

A PRE-PERIOD OF A FINITE DISTRIBUTIVE LATTICE

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Abstract

The notion of a pre-period of a finite bounded distributive lattice (BDL) A is defined by means of the notion of a pre-period of a finite connected monounary algebra: it is the maximum value of the pre-period of an endomorphism and 0-fixing connected mapping of A to A . The main result is that the pre-period of any finite BDL is less than or equal to the length of the lattice; also, necessary and sufficient conditions under which it is equal to the length of the lattice, are shown.

Keywords: distributive lattice, pre-period, connected unary operation, BDLC-algebra.

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1. INTRODUCTION

The aim of the paper is to study some properties of endomorphism of bounded lattices.

An endomorphism f of a structure A can be considered as a unary operation and $\langle A; f \rangle$ is a monounary algebra.

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The importance of theory of unary and monounary algebras is pointed out for example in the monographs [7, 9, 10, 11]. The advantage of monounary algebras is their relatively easy visualization as they can be represented as planar directed graphs. Endomorphism of monounary algebras were investigated, e.g., in [4, 5, 8, 12, 13].

The results of the present paper can be considered as a modest contribution in the direction of studying finite distributive lattices, by applying theory of monounary algebras.

Let $f : A \rightarrow A$ be a unary operation on a set A . Let f^0 be the identity map on A and $\text{Im}(f) := \{f(a) \mid a \in A\}$. A *pre-period* (or *stabilizer*) of f is the least nonnegative integer n satisfying $\text{Im}f^n = \text{Im}f^{n+1}$ and denoted by $\lambda(f)$ (see e.g. [16]). Let us remark that the notion of $\lambda(f)$ was defined for finite monounary algebras only. However, $\lambda(f)$ exists also for some infinite algebras, so we will always mention whether we deal with a finite or an infinite case. An operation f on A is *connected* if for each $a, b \in A$, there exist nonnegative integers n, m such that $f^n(a) = f^m(b)$. The results from [14] and [3] imply that $\lambda(f) \leq |A| - 1$ and if $\lambda(f) = |A| - 1$ then f is connected.

A Boolean algebra is a bounded distributive lattice $\langle A; \vee, \wedge, 0, 1 \rangle$ equipped with an onto operation $f : A \rightarrow A$ which maps x to the complement of x satisfying $x \vee f(x) = 1$ and $x \wedge f(x) = 0$ for all $x \in A$. Since f is onto, $\lambda(f) = 0$; furthermore, f is not connected if $|A| > 2$.

Clearly, all constant functions are connected endomorphisms of $\langle A; \vee, \wedge \rangle$. Several authors focus specially on connected monounary algebras (see e.g., [6, 15]). It will be shown (Lemma 1), that any connected order-preserving mapping f of a bounded poset A has an (obviously, unique) fix-point and also, that $\lambda(f)$ is defined, even in the case when A is infinite.

We are going to investigate bounded distributive lattices (shortly, BDL) $\widehat{A} = \langle A; \vee, \wedge, 0, 1 \rangle$ and connected endomorphisms of $\langle A; \vee, \wedge \rangle$. Moreover, with respect to Lemma 1, let us consider only the endomorphisms fixing the least element 0. If there is an n such that n is the maximum of all $\lambda(f)$, then we set

$$\lambda(\widehat{A}) := n.$$

It is interesting whether for each positive number k , can we find a connected endomorphism f with $\lambda(f) = k$.

Applying some results of [1, 2] we will show that if a BDL is finite, then $\lambda(\widehat{A})$ is less or equal to the length of the lattice. Also, we prove necessary and sufficient conditions under which

$$\lambda(\widehat{A}) = \text{length}(\widehat{A}).$$

2. PRELIMINARIES

Lemma 1. *Let A be a bounded poset and let f be a connected order-preserving mapping of A . Then f has a unique fix-point α and $\lambda(f)$ is the greater number of $\min \{n \in \mathbb{N} \cup \{0\} \mid f^n(1) = \alpha\}$ and $\min \{m \in \mathbb{N} \cup \{0\} \mid f^m(0) = \alpha\}$.*

Proof. Suppose that f is connected and preserves \leq . Then there exist the least nonnegative integers m and n such that $f^m(0) = f^n(1) = \alpha$ and $f^m(0) \leq f^{m+1}(0)$ and $f^{n+1}(1) \leq f^n(1)$ which imply that $f(\alpha) = \alpha$.

Let k be the considered greater number. If $x \in A$, then $0 < x < 1$ yields $\alpha = f^k(0) \leq f^k(x) \leq f^k(1) = \alpha$, hence $\lambda(f) \leq k$. The equality follows from the definition of k . ■

An algebra $\langle A; \vee, \wedge, f, 0, 1 \rangle$ is called a *BDLC-algebra* if $\langle A; \vee, \wedge, 0, 1 \rangle$ is a BDL and f is a connected endomorphism on $\langle A; \vee, \wedge \rangle$ fixed 0. For each $n \in \mathbb{N} \cup \{0\}$, let \mathcal{M}_n be the class of all BDLC-algebras $\langle A; \vee, \wedge, f, 0, 1 \rangle$ whose $\lambda(f) \leq n$ and it is shown in [1] that \mathcal{M}_n is the variety satisfying the following identities:

- $f(a \vee b) \approx f(a) \vee f(b)$,
- $f(a \wedge b) \approx f(a) \wedge f(b)$,
- $f(0) \approx 0$,
- $f^n(1) \approx 0$.

For each positive integer n and BDL $\widehat{A} = \langle A; \vee, \wedge, 0, 1 \rangle$, define $\underline{A}^{*n} := \langle A^n; \vee, \wedge, f, \mathbf{0}, \mathbf{1} \rangle$ whose $\langle A^n; \vee, \wedge, \mathbf{0}, \mathbf{1} \rangle$ is the usual direct product of \widehat{A} and $f : A^n \rightarrow A^n$ is defined by $f(a_1, a_2, \dots, a_n) = (a_2, \dots, a_n, 0)$ for all $a_i \in A$ and $1 \leq i \leq n$. Denote $\mathbf{0} := (\underbrace{0, \dots, 0}_n)$, $\mathbf{1} := (\underbrace{1, \dots, 1}_n)$ and \underline{A}^{*0} to be the trivial

BDLC-algebra. In particular, if \widehat{A} is the 2-element chain then we call it that an *n-cube BDLC-algebra*, denoted by $\underline{2}^{*n}$. In [2], Charoenpol and Ratanaprasert proved the following facts.

Theorem 2 [2]. *Let $\underline{A} = \langle A; \vee, \wedge, f, 0, 1 \rangle$ be a BDLC-algebra with $\lambda(f) = n$. The following are equivalent:*

1. \underline{A} is a subdirectly irreducible algebra,
2. $0 = f^n(1) \prec f^{n-1}(1) \prec \dots \prec f(1) \prec 1$,
3. $\underline{A} \leq \underline{2}^{*n}$.

Theorem 3 [2]. *For each $n \in \mathbb{N}$, \mathcal{M}_n is a variety generated by $\underline{2}^{*n}$.*

3. A REPRESENTATION OF A BDLC-ALGEBRA

For each BDLC-algebra \underline{A} , there is a natural number n such that $\underline{A} \in \mathcal{M}_n$ which implies that \underline{A} is a homomorphic image of subalgebra of direct product of $\underline{2}^{*n}$.

Lemma 4. For each $n \in \mathbb{N}$, $(\underline{2}^{*n})^I \cong (\underline{2}^I)^{*n}$.

Proof. Define a function $\psi : (\underline{2}^{*n})^I \rightarrow (\underline{2}^I)^{*n}$ by $\psi(a) = (\pi_1 \circ a, \pi_2 \circ a, \dots, \pi_n \circ a)$ for all $a \in (\underline{2}^{*n})^I$ where $\pi_i : \{0, 1\}^n \rightarrow \{0, 1\}$ is the i -projection for all $1 \leq i \leq n$. It is routine to show that the mapping ψ is an isomorphism. ■

This theorem implies that for each $\underline{A} \in \mathcal{M}_n$, there exist $\underline{B} \leq (\underline{2}^I)^{*n}$ and homomorphism $h : \underline{B} \rightarrow \underline{A}$ such that $\underline{A} = h(\underline{B})$. So for $a, b \in \underline{A}$, one can see that $a = h(\bar{a}_1, \dots, \bar{a}_n)$ and $b = h(\bar{b}_1, \dots, \bar{b}_n)$ for some $\bar{a}_i, \bar{b}_i \in \underline{2}^I$ (that is, $\bar{a}_i, \bar{b}_i : I \rightarrow \underline{2}$); and hence,

$$a \vee b = h(\bar{a}_1 \vee \bar{b}_1, \dots, \bar{a}_n \vee \bar{b}_n)$$

and

$$a \wedge b = h(\bar{a}_1 \wedge \bar{b}_1, \dots, \bar{a}_n \wedge \bar{b}_n).$$

Moreover,

$$f(a) = h(\bar{a}_2, \dots, \bar{a}_n, \bar{0}), 1_{\underline{A}} = h(\bar{1}, \dots, \bar{1}) \text{ and } 0_{\underline{A}} = h(\bar{0}, \dots, \bar{0})$$

where $\bar{0}$ and $\bar{1}$ are the constant function 0 and 1, respectively. Since h preserves \leq , we have $h(\underbrace{\bar{1}, \dots, \bar{1}}_{n-j}, \underbrace{\bar{0}, \dots, \bar{0}}_j) \leq h(\underbrace{\bar{1}, \dots, \bar{1}}_{n-j+1}, \underbrace{\bar{0}, \dots, \bar{0}}_{j-1})$ for all $1 \leq j \leq n$.

The following theorem shows the classification of j with $h(\underbrace{\bar{1}, \dots, \bar{1}}_{n-j}, \underbrace{\bar{0}, \dots, \bar{0}}_j) = h(\underbrace{\bar{1}, \dots, \bar{1}}_{n-j+1}, \underbrace{\bar{0}, \dots, \bar{0}}_{j-1})$.

Theorem 5. For each BDLC-algebra \underline{A} with $\lambda(f) = m$, if $h : \underline{B} \rightarrow \underline{A}$ is a homomorphism for some $\underline{B} \leq (\underline{2}^I)^{*n}$, then $h(\underbrace{\bar{1}, \dots, \bar{1}}_{n-m+i}, \underbrace{\bar{0}, \dots, \bar{0}}_{m-i}) < h(\underbrace{\bar{1}, \dots, \bar{1}}_{n-m+(i+1)}, \underbrace{\bar{0}, \dots, \bar{0}}_{m-(i+1)})$ for all $0 \leq i \leq m-1$ and $h(\underbrace{\bar{1}, \dots, \bar{1}}_{n-i}, \underbrace{\bar{0}, \dots, \bar{0}}_i) = 0_{\underline{A}}$ for all $m \leq i \leq n$.

Proof. Let $h : \underline{B} \rightarrow \underline{A}$ be a homomorphism for some $\underline{B} \leq (\underline{2}^I)^{*n}$ and $0 \leq i \leq m-1$. Suppose that $h(\underbrace{\bar{1}, \dots, \bar{1}}_{n-m+i}, \underbrace{\bar{0}, \dots, \bar{0}}_{m-i}) = h(\underbrace{\bar{1}, \dots, \bar{1}}_{n-m+(i+1)}, \underbrace{\bar{0}, \dots, \bar{0}}_{m-(i+1)})$. Since h preserves f , we get $h(\underbrace{\bar{1}, \dots, \bar{1}}_{n-m+i-1}, \underbrace{\bar{0}, \dots, \bar{0}}_{m-i+1}) = h(\underbrace{\bar{1}, \dots, \bar{1}}_{n-m+i}, \underbrace{\bar{0}, \dots, \bar{0}}_{m-i})$. By continuity

in this way, this implies that $h(\underbrace{\bar{1}, \dots, \bar{1}}_{n-m+(i+1)}, \underbrace{\bar{0}, \dots, \bar{0}}_{m-(i+1)}) = 0_{\underline{A}}$. So, $f^{m-(i+1)}(1_{\underline{A}}) = f^{m-(i+1)}(h(\underbrace{\bar{1}, \dots, \bar{1}}_n)) = h(f^{m-(i+1)}(\underbrace{\bar{1}, \dots, \bar{1}}_n)) = h(\underbrace{\bar{1}, \dots, \bar{1}}_{n-m+(i+1)}, \underbrace{\bar{0}, \dots, \bar{0}}_{m-(i+1)}) = 0_{\underline{A}}$, a contradict with $\lambda(f) = m$. Therefore, $h(\underbrace{\bar{1}, \dots, \bar{1}}_{n-m+i}, \underbrace{\bar{0}, \dots, \bar{0}}_{m-i}) < h(\underbrace{\bar{1}, \dots, \bar{1}}_{n-m+(i+1)}, \underbrace{\bar{0}, \dots, \bar{0}}_{m-(i+1)})$.
 Let $m \leq i \leq n$. Since $\lambda(f) = m$, we have $0_{\underline{A}} \leq h(\underbrace{\bar{1}, \dots, \bar{1}}_{n-i}, \underbrace{\bar{0}, \dots, \bar{0}}_i) \leq h(\underbrace{\bar{1}, \dots, \bar{1}}_{n-m}, \underbrace{\bar{0}, \dots, \bar{0}}_m) = 0_{\underline{A}}$ which implies that $h(\underbrace{\bar{1}, \dots, \bar{1}}_{n-i}, \underbrace{\bar{0}, \dots, \bar{0}}_i) = 0_{\underline{A}}$. ■

Corollary 6. *For each BDLC-algebra \underline{A} with $\lambda(f) = m$, there exists an $(m+1)$ -element chain as a sublattice of \hat{A} . Moreover, the chain is*

$$0 = h(\underbrace{\bar{1}, \dots, \bar{1}}_{n-m}, \underbrace{\bar{0}, \dots, \bar{0}}_m) < h(\underbrace{\bar{1}, \dots, \bar{1}}_{n-m+1}, \underbrace{\bar{0}, \dots, \bar{0}}_{m-1}) < \dots < h(\bar{1}, \dots, \bar{1}) = 1.$$

4. A PRE-PERIOD OF A FINITE BOUNDED DISTRIBUTIVE LATTICE

Now, our tools are ready to investigate $\lambda(\hat{A})$ for any finite BDL \hat{A} . Since the constant mapping $f(x) = 0$ is a connected endomorphism fixing 0 with $\lambda(f) = 1$, we obtain $\lambda(\hat{A}) \geq 1$.

Theorem 7. *For each finite BDL $\hat{A} = \langle A; \vee, \wedge, 0, 1 \rangle$ and $k \leq \lambda(\hat{A})$, there is a unary operation f_k on A such that $\langle A; \vee, \wedge, f_k, 0, 1 \rangle$ is a BDLC-algebra with $\lambda(f_k) = k$.*

Proof. Suppose that $\lambda(\hat{A}) = m$. Then there is a unary operation f such that $\underline{A} = \langle A; \vee, \wedge, f, 0, 1 \rangle$ is a BDLC-algebra with $\lambda(f) = m$. So, $\underline{A} = h(\underline{B})$ for some $\underline{B} \leq (\underline{2}^I)^{*m}$ and homomorphism h . Let $k \leq m$, define $f_k : A \rightarrow A$ by

$$f_k(h(\bar{a}_1, \dots, \bar{a}_m)) = h(\bar{a}_2, \dots, \bar{a}_k, \bar{0}, \dots, \bar{0})$$

for all $(\bar{a}_1, \dots, \bar{a}_m) \in B$. Since $\underline{B} \leq (\underline{2}^I)^{*m}$, we get

$$\begin{aligned} (\bar{a}_2, \dots, \bar{a}_k, \bar{0}, \dots, \bar{0}) &= (\bar{a}_2, \dots, \bar{a}_m, \bar{0}) \wedge (\underbrace{\bar{1}, \dots, \bar{1}}_{k-1}, \bar{0}, \dots, \bar{0}) \\ &= f_{\underline{B}}(\bar{a}_1, \dots, \bar{a}_m) \wedge f_{\underline{B}}^{k-1}(\bar{1}, \dots, \bar{1}) \in B \end{aligned}$$

for all $(\bar{a}_1, \dots, \bar{a}_m) \in B$. So, f_k is well-defined. It is clear that $\langle A; \vee, \wedge, f_k, 0, 1 \rangle$ is a BDLC-algebra with $\lambda(f_k) = k$. ■

Theorem 8. *Let \hat{A} be a finite BDL. Then*

$$\lambda(\hat{A}) \leq \text{length}(\hat{A}).$$

Proof. The assertion follows from Corollary 6. ■

Example 9. Let $\hat{A} = \langle A; \vee, \wedge, 0, 1 \rangle$ be a BDL which is shown as Figure 1.

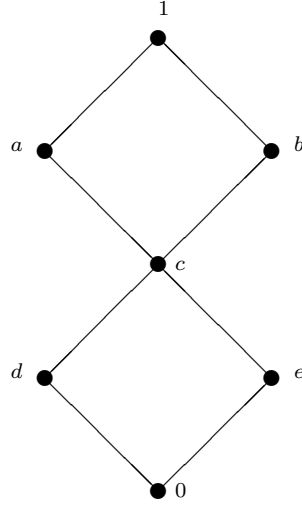


Figure 1. A bounded distributive lattice.

Due to Theorem 8, $\lambda(\hat{A}) \leq 4$.

Suppose that $\lambda(\hat{A}) = 4$. Then we can define f such that $\langle A; \vee, \wedge, f, 0, 1 \rangle$ is a BDLC-algebra with $\lambda(f) = 4$. We may assume that $f(1) = a$, $f(a) = c$, $f(c) = d$ and $f(d) = 0$. Since $a = f(1) = f(a \vee b) = f(a) \vee f(b) = c \vee f(b)$, we get $f(b) = a$ which implies that $d = f(c) = f(a \wedge b) = f(a) \wedge f(b) = c \wedge a = c$, a contradiction. So, $\lambda(\hat{A}) \leq 3$.

Define $f : A \rightarrow A$ by $f(1) = f(b) = c$, $f(a) = f(c) = f(e) = d$ and $f(d) = f(0) = 0$. One can see that f preserves \wedge , \vee and 0 . Hence, $\langle A; \vee, \wedge, f, 0, 1 \rangle$ is a BDLC-algebra with $\lambda(f) = 3$. So, $\lambda(\hat{A}) = 3$.

Theorem 10. *Let \hat{A} be a finite BDL. Then*

$$\lambda(\hat{A}) = \text{length}(\hat{A}) \text{ if and only if } 0 = f^{\lambda(f)}(1) \prec f^{\lambda(f)-1}(1) \prec \dots \prec f(1) \prec 1$$

for some connected endomorphism f on $\langle A; \vee, \wedge \rangle$ fixing 0 .

Proof. Suppose that $\lambda(\hat{A}) = n$ and we choose a connected endomorphism f on $\langle A; \vee, \wedge \rangle$ fixing 0 with $\lambda(f) = n$. Hence, n is the smallest natural number with

$f^n(1) = 0$. Furthermore, $C = \{1 > f(1) > \dots > f^{n-1}(1) > f^n(1) = 0\}$ is a chain with $|C| = n + 1$. Since \widehat{A} is distributive,

$$\begin{aligned} n = \text{length}(\widehat{A}) &\Leftrightarrow C \text{ is a maximal chain} \\ &\Leftrightarrow 0 = f^n(1) \prec f^{n-1}(1) \prec \dots \prec f(1) \prec 1. \quad \blacksquare \end{aligned}$$

Corollary 11. *The pre-period of the directed product $\widehat{2}^n$ of the 2-element chain $\widehat{2}$ is equal to n for all $n \in \mathbb{N}$; that is, $\lambda_0(\widehat{2}^n) = n$.*

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