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A PRE-PERIOD OF A FINITE DISTRIBUTIVE LATTICE

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Abstract

The notion of a pre-preriod of a finite bounded distributive lattice (BDL) A is defined by means of the notion of a pre-period of a finite connected monounary algebra: it is the maximum value of the pre-period of an endomorphism and 0-fixing connected mapping of A to A. The main result is that the pre-period of any finite BDL is less than or equal to the length of the lattice; also, necessary and sufficient conditions under which it is equal to the length of the lattice, are shown.

Keywords: distributive lattice, pre-period, connected unary operation, BDLC-algebra.

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1. Introduction

The aim of the paper is to study some properties of endomorphism of bounded lattices.

An endomorphism f of a structure A can be considered as a unary operation and $\langle A; f\rangle$ is a monounary algebra.

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The importance of theory of unary and monounary algebras is pointed out for example in the monographs [7, 9, 10, 11]. The advantage of monounary algebras is their relatively easy visualization as they can be represented as planar directed graphs. Endomorphism of monounary algebras were investigated, e.g., in [4, 5, 8, 12, 13].

The results of the present paper can be considered as a modest contribution in the direction of studying finite distributive lattices, by applying theory of monounary algebras.

Let $f : A \to A$ be a unary operation on a set A. Let f^0 be the identity map on A and $\mathrm{Im}(f) := \{f(a) \mid a \in A\}$. A *pre-period* (or *stabilizer*) of f is the least nonnegative integer n satisfying $\mathrm{Im} f^n = \mathrm{Im} f^{n+1}$ and denoted by $\lambda(f)$ (see e.g.[16]). Let us remark that the notion of $\lambda(f)$ was defined for finite monounary algebras only. However, $\lambda(f)$ exists also for some infinite algebras, so we will always mention whether we deal with a finite or an infinite case. An operation f on A is *connected* if for each $a, b \in A$, there exist nonnegative integers n, m such that $f^n(a) = f^m(b)$. The results from [14] and [3] imply that $\lambda(f) \leq |A| - 1$ and if $\lambda(f) = |A| - 1$ then f is connected.

A Boolean algebra is a bounded distributive lattice $\langle A; \vee, \wedge, 0, 1\rangle$ equipped with an onto operation $f : A \to A$ which maps x to the complement of x satisfying $x \vee f(x) = 0$ and $x \wedge f(x) = 0$ for all $x \in A$. Since f is onto, $\lambda(f) = 0$; furthermore, f is not connected if $|A| > 2$.

Clearly, all constant functions are connected endomorphisms of $\langle A; \vee, \wedge\rangle$. Several authors focus specially on connected monounary algebras (see e.g., [6, 15]). It will be shown (Lemma 1), that any connected order-preserving mapping f of a bounded poset A has an (obviously, unique) fix-point and also, that $\lambda(f)$ is defined, even in the case when A is infinite.

We are going to investigate bounded distributive lattices (shortly, BDL) $\widehat{A} = \langle A; \vee, \wedge, 0, 1\rangle$ and connected endomorphisms of $\langle A; \vee, \wedge\rangle$. Moreover, with respect to Lemma 1, let us consider only the endomorphisms fixing the least element 0. If there is an n such that n is the maximum of all $\lambda(f)$, then we set

$$\lambda(\widehat{A}) := n.$$

It is interesting whether for each positive number k, can we find a connected endomorphism f with $\lambda(f) = k$.

Applying some results of [1, 2] we will show that if a BDL is finite, then $\lambda(\widehat{A})$ is less or equal to the length of the lattice. Also, we prove necessary and suficient conditions under which

$$\lambda(\widehat{A}) = \mathrm{length}(\widehat{A}).$$

2. Preliminaries

Lemma 1. *Let A be a bounded poset and let f be a connected order-preserving mapping of A. Then f has a unique fix-point α and $\lambda(f)$ is the greater number of* $\min\{n \in \mathbb{N} \cup \{0\} \mid f^n(1) = \alpha\}$ *and* $\min\{m \in \mathbb{N} \cup \{0\} \mid f^m(0) = \alpha\}$.

Proof. Suppose that f is connected and preserves $\leq$. Then there exist the least nonnegative integers m and n such that $f^m(0) = f^n(1) = \alpha$ and $f^m(0) \leq f^{m+1}(0)$ and $f^{n+1}(1) \leq f^n(1)$ which imply that $f(\alpha) = \alpha$.

Let k be the considered greater number. If $x \in A$, then $0 < x < 1$ yields $\alpha = f^k(0) \leq f^k(x) \leq f^k(1) = \alpha$, hence $\lambda(f) \leq k$. The equality follows from the definition of k. ∎

An algebra $\langle A; \vee, \wedge, f, 0, 1\rangle$ is called a *BDLC-algebra* if $\langle A; \vee, \wedge, 0, 1\rangle$ is a BDL and f is a connected endomorphism on $\langle A; \vee, \wedge\rangle$ fixed 0. For each $n \in \mathbb{N} \cup \{0\}$, let $\mathcal{M}_n$ be the class of all BDLC-algebras $\langle A; \vee, \wedge, f, 0, 1\rangle$ whose $\lambda(f) \leq n$ and it is shown in [1] that $\mathcal{M}_n$ is the variety satisfying the following identities:

- $f(a \vee b) \approx f(a) \vee f(b)$,
- $f(a \wedge b) \approx f(a) \wedge f(b)$,
- $f(0) \approx 0$,
- $f^n(1) \approx 0$.

For each positive integer n and BDL $\widehat{A} = \langle A; \vee, \wedge, 0, 1\rangle$, define $\underline{\mathrm{A}}^{*n} := \langle A^n; \vee, \wedge, f, \mathbf{0}, \mathbf{1}\rangle$ whose $\langle A^n; \vee, \wedge, \mathbf{0}, \mathbf{1}\rangle$ is the usual direct product of $\widehat{A}$ and $f : A^n \to A^n$ is defined by $f(a_1, a_2, \ldots, a_n) = (a_2, \ldots, a_n, 0)$ for all $a_i \in A$ and $1 \leq i \leq n$. Denote $\mathbf{0} := (\underbrace{0, \ldots, 0}_{n})$, $\mathbf{1} := (\underbrace{1, \ldots, 1}_{n})$ and $\underline{\mathrm{A}}^{*0}$ to be the trivial BDLC-algebra. In particular, if $\widehat{A}$ is the 2-element chain then we call it that an *n-cube BDLC-algebra*, denoted by $\underline{2}^{*n}$. In [2], Charoenpol and Ratanaprasert proved the following facts.

Theorem 2 [2]**.** *Let $\underline{\mathrm{A}} = \langle A; \vee, \wedge, f, 0, 1\rangle$ be a BDLC-algebra with $\lambda(f) = n$. The following are equivalent:*

1. *$\underline{\mathrm{A}}$ is a subdirectly irreducible algebra,*
2. *$0 = f^n(1) \prec f^{n-1}(1) \prec \ldots \prec f(1) \prec 1$,*
3. *$\underline{\mathrm{A}} \leq \underline{2}^{*n}$.*

Theorem 3 [2]**.** *For each $n \in \mathbb{N}$, $\mathcal{M}_n$ is a variety generated by $\underline{2}^{*n}$.*

3. A REPRESENTATION OF A BDLC-ALGEBRA

For each BDLC-algebra $\underline{\mathrm{A}}$, there is a natural number n such that $\underline{\mathrm{A}} \in \mathcal{M}_n$ which implies that $\underline{\mathrm{A}}$ is a homomorphic image of subalgebra of direct product of $\underline{2}^{*n}$.

Lemma 4. *For each $n \in \mathbb{N}$, $(\underline{2}^{*n})^I \cong (\underline{2}^I)^{*n}$.*

Proof. Define a function $\psi : (\underline{2}^{*n})^I \to (\underline{2}^I)^{*n}$ by $\psi(a) = (\pi_1 \circ a, \pi_2 \circ a, \dots, \pi_n \circ a)$ for all $a \in (\underline{2}^{*n})^I$ where $\pi_i : \{0,1\}^n \to \{0,1\}$ is the $i-$projection for all $1 \le i \le n$. It is routine to show that the mapping ψ is an isomorphism. ∎

This theorem implies that for each $\underline{\mathrm{A}} \in \mathcal{M}_n$, there exist $\underline{\mathrm{B}} \le (\underline{2}^I)^{*n}$ and homomorphism $h : \underline{\mathrm{B}} \to \underline{\mathrm{A}}$ such that $\underline{\mathrm{A}} = h(\underline{\mathrm{B}})$. So for $a, b \in \underline{\mathrm{A}}$, one can see that $a = h(\bar{a}_1, \dots, \bar{a}_n)$ and $b = h(\bar{b}_1, \dots, \bar{b}_n)$ for some $\bar{a}_i, \bar{b}_i \in \underline{2}^I$ (that is, $\bar{a}_i, \bar{b}_i : I \to \underline{2}$); and hence,

$$a \vee b = h(\bar{a}_1 \vee \bar{b}_1, \dots, \bar{a}_n \vee \bar{b}_n)$$

and

$$a \wedge b = h(\bar{a}_1 \wedge \bar{b}_1, \dots, \bar{a}_n \wedge \bar{b}_n).$$

Moreover,

$$f(a) = h(\bar{a}_2, \dots, \bar{a}_n, \bar{0}), 1_{\underline{\mathrm{A}}} = h(\bar{1}, \dots, \bar{1}) \text{ and } 0_{\underline{\mathrm{A}}} = h(\bar{0}, \dots, \bar{0})$$

where $\bar{0}$ and $\bar{1}$ are the constant function 0 and 1, respectively. Since h preserves $\le$, we have $h(\underbrace{\bar{1}, \dots, \bar{1}}_{n-j}, \underbrace{\bar{0}, \dots, \bar{0}}_{j}) \le h(\underbrace{\bar{1}, \dots, \bar{1}}_{n-j+1}, \underbrace{\bar{0}, \dots, \bar{0}}_{j-1})$ for all $1 \le j \le n$. The following theorem shows the classification of j with $h(\underbrace{\bar{1}, \dots, \bar{1}}_{n-j}, \underbrace{\bar{0}, \dots, \bar{0}}_{j}) = h(\underbrace{\bar{1}, \dots, \bar{1}}_{n-j+1}, \underbrace{\bar{0}, \dots, \bar{0}}_{j-1})$.

Theorem 5. *For each BDLC-algebra $\underline{\mathrm{A}}$ with $\lambda(f) = m$, if $h : \underline{\mathrm{B}} \to \underline{\mathrm{A}}$ is a homomorphism for some $\underline{\mathrm{B}} \le (\underline{2}^I)^{*n}$, then $h(\underbrace{\bar{1}, \dots, \bar{1}}_{n-m+i}, \underbrace{\bar{0}, \dots, \bar{0}}_{m-i}) < h(\underbrace{\bar{1}, \dots, \bar{1}}_{n-m+(i+1)}, \underbrace{\bar{0}, \dots, \bar{0}}_{m-(i+1)})$ for all $0 \le i \le m-1$ and $h(\underbrace{\bar{1}, \dots, \bar{1}}_{n-i}, \underbrace{\bar{0}, \dots, \bar{0}}_{i}) = 0_{\underline{\mathrm{A}}}$ for all $m \le i \le n$.*

Proof. Let $h : \underline{\mathrm{B}} \to \underline{\mathrm{A}}$ be a homomorphism for some $\underline{\mathrm{B}} \le (\underline{2}^I)^{*n}$ and $0 \le i \le m-1$. Suppose that $h(\underbrace{\bar{1}, \dots, \bar{1}}_{n-m+i}, \underbrace{\bar{0}, \dots, \bar{0}}_{m-i}) = h(\underbrace{\bar{1}, \dots, \bar{1}}_{n-m+(i+1)}, \underbrace{\bar{0}, \dots, \bar{0}}_{m-(i+1)})$. Since h preserves f, we get $h(\underbrace{\bar{1}, \dots, \bar{1}}_{n-m+i-1}, \underbrace{\bar{0}, \dots, \bar{0}}_{m-i+1}) = h(\underbrace{\bar{1}, \dots, \bar{1}}_{n-m+i}, \underbrace{\bar{0}, \dots, \bar{0}}_{m-i})$. By continuity in this way, this implies that $h(\underbrace{\bar{1},\ldots,\bar{1}}_{n-m+(i+1)},\underbrace{\bar{0},\ldots,\bar{0}}_{m-(i+1)}) = 0_{\underline{A}}$. So, $f^{m-(i+1)}(1_{\underline{A}}) = f^{m-(i+1)}(h(\underbrace{\bar{1},\ldots,\bar{1}}_{n})) = h(f^{m-(i+1)}(\underbrace{\bar{1},\ldots,\bar{1}}_{n})) = h(\underbrace{\bar{1},\ldots,\bar{1}}_{n-m+(i+1)},\underbrace{\bar{0},\ldots,\bar{0}}_{m-(i+1)}) = 0_{\underline{A}}$, a contradict with $\lambda(f) = m$. Therefore, $h(\underbrace{\bar{1},\ldots,\bar{1}}_{n-m+i},\underbrace{\bar{0},\ldots,\bar{0}}_{m-i}) < h(\underbrace{\bar{1},\ldots,\bar{1}}_{n-m+(i+1)},\underbrace{\bar{0},\ldots,\bar{0}}_{m-(i+1)})$.

Let $m \leq i \leq n$. Since $\lambda(f) = m$, we have $0_{\underline{A}} \leq h(\underbrace{\bar{1},\ldots,\bar{1}}_{n-i},\underbrace{\bar{0},\ldots,\bar{0}}_{i}) \leq h(\underbrace{\bar{1},\ldots,\bar{1}}_{n-m},\underbrace{\bar{0},\ldots,\bar{0}}_{m}) = 0_{\underline{A}}$ which implies that $h(\underbrace{\bar{1},\ldots,\bar{1}}_{n-i},\underbrace{\bar{0},\ldots,\bar{0}}_{i}) = 0_{\underline{A}}$. ■

Corollary 6. *For each BDLC-algebra $\underline{A}$ with $\lambda(f) = m$, there exists an $(m+1)$-element chain as a sublattice of $\widehat{A}$. Moreover, the chain is*

$$0 = h(\underbrace{\bar{1},\ldots,\bar{1}}_{n-m},\underbrace{\bar{0},\ldots,\bar{0}}_{m}) < h(\underbrace{\bar{1},\ldots,\bar{1}}_{n-m+1},\underbrace{\bar{0},\ldots,\bar{0}}_{m-1}) < \cdots < h(\bar{1},\ldots,\bar{1}) = 1.$$

4. A Pre-Period of a Finite Bounded Distributive Lattice

Now, our tools are ready to investigate $\lambda(\widehat{A})$ for any finite BDL $\widehat{A}$. Since the constant mapping $f(x) = 0$ is a connected endomorphism fixing 0 with $\lambda(f) = 1$, we obtain $\lambda(\widehat{A}) \geq 1$.

Theorem 7. *For each finite BDL $\widehat{A} = \langle A; \vee, \wedge, 0, 1\rangle$ and $k \leq \lambda(\widehat{A})$, there is a unary operation f_k on A such that $\langle A; \vee, \wedge, f_k, 0, 1\rangle$ is a BDLC-algebra with $\lambda(f_k) = k$.*

Proof. Suppose that $\lambda(\widehat{A}) = m$. Then there is a unary operation f such that $\underline{A} = \langle A; \vee, \wedge, f, 0, 1\rangle$ is a BDLC-algebra with $\lambda(f) = m$. So, $\underline{A} = h(\underline{B})$ for some $\underline{B} \leq (\underline{2}^I)^{*m}$ and homomorphism h. Let $k \leq m$, define $f_k : A \to A$ by

$$f_k(h(\bar{a}_1,\ldots,\bar{a}_m)) = h(\bar{a}_2,\ldots,\bar{a}_k,\bar{0},\ldots,\bar{0})$$

for all $(\bar{a}_1,\ldots,\bar{a}_m) \in B$. Since $\underline{B} \leq (\underline{2}^I)^{*m}$, we get

$$\begin{aligned}(\bar{a}_2,\ldots,\bar{a}_k,\bar{0},\ldots,\bar{0}) &= (\bar{a}_2,\ldots,\bar{a}_m,\bar{0}) \wedge (\underbrace{\bar{1},\ldots,\bar{1}}_{k-1},\bar{0},\ldots,\bar{0}) \\ &= f_{\underline{B}}(\bar{a}_1,\ldots,\bar{a}_m) \wedge f_{\underline{B}}^{m-k+1}(\bar{1},\ldots,\bar{1}) \in B\end{aligned}$$

for all $(\bar{a}_1,\ldots,\bar{a}_m) \in B$. So, f_k is well-defined. It is clear that $\langle A; \vee, \wedge, f_k, 0, 1\rangle$ is a BDLC-algebra with $\lambda(f_k) = k$. ■

Theorem 8. *Let $\widehat{A}$ be a finite BDL. Then*

$$\lambda(\widehat{A}) \leq \text{length}(\widehat{A}).$$

Proof. The assertion follows from Corollary 6. ■

Example 9. Let $\widehat{A} = \langle A; \vee, \wedge, 0, 1\rangle$ be a BDL which is shown as Figure 1.

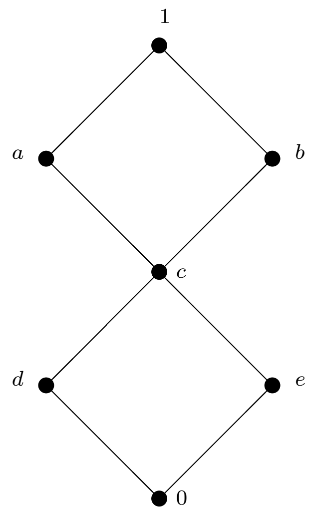

Figure 1. A bounded distributive lattice.

Due to Theorem 8, $\lambda(\widehat{A}) \leq 4$.

Suppose that $\lambda(\widehat{A}) = 4$. Then we can define f such that $\langle A; \vee, \wedge, f, 0, 1\rangle$ is a BDLC-algebra with $\lambda(f) = 4$. We may assume that $f(1) = a$, $f(a) = c$, $f(c) = d$ and $f(d) = 0$. Since $a = f(1) = f(a \vee b) = f(a) \vee f(b) = c \vee f(b)$, we get $f(b) = a$ which implies that $d = f(c) = f(a \wedge b) = f(a) \wedge f(b) = c \wedge a = c$, a contradiction. So, $\lambda(\widehat{A}) \leq 3$.

Define $f : A \to A$ by $f(1) = f(b) = c$, $f(a) = f(c) = f(e) = d$ and $f(d) = f(0) = 0$. One can see that f preserves $\wedge$, $\vee$ and 0. Hence, $\langle A; \vee, \wedge, f, 0, 1\rangle$ is a BDLC-algebra with $\lambda(f) = 3$. So, $\lambda(\widehat{A}) = 3$.

Theorem 10. *Let $\widehat{A}$ be a finite BDL. Then*

$$\lambda(\widehat{A}) = \text{length}(\widehat{A}) \textit{ if and only if } 0 = f^{\lambda(f)}(1) \prec f^{\lambda(f)-1}(1) \prec \cdots \prec f(1) \prec 1$$

for some connected endomorphism f on $\langle A; \vee, \wedge\rangle$ fixing 0.

Proof. Suppose that $\lambda(\widehat{A}) = n$ and we choose a connected endomorphism f on $\langle A; \vee, \wedge\rangle$ fixing 0 with $\lambda(f) = n$. Hence, n is the smallest natural number with

$f^n(1) = 0$. Furthermore, $C = \{1 > f(1) > \cdots > f^{n-1} > f^n(1) = 0\}$ is a chain with $|C| = n + 1$. Since $\widehat{A}$ is distributive,

$$\begin{aligned} n = \operatorname{length}(\widehat{A}) &\Leftrightarrow C \text{ is a maximal chain} \\ &\Leftrightarrow 0 = f^n(1) \prec f^{n-1}(1) \prec \cdots \prec f(1) \prec 1. \end{aligned}$$ ∎

Corollary 11. *The pre-period of the directed product* $\widehat{2}^n$ *of the 2-element chain* $\widehat{2}$ *is equal to* n *for all* $n \in \mathbb{N}$*; that is,* $\lambda_0(\widehat{2}^n) = n$.

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