E$-algebras was introduced and extensively studied by Kim and Kim in [5]. These classes of $B E$-algebras were introduced as a generalization of the class of $B C K$-algebras of Iseki and Tanaka [4]. Some properties of filters of $B E$-algebras were studied by Ahn and Kim in [1] and by Meng in [6]. In [12], Walendziak discussed some significant properties of commutative $B E$-algebras. He also investigated the relationship between $B E$-algebras, implicative algebras and $J$-algebras. In [6], Meng introduced the notion of prime filters in $B C K$-algebras, and then gave a description of the filter generated by a set, and obtained some of fundamental properties of prime filters. In [4], some properties of prime ideals are investigated in $B C K$-algebras. In [8], the author studied some properties of prime filters in $B E$-algebras. In this paper, the author extensively studied the algebraic as well as the topological properties of prime filters of commutative
$B E$-algebras. In [9], the authors introduced the notion of dual annihilators of commutative $B E$-algebra and studied extensively the properties of these dual annihilators. In 2020, the authors introduced the notions of regular filters [10] and O-filters [11] in commutative $B E$-algebras and the interconnection between those two special classes of filters is studied.

In this paper, the concept of $\sigma$-filters is introduced in commutative $B E$ algebras and their properties are studied analogous to that in a distributive lattice [3]. A set of equivalent conditions is given for every filter of a commutative $B E$-algebra to become a $\sigma$-filter. It is observed that every $\sigma$-filter of a commutative $B E$-algebra is a regular filter but not the converse in general. However, some equivalent conditions are proved for every regular filter of a commutative $B E$-algebra to become a $\sigma$-filter. It is also observed that every O-filter of a commutative $B E$-algebra is a $\sigma$-filter but not the converse in general. Some necessary and sufficient conditions are given for every $\sigma$-filter of a commutative $B E$-algebra to become an O-filter. Some equivalent conditions are given to prove that the class of all $\sigma$-filters of a commutative $B E$-algebra to become a sublattice to the lattice of all filters of a commutative $B E$-algebra.

## 1. Preliminaries

In this section, we present certain definitions and results which are taken mostly from the papers [1, 5, 9, 10], and [11] for the ready reference of the reader.
Definition 1.1 [5]. An algebra $(X, *, 1)$ of type $(2,0)$ is called a $B E$-algebra if it satisfies the following properties:
(1) $x * x=1$,
(2) $x * 1=1$,
(3) $1 * x=x$,
(4) $x *(y * z)=y *(x * z)$ for all $x, y, z \in X$.

A $B E$-algebra $X$ is called self-distributive if $x *(y * z)=(x * y) *(x * z)$ for all $x, y, z \in X$. A $B E$-algebra $X$ is called transitive if $y * z \leq(x * y) *(x * z)$ for all $x, y, z \in X$. A $B E$-algebra $X$ is called commutative if $(x * y) * y=(y * x) * x$ for all $x, y \in X$. Every commutative $B E$-algebra is transitive. For any $x, y \in X$, define $x \vee y=(y * x) * x$. If $X$ is commutative then $(X, \vee)$ is a semilattice [12]. We introduce a relation $\leq$ on a $B E$-algebra $X$ by $x \leq y$ if and only if $x * y=1$ for all $x, y \in X$. Clearly $\leq$ is reflexive. If $X$ is commutative, then $\leq$ is transitive, anti-symmetric and hence a partial order on $X$.

Theorem 1.2 [5]. Let $X$ be a transitive $B E$-algebra and $x, y, z \in X$. Then
(1) $1 \leq x$ implies $x=1$,
(2) $y \leq z$ implies $x * y \leq x * z$ and $z * x \leq y * x$.

Definition 1.3 [1]. A non-empty subset $F$ of a $B E$-algebra $X$ is called a filter of $X$ if, for all $x, y \in X$, it satisfies the following properties:
(1) $1 \in F$,
(2) $x \in F$ and $x * y \in F$ imply that $y \in F$.

For any non-empty subset $A$ of a transitive $B E$-algebra $X$, the set $\langle A\rangle=\{x \in$ $X \mid a_{1} *\left(a_{2} *\left(\cdots *\left(a_{n} * x\right) \cdots\right)\right)=1$ for some $\left.a_{1}, a_{2}, \ldots a_{n} \in A\right\}$ is the smallest filter containing $A$. For any $a \in X,\langle a\rangle=\left\{x \in X \mid a^{n} * x=1\right.$ for some $\left.n \in \mathbb{N}\right\}$, where $a^{n} * x=a *(a *(\cdots *(a * x) \cdots))$ with the repetition of $a$ is $n$ times, is called the principal filter generated $a$. Let $F$ be a filter of a transitive $B E$-algebra and $a \in X$, then $\langle F \cup\{a\}\rangle=\left\{x \in X \mid a^{n} * x=1\right.$ for some $\left.n \in \mathbb{N}\right\}$. A proper filter $P$ of a $B E$-algebra is called prime [8] if $F \cap G \subseteq P$ implies $F \in P$ or $G \in P$ for any two proper filters $F, G$ of $X$. A proper filter $P$ of a $B E$-algebra is called prime [8] if $\langle x\rangle \cap\langle y\rangle \subseteq P$ implies $x \in P$ or $y \in P$ for any $x, y \in X$. A proper filter $M$ of a transitive $B E$-algebra $X$ is called maximal if there exist no proper filters $Q$ such that $M \subset Q$. "Every maximal filter of a commutative $B E$-algebra is prime".

Theorem 1.4 [8]. Let $F$ and $G$ be two filters of a transitive BE-algebra X. Then

$$
F \vee G=\{x \in X \mid a *(b * x)=1 \text { for some } a \in F, b \in G\}
$$

is the supremum of $F$ and $G$. Hence the set $\mathcal{F}(X)$ of all filters of $X$ is a lattice.
Lemma 1.5 [9]. Let $X$ be a commutative BE-algebra. Then for any $x, y, a \in X$
(1) $y * z \leq(z * x) *(y * x)$,
(2) $(x * y) \vee a \leq(x \vee a) *(y \vee a)$.

For any non-empty subset $A$ of a $B E$-algebra $X$, the dual annihilator [9] of $A$ is defined as $A^{+}=\{x \in X \mid x \vee a=1$ for all $a \in A\}$. In a commutative $B E$-algebra $X$, the set $A^{+}$forms a filter of $X$ such that $A \cap A^{+}=\{1\}$. In case of $A=\{a\}$, we have $(a)^{+}=\{x \in X \mid a \vee x=1\}$. For $a \in X$, the set $(a)^{+}$is called the dual annulet of $a$. Clearly $X^{+}=\{1\}$ and $\{1\}^{+}=X$.

Proposition 1.6 [9]. Let $X$ be a commutative $B E$-algebra and $\emptyset \neq A, B \subseteq X$. Then
(1) if $A \subseteq B$, then $B^{+} \subseteq A^{+}$,
(2) $A \subseteq A^{++}$,
(3) $A^{+}=A^{+++}$.

Proposition 1.7 [9]. Let $F$ and $G$ be two filters of a commutative BE-algebra $X$. Then
(1) $F \cap G=\{1\}$ if and only if $F \subseteq G^{+}$,
(2) $(F \vee G)^{+}=F^{+} \cap G^{+}$,
(3) $(F \cap G)^{++}=F^{++} \cap G^{++}$.

Proposition 1.8 [9]. Let $X$ be a commutative $B E$-algebra and $a, b \in X$. Then we have
(1) $\langle a\rangle \subseteq(a)^{++}$,
(2) $a \leq b$ implies $(a)^{+} \subseteq(b)^{+}$,
(3) $a \in(b)^{++}$implies $(b)^{+} \subseteq(a)^{+}$.

A filter $F$ of a commutative $B E$-algebra $X$ is called a dual annihilator filter [9] if $F=F^{++}$. A filter $F$ of a commutative $B E$-algebra $X$ is called a regular filter [10] if $(x)^{++} \subseteq F$ whenever $x \in F$. A filter $F$ of a commutative $B E$-algebra $X$ is called an $O$-filter [11] if $F=O(S)$ for some $\vee$-closed subset $S$ of $X$, where $O(S)=\{x \in X \mid x \vee s=1$ for some $s \in S\}$. Every O-filter of a commutative $B E$-algebra is a regular filter.

## 2. $\sigma$-FILTERS OF $B E$-algebras

In this section, the concept of $\sigma$-filters is introduced in commutative $B E$-algebras. Some properties of $\sigma$-filters are proved. A set of equivalent conditions is given for every prime filter of a commutative $B E$-algebra to become a $\sigma$-filter. Interconnections among $\sigma$-filters, regular filters, O-filters of commutative $B E$-algebras are established.

Lemma 2.1. Let $X$ be a commutative $B E$-algebra. For any $x, y \in X$, we have
(1) $(x)^{+} \cap(x * y)^{+} \subseteq(y)^{+}$,
(2) $(x \vee y)^{++}=(x)^{++} \cap(y)^{++}$,
(3) $(x)^{+} \cap(y)^{+}=\{1\}$ if and only if $(x)^{+} \subseteq(y)^{++}$,
(4) $x \in(y)^{+}$if and only if $(x)^{++} \subseteq(y)^{+}$.

Proof. (1) Let $a \in(x)^{+} \cap(x * y)^{+}$. Then $x \vee a=1$ and $(x * y) \vee a=1$. Hence

$$
\begin{aligned}
1 & =(x * y) \vee a \\
& \leq(x \vee a) *(y \vee a) \quad \text { by Lemma } 1.5(2) \\
& =1 *(y \vee a) \\
& =y \vee a
\end{aligned}
$$

which means $y \vee a=1$. Hence $a \in(y)^{+}$. Therefore $(x)^{+} \cap(x * y) \subseteq(y)^{+}$.
(2) Let $x, y \in X$. Since $x, y \leq x \vee y$, we get $(x)^{+},(y)^{+} \subseteq(x \vee y)^{+}$. Hence $(x \vee y)^{++} \subseteq(x)^{++},(y)^{++}$. Thus $(x \vee y)^{++} \subseteq(x)^{++} \cap(y)^{++}$. Conversely, let $a \in$ $(x)^{++} \cap(y)^{++}$. Suppose $b \in(x \vee y)^{+}$be an arbitrary element. Since $b \in(x \vee y)^{+}$, we get

$$
\begin{aligned}
b \vee(x \vee y)=1 & \Rightarrow b \vee x \in(y)^{+} \\
& \Rightarrow a \vee b \vee x=1 \quad \text { since } a \in(y)^{++} \\
& \Rightarrow a \vee b \in(x)^{+} \\
& \Rightarrow a \vee(a \vee b)=1 \quad \text { since } a \in(x)^{++} \\
& \Rightarrow a \vee b=1 \text { for all } b \in(x \vee y)^{+}
\end{aligned}
$$

which means that $a \in(x \vee y)^{++}$. Therefore $(x)^{++} \cap(y)^{++} \subseteq(x \vee y)^{++}$.
(3) Let $x, y \in X$. Assume that $(x)^{+} \cap(y)^{+}=\{1\}$. Let $a \in(x)^{+}$. Let $b \in(y)^{+}$ be any element. Then, we get that $a \vee b \in(x)^{+} \cap(y)^{+}=\{1\}$. Hence $a \in(b)^{+}$for all $b \in(y)^{+}$. Therefore $a \in(y)^{++}$, which gives that $(x)^{+} \subseteq(y)^{++}$. Conversely, suppose that $(x)^{+} \subseteq(y)^{++}$. Then $(x)^{+} \cap(y)^{+} \subseteq(y)^{++} \cap(y)^{+}=\{1\}$. Therefore $(x)^{+} \cap(y)^{+}=\{1\}$.
(4) Let $x, y \in X$. Suppose $x \in(y)^{+}$. Then $x \vee y=1$. Hence $(x)^{++} \cap(y)^{++}=$ $(x \vee y)^{++}=(1)^{++}=\{1\}$. Thus by (3), we get $(x)^{++} \subseteq(y)^{+++}=(y)^{+}$. Converse is clear.

Definition 2.2. For any prime filter $P$ of a commutative $B E$-algebra $X$, define $O(P)=\left\{x \in X \mid(x)^{+} \nsubseteq P\right\}$.

Proposition 2.3. For any prime filter $P$ of a commutative $B E$-algebra $X$, the set $O(P)$ is a filter of $X$ such that $O(P) \subseteq P$.

Proof. Clearly $1 \in O(P)$. Suppose $x, x * y \in O(P)$. Then $(x)^{+} \nsubseteq P$ and $(x * y)^{+} \nsubseteq P$. Since $P$ is prime, we get $(x)^{+} \cap(x * y)^{+} \nsubseteq P$. By Lemma 2.1(1), we get $(y)^{+} \nsubseteq P$. Hence $y \in O(P)$. Therefore $O(P)$ is a filter of $X$. Again, let $x \in O(P)$. Then $(x)^{+} \nsubseteq P$. Then there exists $y \in(x)^{+}$such that $y \notin P$. Since $y \in(x)^{+}$, we get $x \vee y=1$. Hence $(x)^{++} \cap(y)^{++}=(x \vee y)^{++}=\{1\}^{++}=\{1\} \subseteq P$. Since $P$ is prime, we get $(x)^{++} \subseteq P$ or $(y)^{++} \subseteq P$. Suppose $(y)^{++} \subseteq P$. Since $y \in(y)^{++}$, we get $y \in P$ which is a contradiction. Hence $(x)^{++} \subseteq P$, which means $x \in P$. Therefore $O(P) \subseteq P$.

Definition 2.4. Let $X$ be a commutative $B E$-algebra. For any filter $F$ of $X$, define

$$
\sigma(F)=\left\{x \in X \mid(x)^{+} \vee F=X\right\} .
$$

Clearly $\sigma(X)=X$. For $F=\{1\}$, obviously we get $\sigma(\{1\})=\{1\}$.

Lemma 2.5. For any filter $F$ of a commutative $B E$-algebra $X, \sigma(F)$ is a filter of $X$.

Proof. Clearly $1 \in \sigma(F)$. Let $x, x * y \in \sigma(F)$. Then $(x)^{+} \vee F=X$ and $(x * y)^{+} \vee F=X$. Hence

$$
\begin{aligned}
X & =X \cap X \\
& =\left\{(x)^{+} \vee F\right\} \cap\left\{(x * y)^{+} \vee F\right\} \\
& =\left\{(x)^{+} \cap(x * y)^{+}\right\} \vee F \\
& \subseteq(y)^{+} \vee F
\end{aligned}
$$

which gives $(y)^{+} \vee F=X$. Hence $y \in \sigma(F)$. Therefore $\sigma(F)$ is a filter of $X$.
In the following result, some elementary properties of $\sigma(F)$ are derived.
Lemma 2.6. For any two filters $F, G$ of a commutative $B E$-algebra $X$, we have
(1) $\sigma(F) \subseteq F$,
(2) $F \subseteq G$ implies $\sigma(F) \subseteq \sigma(G)$,
(3) $\sigma(F \cap G)=\sigma(F) \cap \sigma(G)$,
(4) $\sigma(F) \vee \sigma(G) \subseteq \sigma(F \vee G)$.

Proof. (1) Let $x \in \sigma(F)$. Then $(x)^{+} \vee F=X$. Hence $a *(b * x)=1$ for some $a \in(x)^{+}$and $b \in F$. Since $a \in(x)^{+}$, we get $(a * x) * x=a \vee x=1$. Since $X$ is commutative, we get $1=a *(b * x)=b *(a * x) \leq((a * x) * x) *(b * x)=1 *(b * x)=b * x$. Hence $b * x=1$, which gives $b \leq x$. Since $b \in F$ and $F$ is a filter, it concludes that $x \in F$. Therefore $\sigma(F) \subseteq F$.
(2) Suppose $F \subseteq G$. Let $x \in \sigma(F)$. Then $X=(x)^{+} \vee F \subseteq(x)^{+} \vee G$. Therefore $x \in \sigma(G)$.
(3) Clearly $\sigma(F \cap G) \subseteq \sigma(F) \cap \sigma(G)$. Conversely, let $x \in \sigma(F) \cap \sigma(G)$. Then $(x)^{+} \vee F=(x)^{+} \vee G=X$. Now $(x)^{+} \vee(F \cap G)=\left\{(x)^{+} \vee F\right\} \cap\left\{(x)^{+} \vee G\right\}=$ $X \cap X=X$. Hence $x \in \sigma(F \cap G)$. Thus $\sigma(F) \cap \sigma(G) \subseteq \sigma(F \cap G)$. Therefore $\sigma(F \cap G)=\sigma(F) \cap \sigma(G)$.
(4) $\mathrm{By}(2)$, it is obvious.

Proposition 2.7. Let $P$ be a proper filter of a commutative $B E$-algebra $X$. Then
(1) if $P$ is prime, then $\sigma(P) \subseteq O(P)$,
(2) if $P$ is maximal, then $\sigma(P)=O(P)$.

Proof. (1) Let $x \in \sigma(P)$. Then $(x)^{+} \vee P=X$. Suppose that $(x)^{+} \subseteq P$. Then we get $P=X$, which is a contradiction. Hence $(x)^{+} \nsubseteq P$. Thus $x \in O(P)$. Therefore $\sigma(P) \subseteq O(P)$.
(2) Since every maximal filter is prime, we get $\sigma(P) \subseteq O(P)$. Conversely, let $x \in O(P)$. Then $a \vee x=1$ for some $a \notin P$. Thus there exists $a \in(x)^{+}$and $a \notin P$. Hence $(x)^{+} \nsubseteq P$. Since $P$ is maximal, we get $(x)^{+} \vee P=X$. Thus $x \in \sigma(P)$. Therefore $\sigma(P)=O(P)$.

Definition 2.8. A filter $F$ of a $B E$-algebra $X$ is called a $\sigma$ - filter if $F=\sigma(F)$.
Clearly the improper filters $\{1\}$ and $X$ are trivial $\sigma$-filters of $X$. In the following, we observe a non-trivial example for $\sigma$-filters of a $B E$-algebra.

Example 2.9. Let $X=\{a, b, c, d, 1\}$ be a set. Define a binary operation $*$ on $X$ as

| $*$ | 1 | $a$ | $b$ | $c$ | $d$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| 1 | 1 | $a$ | $b$ | $c$ | $d$ |
| $a$ | 1 | 1 | 1 | 1 | $d$ |
| $b$ | 1 | $c$ | 1 | $c$ | $d$ |
| $c$ | 1 | $b$ | $b$ | 1 | $d$ |
| $d$ | 1 | $a$ | $b$ | $c$ | 1 |


| $\vee$ | 1 | $a$ | $b$ | $c$ | $d$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | 1 | 1 | 1 | 1 |
| $a$ | 1 | $a$ | $b$ | $c$ | 1 |
| $b$ | 1 | $b$ | $b$ | 1 | 1 |
| $c$ | 1 | $c$ | 1 | $c$ | 1 |
| $d$ | 1 | 1 | 1 | $c$ | 1 |

Clearly $(X, *, \vee, 1)$ is a commutative $B E$-algebra. Consider the filter $F=\{1, a$, $b, c\}$. It can be easily verified that $(a)^{+}=\{1, d\},(b)^{+}=\{1, c, d\},(c)^{+}=\{1, b, d\}$ and $(d)^{+}=\{1, a, b, c\}$. Clearly $(1)^{+} \vee F=X$. Observe that $(a)^{+} \vee F=(b)^{+} \vee F=$ $(c)^{+} \vee F=X$. Thus $\sigma(F)=\{1, a, b, c\}=F$. Therefore $F$ is a $\sigma$-filter of $X$.

It is observed that a proper $\sigma$-filter of a commutative $B E$-algebra contains no dual dense elements (an element $x$ of a commutative $B E$-algebra is called dual dense if $\left.(x)^{+}=\{1\}\right)$ and the converse is not true. For this, consider the following example.

Example 2.10. Let $X=\{1, a, b, c, d\}$ be a set. Define a binary operation $*$ on $X$ as

| $*$ | 1 | $a$ | $b$ | $c$ | $d$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| 1 | 1 | $a$ | $b$ | $c$ | $d$ |
| $a$ | 1 | 1 | $a$ | $c$ | $d$ |
| $b$ | 1 | 1 | 1 | $c$ | $d$ |
| $c$ | 1 | $a$ | $b$ | 1 | $d$ |
| $d$ | 1 | $a$ | $b$ | $c$ | 1 |


| $*$ | 1 | $a$ | $b$ | $c$ | $d$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| 1 | 1 | 1 | 1 | 1 | 1 |
| $a$ | 1 | $a$ | $a$ | 1 | 1 |
| $b$ | 1 | $a$ | $b$ | 1 | 1 |
| $c$ | 1 | 1 | 1 | $c$ | 1 |
| $d$ | 1 | 1 | 1 | 1 | $d$ |

Clearly $(X, *, \vee, 1)$ is a commutative $B E$-algebra. Now $(a)^{+}=\{1, c, d\} ;(b)^{+}=$ $\{1, c, d\} ;(c)^{+}=\{1, a, b, d\}$ and $(d)^{+}=\{1, a, b, d\}$. Consider the filter $F=\{1, d\}$ of $X$ which is not containing dual dense elements. Hence $(a)^{+} \vee F=(b)^{+} \vee F=$ $\{1, c, d\},(c)^{+} \vee F=F$ and $(d)^{+} \vee F=F$. Thus $\sigma(F)=\{1\}$. Therefore $F$ is not a $\sigma$-filter of $X$.

Theorem 2.11. Following assertions are equivalent in a commutative BE-algebra $X$ :
(1) every filter is a $\sigma$-filter;
(2) every prime filter is a $\sigma$-filter;
(3) for every prime filter $P, O(P)=P$.

Proof. $(1) \Rightarrow(2)$ : It is clear.
$(2) \Rightarrow(3)$ : Assume that every prime filter is a $\sigma$-filter. Let $P$ be a prime filter of $X$. Since $P$ is proper, there exists $c \in X$ such that $c \notin P$. Since by (2), $P$ is a $\sigma$-filter of $X$, we have $\sigma(P)=P$. Clearly $O(P) \subseteq P$. Conversely, let $x \in P=\sigma(P)$. Then $(x)^{+} \vee P=X$. Since $c \in X$, we get $c \in(x)^{+} \vee P$. Then $a *(b * c)=1$ for some $a \in(x)^{+}$and $b \in P$. Hence $a \leq b * c$. Suppose $a \in P$. Then $b * c \in P$. Since $b \in P$, we get $c \in P$, which is a contradiction. Thus $a \notin P$. Hence $a \vee x=1$ for some $a \notin P$. Therefore $x \in O(P)$, which gives that $P=O(P)$.
$(3) \Rightarrow(1)$ : Assume that $O(P)=P$ for every prime filter of $X$. Let $F$ be an arbitrary filter of $X$. By Lemma $2.6(1), \sigma(F) \subseteq F$. Conversely, let $x \in F$. Suppose $(x)^{+} \vee F \neq X$. Then there exists a maximal filter $P$ such that $(x)^{+} \vee F \subseteq$ $P$. Since every maximal filter is prime, we get that $P$ is prime. Hence $(x)^{+} \subseteq P$ and $F \subseteq P$. Since $(x)^{+} \subseteq P$, we get that $x \notin O(P)=P$. Since $x \in F$, we get $x \in P$ which is a contradiction. Hence $(x)^{+} \vee F=X$. Therefore $F$ is a $\sigma$-filter of $X$.

In [10], the class of all regular filters of a commutative $B E$-algebra $X$ is characterized in terms of dual annihilators. In the following theorem, it is proved that the class of all regular filters of $X$ contains properly the class of all $\sigma$-filters of $X$.

Proposition 2.12. Every $\sigma$-filter of a commutative $B E$-algebra is a regular filter.
Proof. Let $F$ be a $\sigma$-filter of a commutative $B E$-algebra $X$. Then $\sigma(F)=F$. Let $x \in F$. Then $(x)^{+} \vee F=X$. Now, let $t \in(x)^{++}$. Then, by Proposition $1.8(3),(x)^{+} \subseteq(t)^{+}$. Hence $X=(x)^{+} \vee F \subseteq(t)^{+} \vee F$. Thus $t \in \sigma(F)=F$. Thus $(x)^{++} \subseteq F$. Therefore $F$ is a regular filter of $X$.

The converse of the above proposition is not true, i.e., every regular filter of a commutative $B E$-algebra need not be a $\sigma$-filter. Indeed, consider Example 2.9. Here, $F=\{1, d\}$ is clearly a regular filter, because $(d)^{++} \subseteq F$. But $F$ is not a $\sigma$-filter of $X$, because of $(d)^{+} \vee F \neq X$. However, some equivalent conditions are given for every regular filter of a commutative $B E$-algebra to become a $\sigma$-filter.

Theorem 2.13. Following assertions are equivalent in a commutative BE-algebra $X$ :
(1) every regular filter is a $\sigma$-filter;
(2) every dual annihilator filter is a $\sigma$-filter;
(3) for each $x \in X,(x)^{++}$is a $\sigma$-filter;
(4) for each $x \in X,(x)^{+} \vee(x)^{++}=X$.

Proof. (1) $\Rightarrow$ (2): Since every dual annihilator filter is a regular filter, it is clear.
$(2) \Rightarrow(3)$ : Since each $(x)^{++}$is a dual annihilator filter, it is clear.
$(3) \Rightarrow(4)$ : Assume the statement (3). Let $x \in X$. Since $(x)^{++}$is a $\sigma$ filter of $X$, we get $(x)^{++}=\sigma\left((x)^{++}\right)$. Clearly $x \in(x)^{++}=\sigma\left((x)^{++}\right)$. Hence $(x)^{+} \vee(x)^{++}=X$.
(4) $\Rightarrow$ (1): Assume that $(x)^{+} \vee(x)^{++}=X$ for each $x \in X$. Let $F$ be a regular filter of $X$. Clearly $\sigma(F) \subseteq F$. Conversely, let $x \in F$. Since $F$ is a regular filter, we get $(x)^{++} \subseteq F$. Hence $X=(x)^{+} \vee(x)^{++} \subseteq(x)^{+} \vee F$. Thus $x \in \sigma(F)$. Therefore $F$ is a $\sigma$-filter of $X$.

Recall that a filter $F$ of a commutative $B E$-algebra $X$ is called an O-filter if $F=O(S)$ for some $\vee$-closed subset $S$ of $X$. In [11], authors studied the properties of O-filters and proved that every O-filter of a self-distributive and commutative $B E$-algebra is the intersection of all minimal prime filters containing it. In the following result, it is proved that the class of all $\sigma$-filters of a commutative $B E$ algebra $X$ is properly contained in the class of all O-filters of $X$.

Theorem 2.14. Suppose $X$ is a commutative BE-algebra with a dual-dense element (i.e., $\left.(x)^{+}=\{1\}\right)$. Then every $\sigma$-filter of $X$ is an $O$-filter.

Proof. Let $F$ be a $\sigma$-filter of $X$. Then $\sigma(F)=F$. Consider the set $S=\{x \in$ $\left.X \mid(x)^{++} \vee F=X\right\}$. It can be easily verified, by using Lemma 2.1(2), that $S$ is a $\vee$-closed subset of $X$. We now show that $F=O(S)$. Let $x \in O(S)$. Then $x \vee y=1$ for some $y \in S$. Now

$$
\begin{aligned}
x \vee y=1 & \Rightarrow y \in(x)^{+} & & \\
& \Rightarrow(y)^{++} \subseteq(x)^{+} & & \text {by Lemma } 2.1(4) \\
& \Rightarrow X=(y)^{++} \vee F \subseteq(x)^{+} \vee F & & \text { since } y \in S \\
& \Rightarrow x \in \sigma(F)=F & & \text { since } F \text { is a } \sigma \text {-filter }
\end{aligned}
$$

which concludes that $O(S) \subseteq F$. Conversely, let $x \in F=\sigma(F)$ and $d$ a dualdense element of $X$. Then $(x)^{+} \vee \sigma(F)=X$. Therefore $d \in(x)^{+} \vee \sigma(F)$. Hence $a *(b * d)=1$ for some $a \in(x)^{+}$and $b \in \sigma(F)$. Thus $a \vee x=1$ and $(b)^{+} \vee F=X$. Now

$$
\begin{aligned}
a *(b * d)=1 & \Rightarrow a \leq b * d \\
& \Rightarrow(a)^{+} \subseteq(b * d)^{+}
\end{aligned}
$$

$$
\begin{aligned}
& \Rightarrow(a)^{+} \cap(b)^{+} \subseteq(b)^{+} \cap(b * d)^{+} \\
& \Rightarrow(a)^{+} \cap(b)^{+} \subseteq(d)^{+}=\{1\} \\
& \Rightarrow(b)^{+} \subseteq(a)^{++} \\
& \Rightarrow X=(b)^{+} \vee F \subseteq(a)^{++} \vee F \\
& \text { by Lemma } 2.1(1) \\
& \Rightarrow a \in S \text { and } \quad \text { since } b \in \sigma(F) \\
& \Rightarrow x \in O(S)
\end{aligned}
$$

which gives $F=\sigma(F) \subseteq O(S)$. Hence $F=O(S)$. Therefore $F$ is an O-filter of $X$.

The converse of the above theorem is not true, i.e., every O-filter of a commutative $B E$-algebra need not be a $\sigma$-filter. For, consider the following example.

Example 2.15. Let $X=\{1, a, b, c\}$ be a set. Define a binary operation $*$ on $X$ as

| $*$ | 1 | $a$ | $b$ | $c$ |
| :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | $a$ | $b$ | $c$ |
| $a$ | 1 | 1 | $a$ | $c$ |
| $b$ | 1 | 1 | 1 | $c$ |
| $c$ | 1 | $a$ | $b$ | 1 |


| $\vee$ | 1 | $a$ | $b$ | $c$ |
| :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | 1 | 1 | 1 |
| $a$ | 1 | $a$ | $a$ | 1 |
| $b$ | 1 | $a$ | $b$ | 1 |
| $c$ | 1 | 1 | 1 | $c$ |

It can be routinely verified that $(X, *, \vee, 1)$ is a commutative $B E$-algebra. Observe that $(a)^{+}=(b)^{+}=\{1, c\}$, and $(c)^{+}=\{1, a, b\}$. Consider the filter $F=\{1, c\}$ of $X$. Clearly $S=\{a, b\}$ is a $\vee$-closed subset of $X$. It is easy to observe that $F=O(S)$. Hence $F$ is an O-filter of $X$. Now $\sigma(F)=\{1\} \subset F$. Therefore $F$ is not a $\sigma$-filter of $X$.

Lemma 2.16. In a commutative BE-algebra, every dual annulet is an $O$-filter.
Proof. Let $X$ be a commutative $B E$-algebra and $a \in X$. Consider $[a]=\{x \in$ $X \mid x \leq a\}$. Let $x, y \in[a]$. Then $x \leq a$ and $y \leq a$. Since $X$ is commutative, it is partially ordered. Hence $x \vee y \leq a$, which gives that $x \vee y \in[a]$. Therefore $[a]$ is a $\vee$-closed subset of $X$. We now show that $(a)^{+}=O([a])$. Let $x \in(a)^{+}$. Then $a \vee x=1$ and $a \in[a]$. Hence $x \in O([a])$, which gives that $(a)^{+} \subseteq O([a])$. Conversely, let $x \in O([a])$. Then $x \vee y=1$ for some $y \in[a]$. Since $y \in[a]$, we get $y \leq a$. Hence $1=x \vee y \leq x \vee a$. Thus $x \in(a)^{+}$. Hence $O([a]) \subseteq(a)^{+}$. Therefore $(a)^{+}$is an $O$-filter of $X$.

Theorem 2.17. Following assertions are equivalent in a commutative BE-algebra $X$ :
(1) every $O$-filter is a $\sigma$-filter;
(2) each dual annulet is a $\sigma$-filter;
(3) for any $x, y \in X, x \vee y=1$ implies $(x)^{+} \vee(y)^{+}=X$.

Proof. (1) $\Rightarrow$ (2): Since each dual annulet is an O-filter, it is clear.
$(2) \Rightarrow(3)$ : Assume that each dual annulet is a $\sigma$-filter of $X$. Let $x, y \in X$ be such that $x \vee y=1$. Hence $x \in(y)^{+}$. By (2), we get that $(y)^{+}$is a $\sigma$-filter of $X$. Hence $x \in(y)^{+}=\sigma\left((y)^{+}\right)$. Thus we get $(x)^{+} \vee(y)^{+}=X$. Therefore, condition (3) is proved.
$(3) \Rightarrow(1)$ : Assume that condition (3) holds. Let $F$ be an O-filter of $X$. Then $F=O(S)$ for some $\vee$-closed subset $S$ of $X$. Clearly $\sigma(F) \subseteq F$. We claim that $O(S) \subseteq \sigma(F)$. Now

$$
\begin{array}{rlrl}
x \in O(S) & \Rightarrow x \vee y=1 \text { for some } y \in S & \\
& \Rightarrow(x)^{+} \vee(y)^{+}=X & & \text { by }(3) \\
& \Rightarrow X=(x)^{+} \vee(y)^{+} \subseteq(x)^{+} \vee O(S) & & \text { since } y \in S \\
& \Rightarrow x \in \sigma(O(S))=\sigma(F) & &
\end{array}
$$

Hence $O(S) \subseteq \sigma(F)$, which gives $F=O(S)=\sigma(F)$. Therefore $F$ is a $\sigma$-filter of $X$.

Theorem 2.18. Let $P$ be a prime filter of a commutative $B E$-algebra $X$ such that $P=O(P)$. If $X$ satisfies any one assertions of the above theorem, then $P$ is a $\sigma$-filter.

Proof. Assume that $X$ satisfies condition (3) of the above theorem. Let $P$ be a prime filter of $X$ such that $P=O(P)$. By Proposition 2.7(1), we have $\sigma(P) \subseteq O(P)=P$. Conversely, let $x \in O(P)$. Then there exists $y \notin P$ such that $x \vee y=1$. Since $y \notin O(P)$, we get $(y)^{+} \subseteq P$. By (3) of the above theorem, we get that $(x)^{+} \vee(y)^{+}=X$. Hence $X=(x)^{+} \vee(y)^{+} \subseteq(x)^{+} \vee P$. Thus $x \in \sigma(P)$. Hence $P$ is a $\sigma$-filter of $X$.

Let us denote by $\mu$ the set of all maximal filters of a $B E$-algebra $X$. For any filter $F$ of a $B E$-algebra $X$, we also denote $\mu(F)=\{M \in \mu \mid F \subseteq M\}$. Since every maximal filter of a commutative $B E$-algebra is prime, by Proposition 2.3, we conclude that $O(M)$ is a filter such that $O(M) \subseteq M$ for every $M \in \mu$. Then we have the following result.

Theorem 2.19. For any filter $F$ of a commutative BE-algebra $X, \sigma(F)=$ $\bigcap_{M \in \mu(F)} O(M)$.

Proof. Let $x \in \sigma(F)$ and $F \subseteq M$ where $M \in \mu$. Then $X=(x)^{+} \vee F \subseteq$ $(x)^{+} \vee M$. Suppose $(x)^{+} \subseteq M$, then $M=X$, which is a contradiction. Hence $(x)^{+} \nsubseteq M$. Thus $x \in O(M)$ for all $M \in \mu(F)$. Therefore $\sigma(F) \subseteq \bigcap_{M \in \mu(F)} O(M)$. Conversely, let $x \in \bigcap_{M \in \mu(F)} O(M)$. Then $x \in O(M)$ for all $M \in \mu(F)$. Suppose
$(x)^{+} \vee F \neq X$. Then there exists a maximal filter $M_{0}$ such that $(x)^{+} \vee F \subseteq M_{0}$. Hence $(x)^{+} \subseteq M_{0}$ and $F \subseteq M$. Since $F \subseteq M_{0}$, by hypothesis, we get $x \in O\left(M_{0}\right)$. Hence $(x)^{+} \nsubseteq M_{0}$, which is a contradiction. Therefore $(x)^{+} \vee F=X$. Thus $x \in \sigma(F)$. Hence $\bigcap_{M \in \mu(F)} O(M) \subseteq \sigma(F)$.

From the above theorem, it can be easily observed that $\sigma(F) \subseteq O(M)$ for every $M \in \mu(F)$. Now, in the following, a set of equivalent conditions is given for the class of all $\sigma$-filters of a commutative $B E$-algebra to become a sublattice to the lattice $\mathcal{F}(X)$ of all filters of the commutative $B E$-algebra $X$.

Theorem 2.20. The following assertions are equivalent in a commutative BEalgebra $X$ :
(1) for any $M \in \mu, O(M)$ is maximal;
(2) for any $F, G \in \mathcal{F}(X), F \vee G=X$ implies $\sigma(F) \vee \sigma(G)=X$;
(3) for any $F, G \in \mathcal{F}(X), \sigma(F) \vee \sigma(G)=\sigma(F \vee G)$;
(4) for any two distinct maximal filters $M$ and $N, O(M) \vee O(N)=X$;
(5) for any $M \in \mu, M$ is the unique member of $\mu$ such that $O(M) \subseteq M$.

Proof. (1) $\Rightarrow$ (2): Assume the condition (1). Then clearly $O(M)=M$ for all $M \in \mu$. Let $F, G \in \mathcal{F}(X)$ be such that $F \vee G=X$. Suppose $\sigma(F) \vee \sigma(G) \neq X$. Then there exists a maximal filter $M$ such that $\sigma(F) \vee \sigma(G) \subseteq M$. Hence $\sigma(F) \subseteq M$ and $\sigma(G) \subseteq M$. Now

$$
\begin{array}{rlrl}
\sigma(F) \subseteq M & \Rightarrow \bigcap_{M_{i} \in \mu(F)} O\left(M_{i}\right) \subseteq M \\
& \Rightarrow O\left(M_{i}\right) \subseteq M & \text { for some } M_{i} \in \mu(F) \text { (since } M \text { is prime) } \\
& \Rightarrow M_{i} \subseteq M & & \text { by condition (1) } \\
& \Rightarrow F \subseteq M & & \text { since } F \subseteq M_{i} .
\end{array}
$$

Similarly, we can obtain that $G \subseteq M$. Hence $X=F \vee G \subseteq M$, which is a contradiction to the maximality of $M$. Therefore $\sigma(F) \vee \sigma(G)=X$.
$(2) \Rightarrow(3)$ : Assume the condition (2). Let $F, G \in \mathcal{F}(X)$. Clearly $\sigma(F) \vee$ $\sigma(G) \subseteq \sigma(F \vee G)$. Let $x \in \sigma(F \vee G)$. Then $\left\{(x)^{+} \vee F\right\} \vee\left\{(x)^{+} \vee G\right\}=(x)^{+} \vee F \vee G=$ $X$. Hence by condition (2), we get $\sigma\left((x)^{+} \vee F\right) \vee \sigma\left((x)^{+} \vee G\right)=X$. Thus $x \in \sigma\left((x)^{+} \vee F\right) \vee \sigma\left((x)^{+} \vee G\right)$. Hence $r *(s * x)=1$ for some $r \in \sigma\left((x)^{+} \vee F\right)$ and $s \in \sigma\left((x)^{+} \vee G\right)$. Now

$$
\begin{aligned}
r \in \sigma\left((x)^{+} \vee F\right) & \Rightarrow(r)^{+} \vee\left\{(x)^{+} \vee F\right\}=X \\
& \Rightarrow X=\left\{(r)^{+} \vee(x)^{+}\right\} \vee F \subseteq(r \vee x)^{+} \vee F \\
& \Rightarrow(r \vee x)^{+} \vee F=X \\
& \Rightarrow r \vee x \in \sigma(F) .
\end{aligned}
$$

Similarly, we can get $s \vee x \in \sigma(G)$. Now, we have the following consequence:

$$
\begin{aligned}
r *(s * x)=1 & \Rightarrow r \leq s * x \\
& \Rightarrow r \vee x \leq(s * x) \vee x \leq(s \vee x) *(x \vee x) \\
& \Rightarrow r \vee x \leq(s \vee x) *(x \vee x) \\
& \Rightarrow r \vee x \leq(s \vee x) * x \\
& \Rightarrow(r \vee x) *((s \vee x) * x)=1
\end{aligned}
$$

where $r \vee x \in \sigma(F)$ and $s \vee x \in \sigma(G)$. Hence $x \in \sigma(F) \vee \sigma(G)$. Thus $\sigma(F \vee G) \subseteq$ $\sigma(F) \vee \sigma(G)$. Therefore $\sigma(F) \vee \sigma(G)=\sigma(F \vee G)$.
(3) $\Rightarrow$ (4): Assume the condition (3). Let $M, N$ be two distinct maximal filters of $X$. Choose $x \in M-N$ and $y \in N-M$. Since $x \notin N$, we get $N \vee\langle x\rangle=X$. Since $y \notin M$, we get $M \vee\langle y\rangle=X$. Now

$$
\begin{array}{rlr}
X & =\sigma(X) & \\
& =\sigma(X \vee X) & \\
& =\sigma(\{N \vee\langle x\rangle\} \vee\{M \vee\langle y\rangle\}) \\
& =\sigma(\{M \vee\langle x\rangle\} \vee\{N \vee\langle y\rangle\}) \\
& =\sigma(M \vee N) & \text { since } x \in M \text { and } y \in N \\
& =\sigma(M) \vee \sigma(N) & \text { by condition (3) } \\
& \subseteq O(M) \vee O(N) & \text { by Proposition 2.7(1). }
\end{array}
$$

Therefore $O(M) \vee O(N)=X$.
(4) $\Rightarrow$ (5): Assume condition (4). Let $M \in \mu$. Suppose $N \in \mu$ such that $N \neq M$ and $O(N) \subseteq M$. Since $O(M) \subseteq M$, by hypothesis, we get $X=O(M) \vee$ $O(N)=M$, which is a contradiction. Hence $M$ is the unique maximal filter such that $O(M) \subseteq M$.
(5) $\Rightarrow(1)$ : Let $M \in \mu$. Suppose $O(M)$ is not maximal. Let $M_{0}$ be a maximal filter of $X$ such that $O(M) \subseteq M_{0}$. We have always $O\left(M_{0}\right) \subseteq M_{0}$, which is a contradiction.

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