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(f, g)-DERIVATION OF ORDERED TERNARY SEMIRINGS

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Abstract

In this paper, we introduce the concept of an (f, g)-derivation of ternary semirings and we study its properties in ordered ternary semirings. We prove that if d is an (f, g)-derivation of an ordered ternary semiring S, then the kernel of d is a k-ideal of S. Moreover, we show that the kernel and the set of all fixed points of d are m-k-ideals of S.

Keywords: ordered ternary semiring, derivation, integral ordered ternary semiring.

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1. Introduction and preliminaries

The notion of semirings was introduced by Vandiver [11] in 1934. The semiring theory is useful to many areas of mathematics and theoretical computer science. There are several authors investigated the relationship between the commutativity of a ring R and the existence of certain specified derivations of R. Bresar and Vukman [1] established that a prime ring admits a nonzero derivation in 1990. In the year 2016, Murali Krishna Rao and Venkateswarlu [6] introduced the notion of generalized right derivations of Γ-inclines and right derivations of ordered Γ-semirings. In 2017–2019, Murali Krishna Rao [7, 8, 9] studied ideals of an ordered Γ-semiring and introduced the concept of (f, g)-derivations which is a generalization of f-derivations and derivations of ordered semirings.

The notion of ternary algebraic structures was introduced by Lehmer [4] in 1932, but earlier such structure was studied by Kasner [3] in 1904 and Prüfer [10] in 1924. Lehmer investigated certain ternary algebraic system called triplexes. In the year 1971, Lister [5] characterized additive semigroups of rings which are closed under the triple ring product and it is called a ternary ring. In 2003, Dutta and Kar [2] introduced a notion of ternary semirings which is a generalization of ternary rings and semirings.

In this paper, we introduce the concept of (f, g)-derivations of ordered ternary semirings and study some properties of these derivations. Firstly, we will recall some of the fundamental concepts and definitions, these are necessary for this paper. A nonempty set S together with a binary operation and a ternary opearation are called addition $+$ and ternary multiplication, respectively, is said to be a ternary semiring if $(S, +)$ is a commutative semigroup satisfying the following conditions: for all $a, b, c, d, e \in S$,

(i) $(abc)de = a(bcd)e = ab(cde)$,

(ii) $(a + b)cd = acd + bcd$, $a(b + c)d = abd + acd$, $ab(c + d) = abc + abd$.

A ternary semiring S is said to have a zero element if there exists element $0 \in S$ such that $0 + a = a + 0 = a$, $0ab = a0b = ab0 = 0$ for all $a, b \in S$. A ternary semiring S is said to be commutative if $abc = acb = bac = bca = cab = cba$ for all $a, b, c \in S$. An element $a \in S$ is said to be a multiplicatively selfpotent element of S if $aaa = a$ (an additively selfpotent element if $a + a = a$). An element $1 \in S$ is said to be unity if $a11 = 1a1 = 11a = a$ for all $a \in S$.

Example 1.1. (1) Every semiring $(S, +, \cdot)$ can be considered to be a ternary semiring by the addition $+$ and ternary multiplication defined by $abc = a \cdot b \cdot c$.

(2) $\mathbb{Z}^-$ under the usual addition $+$ and ternary multiplication defined by $abc = a \cdot b \cdot c$ is a ternary semiring but $(\mathbb{Z}^-, +, \cdot)$ is not a semiring.

Then we can see that the structure of ternary semirings is a generalization of semirings.

A ternary semiring S is called an ordered ternary semiring if it admits a compatible relation $\leq$, i.e., $\leq$ is a partial order on S that satisfies the following conditions. If $a \leq b, c \leq d$ and $x, y \in S$ then

(i) $a + c \leq b + d, c + a = d + b$,

(ii) $axy \leq bxy, xay \leq xby, xya \leq xyb$.

A non-zero element a of an ordered ternary semiring S is called a zero divisor if there exist nonzero elements $b, c \in S$ such that $abc = acb = bac = bca = cab = cba = 0$. If S is an ordered ternary semiring with unity 1, zero element 0 and it has no zero divisor, then S is called an integral ordered ternary semiring.

Example 1.2. (1) Every ternary semiring can be considered to be an ordered ternary semiring by a partial order is an identity relation. Then the structure of ordered ternary semirings is a generalization of ternary semirings.

(2) The ternary semiring $\mathbb{Z}^-$ under the usual addition $+$ and ternary multiplication defined by $abc = a \cdot b \cdot c$ and the less than or equal relation $\leq$ is an ordered ternary semiring.

Let S be an ordered ternary semiring and let $a, b, c \in S$. The semigroup $(S, +)$ is said to be positively ordered, if $a \leq a + b$ and $b \leq a + b$. The ternary semigroup $(S, \cdot)$ is said to be negatively ordered, if $abc \leq a, abc \leq b$ and $abc \leq c$.

An additive subsemigroup A of an ordered ternary semiring S is called a left (resp. right, lateral) ideal of S if $xya \in A$ (resp. $axy \in A, xay \in A$) and $a \leq x$ then $a \in A$, for all $x, y \in S$ and $a \in A$. If A is a left, right, lateral ideal of S, then A is called an ideal of S. An ideal A of S is a $k-$ideal if for all $x, y \in S, x + y \in A, y \in A$, then $x \in A$.

Let S be an ordered ternary semiring. A surjective mapping $f : S \to S$ is called an endomorphism if

(i) $f(x + y) = f(x) + f(y)$,

(ii) $f(xyz) = f(x)f(y)f(z)$ for all $x, y, z \in S$.

2. Main results

In this section, we introduce the concept of an (f,g)-derivation of an ordered ternary semiring S, where f and g are endomorphisms of S. We shall always assume that $a \leq b$ if and only if $a + b = b$ for all $a, b \in S$.

Definition 2.1. Let S be a ternary semiring or an ordered ternary semiring. A mapping $d : S \to S$ is called a derivation if it satisfies

(i) $d(x + y) = d(x) + d(y)$,

(ii) $d(xyz) = d(x)yz + xd(y)z + xyd(z)$ for all $x, y, z \in S$.

Example 2.1. (1) Let $D = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is a differentiable function}\}$. Then D is a ternary semiring under the addition of functions and the ternary multiplication defined by $abc = a \cdot b \cdot c$ where $\cdot$ is the multiplication of functions.

Define $d : D \to D$ by $d(f(x)) = f'(x)$ for all $f(x) \in D$. Then d is a derivation. Note that D is a ring.

(2) Let $\mathbb{Z}_0^-[X]$ be the set of all polynomials over $\mathbb{Z}_0^-$ where $\mathbb{Z}_0^-$ is the set of all non-positive integers. Let the formal derivative d be an operation on elements of $\mathbb{Z}_0^-[X]$, where if

$$f(x) = a_n x^n + \cdots + a_1 x + a_0,$$

then its formal derivative is

$$d(f(x)) = n a_n x^{n-1} + \cdots + 2a_2 x + a_1.$$

Then d is a derivation. Note that $\mathbb{Z}_0^-[X]$ is a ternary semiring but not a semiring.

Definition 2.2. Let S be a ternary semiring or an ordered ternary semiring and f be an endomorphism on S. A mapping $d : S \to S$ is called an f-derivation if it satisfies

(i) $d(x + y) = d(x) + d(y)$,

(ii) $d(xyz) = d(x)f(y)f(z) + f(x)d(y)f(z) + f(x)f(y)d(z)$ for all $x, y, z \in S$.

Definition 2.3. Let S be a ternary semiring or an ordered ternary semiring and f, g be two endomorphisms on S. A mapping $d : S \to S$ is called an (f, g)-derivation if it satisfies

(i) $d(x + y) = d(x) + d(y)$,

(ii) $d(xyz) = d(x)f(y)f(z) + g(x)d(y)f(z) + g(x)g(y)d(z)$ for all $x, y, z \in S$.

It is easy to see that the concept of (f, g)-derivations generalizes the concepts of derivations defined in Definition 2.1 and 2.2. In this paper, we will give properties of derivations in case of ordered ternary semirings. For the case of ternary semirings, it is a special case of results in this paper.

Proposition 2.1. *Let d be an (f, g)-derivation of a selfpotent commutative ordered ternary semiring S and assume that $(S, \cdot)$ is a negatively ordered ternary semigroup. If $f(x) \leq g(x)$, then $d(x) \leq g(x)$ for all $x \in S$.*

Proof. Assume that $f(x) \leq g(x)$, for all $x \in S$. Then $f(x) + g(x) = g(x)$. Let x be a selfpotent element. Then $d(x) = d(xxx) = d(x)f(x)f(x) + g(x)d(x)f(x) + g(x)g(x)d(x) = d(x)f(x)[f(x) + g(x)] + g(x)g(x)d(x) = d(x)f(x)g(x) + g(x)g(x)d(x) = d(x)g(x)[f(x) + g(x)] = d(x)g(x)g(x) \leq g(x)$. ■

Proposition 2.2. *Let d be an (f, g)-derivation of an ordered ternary semiring S. If $f(0) = g(0) = 0$, then $d(0) = 0$.*

Proof. It is obvious. ■

Proposition 2.3. *Let S be an ordered ternary semiring and assume that $(S,+)$ is a band. Let f,g be endomorphisms on selfpotents of S. If $g(x) \leq f(x)$ for all $x \in S$, then f is an (f,g)-derivation of S.*

Proof. Let $x,y,z \in S$. Assume that $g(x) \leq f(x), g(y) \leq f(y)$ and $g(z) \leq f(z)$, i.e., $g(x)+f(x) = f(x), g(y)+f(y) = f(y)$ and $g(z)+f(z) = f(z)$. Consider

$$\begin{aligned} f(xyz) &= f(x)f(y)f(z) \\ &= f(x)f(y)f(z) + f(x)f(y)f(z) \\ &= f(x)f(y)f(z) + [g(x)+f(x)][g(y)+f(y)]f(z) \\ &= f(x)f(y)f(z) + g(x)[g(y)+f(y)]f(z) + f(x)[g(y)+f(y)]f(z) \\ &= f(x)f(y)f(z) + g(x)g(y)f(z) + g(x)f(y)f(z) + f(x)[g(y)+f(y)]f(z) \\ &= f(x)f(y)f(z) + g(x)g(y)f(z) + g(x)f(y)f(z) + f(x)f(y)f(z) \\ &= f(x)f(y)f(z) + g(x)f(y)f(z) + g(x)g(y)f(z). \end{aligned}$$

Thus f is an (f,g)-derivation of S. ■

Proposition 2.4. *Let A be a non-zero ideal of an integral ordered ternary semiring S and assume that $(S,\cdot)$ is a negatively ordered ternary semigroup. If d is a non-zero (f,g)-derivation on S with g is a non-zero function on A, then d is a non-zero (f,g)-derivation on A.*

Proof. Let d be a non-zero (f,g)-derivation on S in which g is a non-zero function on A. Assume that d is a zero (f,g)-derivation on A. Let $x \in A$ and $y \in S$. Then $d(x) = 0$ and $g(x) \neq 0$. Since $(S,\cdot)$ is a negatively ordered semigroup, $xxy \leq x$ and we have $d(xxy) \leq d(x) = 0$. Thus $0 = d(x) = d(x) + d(xxy) = 0 + d(xxy) = d(xxy) = d(x)f(x)f(y) + g(x)d(x)f(y) + g(x)g(x)d(y) = 0 + 0 + g(x)g(x)d(y) = g(x)g(x)d(y)$. This implies that $d(y) = 0$, a contradiction. Hence d is a non-zero (f,g)-derivation on A. ■

Proposition 2.5. *Let S be a selfpotent ordered ternary semiring and d be an (f,g)-derivation on S. If $d \circ d = d$ and $f \circ d = f$, then $d(xxd(x)) = d(x)$ for all $x \in S$.*

Proof. Let $x \in S$. Then

$$\begin{aligned} d(xxd(x)) &= d(x)f(x)f(d(x)) + g(x)d(x)f(d(x)) + g(x)g(x)d(d(x)) \\ &= d(x)f(x)f(x) + g(x)d(x)f(x) + g(x)g(x)d(x) = d(xxx) = d(x). \end{aligned}$$ ■

An ordered ternary semiring S is called a prime ordered ternary semiring if $aSb = 0$ implies $a = 0$ or $b = 0$.

Proposition 2.6. *Let S be a prime commutative ordered ternary semiring and A be a non-zero ideal of S. If there exists an (f,g)-derivation d on S such that $g(x) = x$ for all $x \in S$ and $d(A)Ax = 0$, then $x = 0$.*

Proof. Let $x \in S$. Suppose that $d(A)Ax = 0$. Then for all $a \in A, y \in S$, we obtain that

$$\begin{aligned} 0 &= d(aay)ax \\ &= [d(a)f(a)f(y) + g(a)d(a)f(y) + g(a)g(a)d(y)]ax \\ &= d(a)f(a)f(y)ax + g(a)d(a)f(y)ax + g(a)g(a)d(y)ax \\ &= g(a)g(a)d(y)ax = aad(y)ax. \end{aligned}$$

Let $z \in S$. Replacing y with yyz, then

$$\begin{aligned} 0 &= aad(yyz)ax \\ &= aa[d(y)f(y)f(z) + g(y)d(y)f(z) + g(y)g(y)d(z)]ax \\ &= aag(y)g(y)d(z)ax \\ &= aayyd(z)ax. \end{aligned}$$

Thus $0 = d(z)ax = ad(z)x$ and $d \neq 0$, $a \neq 0$, this implies that $x = 0$. ■

Proposition 2.7. *Let S be a prime commutative ordered ternary semiring and d be an (f,g)-derivation of S where $f \circ d = d \circ f$ and $g(x) = x$ for all $x \in S$. If $d^2 = 0$, then $d = 0$.*

Proof. Suppose that $d^2 = 0$. Let $x \in S$. Then

$$\begin{aligned} 0 &= d^2(xxx) = d[d(xxx)] \\ &= d[d(x)f(x)f(x) + g(x)d(x)f(x) + g(x)g(x)d(x)] \\ &= d[d(x)f(x)f(x)] + d[g(x)d(x)f(x)] + d[g(x)g(x)d(x)] \\ &= [d(d(x))f(f(x))f(f(x)) + g(d(x))d(f(x))f(f(x)) + g(d(x))g(f(x))d(f(x))] \\ &+ [d(g(x))f(d(x))f(f(x)) + g(g(x))d(d(x))f(f(x)) + g(g(x))g(d(x))d(f(x))] \\ &+ [d(g(x))f(g(x))f(d(x)) + g(g(x))d(g(x))f(d(x)) + g(g(x))g(g(x))d(d(x))] \\ &= d(x)d(f(x))f(f(x)) + d(x)f(x)d(f(x)) + d(x)f(d(x))f(f(x)) + xd(x)d(f(x)) \\ &+ d(x)f(x)f(d(x)) + xd(x)f(d(x)) \\ &= d(x)[f(f(x)) + f(x) + f(f(x)) + x + f(x) + x]d(f(x)). \end{aligned}$$

Since S is a prime ordered ternary semiring, $d(x) = 0$ or $0 = d(f(x)) = d(z)$ for some $z \in S$. Therefore $d = 0$. ■

Corollary 2.8. *Let d be an (f,g)-derivation on a prime commutative ordered ternary semiring S and $g(x) = x$ for all $x \in S$. If $a \in S$, $aad(x) = 0$ or $d(x)aa = 0$, then $a = 0$ or $d = 0$.*

Proof. Suppose that $aad(x) = 0$ for all $x \in S$. Let $a, x, y \in S$. Since $aad(xxy) = 0$,

$$\begin{aligned} 0 = aad(xxy) &= aa[d(x)f(x)f(y) + g(x)d(x)f(y) + g(x)g(x)d(y)] \\ &= aad(x)f(x)f(y) + aag(x)d(x)f(y) + aag(x)g(x)d(y) = a(axx)d(y). \end{aligned}$$

Hence $a = 0$ or $d = 0$. In a similar way, if $d(x)aa = 0$, then $a = 0$ or $d = 0$. ∎

Proposition 2.9. *Let d be an (f, g)-derivation of a selfpotent ordered ternary semiring S. If $d \circ d = d$ and $f \circ d = f$, then $d(xxd(xxx)) = d(x)$ for all $x \in S$.*

Proof. Assume that d is an (f, g)-derivation of a selfpotent ordered ternary semiring S such that $d \circ d = d, f \circ d = f$. Let $x \in S$. Then

$$\begin{aligned} d(xxd(xxx)) &= d(x)f(x)f(d(xxx)) + g(x)d(x)f(d(xxx)) + g(x)g(x)d(d(xxx)) \\ &= d(x)f(x)f(xxx) + g(x)d(x)f(xxx) + g(x)g(x)d(xxx) \\ &= d(x)f(x)f(x) + g(x)d(x)f(x) + g(x)g(x)d(x) = d(xxx) = d(x). \end{aligned}$$

Therefore $d(xxd(xxx)) = d(x)$. ∎

Theorem 2.10. *Let S be a commutative ordered ternary semiring and d_1, d_2 be (f, g)-derivations of S where $g \circ d_2 = g \circ d_1, d_1 \circ g = d_2 \circ g, f \circ d_2 = f \circ d_1, d_1 \circ f = d_2 \circ f, f \circ f = f$ and $g \circ g = g$. If $d_1 \circ d_2 = 0$, then $d_2 \circ d_1$ is an (f, g)-derivation of S.*

Proof. Assume $d_1 \circ d_2 = 0$. Let $x, y, z \in S$. Then

$$\begin{aligned}
&0 = d_1 \circ d_2(xyz) = d_1[d_2(xyz)] \\
&= d_1[d_2(x)f(y)f(z) + g(x)d_2(y)f(z) + g(x)g(y)d_2(z)] \\
&= [d_1(d_2(x))f(f(y))f(f(z)) + g(d_2(x))d_1(f(y))f(f(z)) + g(d_2(x))g(f(y))d_1(f(z))] \\
&+ [d_1(g(x))f(d_2(y))f(f(z)) + g(g(x))d_1(d_2(y))f(f(z)) + g(g(x))g(d_2(y))d_1(f(z))] \\
&+ [d_1(g(x))f(g(y))f(d_2(z)) + g(g(x))d_1(g(y))f(d_2(z)) + g(g(x))g(g(y))d_1(d_2(z))] \\
&= [g(d_2(x))d_2(f(y))f(f(z)) + g(d_1(x))g(f(y))d_1(f(z))] \\
&+ [d_2(g(x))f(d_1(y))f(f(z)) + g(x)g(d_1(y))d_2(f(z))] \\
&+ [d_2(g(x))f(g(y))f(d_1(z)) + g(x)d_2(g(y))f(d_2(z))] \quad (*) \\
&\text{and consider } d_2 \circ d_1(xyz) = d_2(d_1(xyz)) \\
&= d_2[d_1(x)f(y)f(z) + g(x)d_1(y)f(z) + g(x)g(y)d_1(z)] \\
&= d_2(d_1(x))f(f(y))f(f(z)) + g(d_1(x))d_2(f(y))f(f(z)) + g(d_1(x))g(f(y))d_2(f(z)) \\
&+ d_2(g(x))f(d_1(y))f(f(z)) + g(g(x))d_2(d_1(y))f(f(z)) + g(g(x))g(d_1(y))d_2(f(z)) \\
&+ d_2(g(x))f(g(y))f(d_1(z)) + g(g(x))d_2(g(y))f(d_1(z)) + g(g(x))g(g(y))d_2(d_1(z))
\end{aligned}$$

$= [g(d_1(x))d_2(f(y))f(f(z)) + g(d_1(x))g(f(y))d_2(f(z)) + d_2(g(x))f(d_1(y))f(f(z))$
$+ g(g(x))g(d_1(y))d_2(f(z)) + d_2(g(x))f(g(y))f(d_1(z)) + g(g(x))d_2(g(y))f(d_1(z))]$
$+ [d_2(d_1(x))f(f(y))f(f(z)) + g(g(x))d_2(d_1(y))f(f(z)) + g(g(x))g(g(y))d_2(d_1(z))]$
$= [g(d_2(x))d_2(f(y))f(f(z)) + g(d_1(x))g(f(y))d_1(f(z))]$
$+ [d_2(g(x))f(d_1(y))f(f(z)) + g(x)g(d_1(y))d_2(f(z))]$
$+ [d_2(g(x))f(g(y))f(d_1(z)) + g(x)d_2(g(y))f(d_2(z))]$
$+ [d_2(d_1(x))f(f(y))f(f(z)) + g(g(x))d_2(d_1(y))f(f(z)) + g(g(x))g(g(y))d_2(d_1(z))]$
$= d_2(d_1(x))f(f(y))f(f(z)) + g(g(x))d_2(d_1(y))f(f(z))$
$+ g(g(x))g(g(y))d_2(d_1(z))$ by$(*)$,
$= d_2(d_1(x))f(y)f(z) + g(x)d_2(d_1(y))f(z) + g(x)g(y)d_2(d_1(z))$.

Therefore $d_2 \circ d_1$ is an (f, g)-derivation of S. ■

Lemma 2.11. *Let d be an (f, g)-derivation of an ordered ternary semiring S with unity and assume that $(S, +)$ is a positively ordered semigroup. If $d(1) = 1$ and $g(x) = x$, then $x \leq d(x)$ for all $x \in S$.*

Proof. Let $x \in S$. Assume that $d(1) = 1$ and $g(x) = x$. Then $d(x) = d(x11) = d(x)f(1)f(1) + g(x)d(1)f(1) + g(x)g(1)d(1) \geq g(x)d(1)f(1) + g(x)g(1)d(1) \geq g(x)g(1)d(1) = x11 = x$. Thus $x \leq d(x)$. ■

Lemma 2.12. *Let d be an (f, g)-derivation of a selfpotent ordered ternary semiring S with unity and assume that $(S, \cdot)$ is a negatively ordered ternary semigroup. If $f(x) \leq x, g(x) = x$, then $d(x) \leq x$ for all $x \in S$.*

Proof. Let $x \in S$. Assume that $f(x) \leq x, g(x) \leq x$. Then we obtain that $d(x) = d(xxx) = d(x)f(x)f(x) + g(x)d(x)f(x) + g(x)g(x)d(x) \leq f(x) + f(x) + g(x) \leq x + x + x = x$. Thus $d(x) \leq x$. ■

Lemma 2.13. *Let S be a selfpotent ordered ternary semiring with unity in which a semigroup $(S, +)$ is positively ordered and a ternary semigroup $(S, \cdot)$ is negatively ordered. Let d be an (f, g)-derivation and assume that $f(x) \leq x, g(x) \leq x$ for all $x \in S$. Then $d(1) = 1$ if and only if $d(x) = x$.*

Proof. Assume that $d(1) = 1$. By Lemma 2.11, $x \leq d(x)$ and by Lemma 2.12, $d(x) \leq x$. Thus $d(x) = x$. Conversely, it is clear. ■

Lemma 2.14. *Let S be an ordered ternary semiring with unity in which a semigroup $(S, +)$ is positively ordered and a ternary semigroup $(S, \cdot)$ is negatively ordered. If d is an (f, g)-derivation such that $d(1) = 1, f(x) \leq x, g(x) = x$ for all $x \in S$, then the following statements hold for all $x, y, z \in S$.*

(i) $d(xyz) \leq d(x)$,

(ii) $d(xyz) \leq d(y)$,

(iii) $d(xyz) \leq d(z)$,

(iv) *d is an isotone derivation.*

Proof. Let $x, y, z \in S$.

(i) $d(xyz) = d(x)f(y)f(z)+g(x)d(y)f(z)+g(x)g(y)d(z) \leq d(x)yz+xd(y)z+xyd(z) \leq d(x)+x+x$. Thus by Lemma 2.11, $d(xyz) \leq d(x)+d(x)+d(x) = d(x)$. The proofs of (ii) and (iii) are similar to that of (i).

(iv) Assume that $x \leq y$. Then $x+y = y$, this implies that $d(x)+d(y) = d(y)$. Therefore $d(x) \leq d(y)$. ∎

Lemma 2.15. *Let S be a selfpotent ordered ternary semiring with unity in which a semigroup $(S,+)$ is positively ordered and a ternary semigroup $(S,\cdot)$ is negatively ordered. Let d be an (f,g)-derivation of S such that $d(1) = 1$. Then $f(x) \leq x, g(x) = x$ if and only if $d(x) = x$ for all $x \in S$.*

Proof. Let d be an (f,g)-derivation of S such that $d(1) = 1$. Assume that $f(x) \leq x$ and $g(x) = x$ for all $x \in S$. Then by Lemma 2.13, $d(x) = x$. Conversely, suppose that $d(x) = x$ for all $x \in S$. Then $x = d(x) = d(xxx) = d(x)f(x)f(x) + g(x)d(x)f(x) + g(x)g(x)d(x) = xf(x)f(x) + g(x)xf(x) + g(x)g(x)x$. Thus $x + x + x \leq f(x) + x + x$, this implies that $x \leq f(x)$. On the other hand, $xxx = x \geq xxf(x)$. Thus $x \geq f(x)$. Hence $f(x) = x$. In a similar way, we obtain that $g(x) = x$. ∎

Lemma 2.16. *Let d be an (f,g)-derivation of an ordered ternary semiring S and assume that $(S,\cdot)$ is a negatively ordered ternary semigroup. Then $ker\ d$ is a k-ideal of S.*

Proof. Let $x, y, z \in ker\ d$. Then $d(x) = d(y) = d(z) = 0$. Thus $d(x+y) = d(x)+d(y) = 0+0 = 0$ and $d(xyz) = d(x)f(y)f(z)+g(x)d(y)f(z)+g(x)g(y)d(z) = 0+0+0 = 0$. This implies that $x+y, xyz \in ker\ d$. Therefore $ker\ d$ is a ternary subsemiring of S. Let $a, b \in S$. Then $abx \leq x$, it implies that $d(abx) \leq d(x) = 0$. Hence $abx \in ker\ d$. Suppose that $a \leq x$, then $a + x = x$, $d(a+x) = d(x)$. Thus $0 = d(x) = d(a+x) = d(a)+d(x) = d(a)+0 = d(a)$. Therefore $a \in ker\ d$. Assume that $a + x \in ker\ d$. Then $d(a+x) = 0$, this implies that $0 = d(a+x) = d(a)+d(x) = d(a)+0 = d(a)$. Thus $a \in ker\ d$. Hence $ker\ d$ is a k-ideal of S. ∎

Definition 2.4. An ideal A of an ordered ternary semiring S is said to be an m-k-ideal if $abx \in A$ and $a, b \in A, x \in S$, then $x \in A$.

Theorem 2.17. *Let d be an (f,g)-derivation of an integral ordered ternary semiring S and assume that $(S,\cdot)$ is a negatively ordered ternary semigroup. Let f,g be non-zero endomorphisms of S. Then $ker\ d$ is an m-k-ideal of S.*

Proof. Let $x,y \in ker\ d$ and $z \in S$. Assume that $xyz \in ker\ d$. Then $d(xyz) = d(x)f(y)f(z) + g(x)d(y)f(z) + g(x)g(y)d(z)$, it implies that $0 = g(x)d(y)d(z)$. Since f,g are non-zero endomorphisms of S, $d(z) = 0$. Therefore $z \in ker\ d$ and by Lemma 2.15, $ker\ d$ is an ideal of S. This implies that $ker\ d$ is an m-k-ideal of S. ■

Theorem 2.18. *Let S be a selfpotent additively cancellative ordered ternary semiring with unity in which a semigroup $(S,+)$ is positively ordered and a ternary semigroup $(S,\cdot)$ is negatively ordered. Let d be an (f,g)-derivation of S and $d(1) = 1$. Define the following set*

$$Fix_d(S) := \{x \in S \mid d(x) = x\}.$$

Then $Fix_d(S)$ is an m-k-ideal of S.

Proof. Let $x,y,z \in Fix_d(S)$. Then $d(x) = x, d(y) = y, d(z) = z$, so by Lemma 2.14, $f(x) \leq x, f(y) \leq y, f(z) \leq z$ and $g(x) = x, g(y) = y, g(z) = z$. Thus we obtain $d(x+y) = d(x)+d(y) = x+y$ and $d(xyz) = d(x)f(y)f(z) + g(x)d(y)f(z) + g(x)g(y)d(z) \leq xyz + xyz + xyz$, we have $d(xyz) \leq xyz$. Since $g(xyz) = g(x)g(y)g(z) = xyz$, by Lemma 2.12, $xyz \leq d(xyz)$. Thus $d(xyz) = xyz$. Therefore $x+y, xyz \in Fix_d(S)$.

Let $a \in S$. Assume that $a \leq x$. Then $a + x = x$. We now have $d(a) + d(x) = d(x)$. So $d(a) + x = x = a + x$, this implies that $d(a) = a$. Thus $a \in Fix_d(S)$. Hence $Fix_d(S)$ is a k-ideal of S. Suppose that $xya \in Fix_d(S)$. Then by Lemma 2.14, $g(xya) = xya$. Thus $xyg(a) = g(x)g(y)g(a) = g(xya) = xya$, this implies that $g(a) = a$. By Lemma 2.11, $a \leq d(a)$. On the other hand, since by Lemma 2.14, $f(xya) \leq xya$, we have $f(xya) + xya = xya$ and $f(x) + x = x, f(y) + y = y$. Then $f(x)f(y)f(a) + [f(x)+x][f(y)+y]a = [f(x)+x][f(y)+y]a$, we obtain that $f(x)f(y)f(a) + f(x)f(y)a + f(x)ya + xf(y)a + xya = f(x)f(y)a + f(x)ya + xf(y)a + xya$. So $f(x)f(y)[f(a)+a] = f(x)f(y)a$. Thus $f(a) + a = a$. It follows that $f(a) \leq a$. Then by Lemma 2.11, $d(a) \leq a$. Hence $d(a) = a$, i.e., $a \in Fix_d(S)$. Therefore $Fix_d(S)$ is an m-k-ideal of S. ■

Note that if S is a ternary semiring with zero but not a ring, then $S[X]$, the set of all polynomials over S, is a ternary semiring but not a ring. The formal derivative over $S[X]$ is a derivation. Some properties of formal derivatives over $S[X]$ do not seem to be properties of formal derivatives of the polynomial ring. All results in this paper will hold for $S[X]$.

3. Conclusion

We have introduced the concept of (f,g)-derivations of an ordered ternary semiring S. We have studied some properties of (f,g)-derivations, such as some relations of images of ideals, selfpotent elements, zero and unity elements of S under (f,g)-derivations. Beside that, we give conditions for the composition of two (f,g)-derivations to be an (f,g)-derivation. Finally, we characterize a fixed point of an (f,g)-derivation, it follows that the kernel and the set of all fixed points of an (f,g)-derivation are m-k-ideals of S. Note that if S is an n-ary semiring with zero, then $S[X]$ will be also an n-ary semiring. Some of the issues for further study in this direction may be to study the concept of (f,g)-derivations of n-ary semirings and ordered n-ary semirings.

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