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QUASI-PRIMARY IDEALS IN COMMUTATIVE SEMIRINGS

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Abstract

In this paper, we define quasi-primary ideals in commutative semirings S with $1 \neq 0$ which is a generalization of primary ideals. A proper ideal I of a semiring S is said to be a quasi-primary ideal of S if $ab \in \sqrt{I}$ implies $a \in \sqrt{I}$ or $b \in \sqrt{I}$. We also introduce the concept of 2-absoring quasi-primary ideal of a semiring S which is a generalization of quasi-primary ideal of S. A proper ideal I of a semiring S is said to be a 2-absorbing quasi-primary ideal if $abc \in \sqrt{I}$ implies $ab \in \sqrt{I}$ or $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$. Some basic results related to 2-absorbing quasi-primary ideal have also been given.

Keywords: semiring, subtractive ideal, primary ideal, quasi-primary ideal, 2-absorbing quasi-primary ideal, Q-ideal.

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1. Introduction

The algebraic structure of semiring plays a prominent role in various branches of mathematics as well as some other branches of applied science. The concept of semiring was first introduced by Vandiver [11] in 1934 and has since then been studied by many authors. The structure of prime ideals in semiring theory has gained importance and many mathematicians have exploited its usefulness in algebraic systems over the decades.

A commutative semiring is a commutative semigroup $(S, \cdot)$ and a commutative monoid $(S, +, 0_S)$ in which 0_S is the additive identity and $0_S \cdot x = x \cdot 0_S = 0_S$ for all $x \in S$, both are connected by ring like distributivity. A non-empty subset I of a semiring S is called an ideal of S if $a, b \in I$ and $r \in S$, $a + b \in I$ and $ra, ar \in I$. An ideal I of a semiring S is called subtractive if $a, a + b \in I$, $b \in S$, then $b \in I$. Let I be an ideal of S, then $(I : x) = \{a \in S : ax \in I\}$. Let I be an ideal of S. Radical of I is defined as $\mathrm{Rad}(I) = \sqrt{I} = \{a \in S : a^n \in I$ for some positive integer $n\}$. Annihilator of an element a of a semiring S is defined as $Ann(a) = \{x \in S : ax = 0\}$. The notion of quasi-primary ideals in commutative rings were introduced by Fuchs in [8]. An ideal I of a ring R is called a quasi-primary ideal if $\sqrt{I}$ is a prime ideal in R. In this paper, we introduce quasi-primary ideals in commutative semirings and some properties of it. A proper ideal I of a semiring S is said to be quasi-primary ideal of S if $\sqrt{I}$ is a prime ideal of S. We also introduce the concept of 2-absorbing quasi primary ideal of a semiring S which is a generalization of quasi-primary ideal of S. A proper ideal I of a semiring S is said to be a 2-absorbing quasi-primary ideal if $abc \in \sqrt{I}$ implies $ab \in \sqrt{I}$ or $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$. Throughout this paper, semiring S is considered as commutative with identity $1 \neq 0$.

2. QUASI-PRIMARY IDEALS

In this section, we introduce the concept of quasi-primary ideal of a semiring S and prove some results related to it.

Definition 2.1. Let S be a commutative semiring and I be a proper ideal of S. Then I is said to be quasi-primary ideal of S if $\sqrt{I}$ is a prime ideal of S.

Example 2.2. Let $n \geq 2$, $n \in N$ and $0 \leq i \leq n$ and $m = n - i$. Then $B(n, i) = \{0, 1, 2, \ldots, n-1\}$ forms a semiring under the following operations:

$$x +_{B(n,i)} y = \begin{cases} x + y & \text{if } x + y \leq n - 1 \\ l & \text{if } x + y \geq n \\ & \text{where } l \equiv (x + y) \bmod m \\ & \text{and } i \leq l \leq n - 1 \end{cases}$$

$$x_{B(n,i)} y = \begin{cases} xy & \text{if } xy \leq n - 1 \\ l & \text{if } xy \geq n \\ & \text{where } l \equiv (xy) \bmod m \\ & \text{and } i \leq l \leq n - 1. \end{cases}$$

If we take $S = B(4, 3) = \{0, 1, 2, 3\}$ and $I = \{0, 3\}$ be an ideal of S. Then its $\sqrt{I} = \{0, 2, 3\}$ is a prime ideal of S.

Example 2.3. Consider a semiring $S = Z_0^+$ under usual addition and multiplication. Take an ideal $I = 4Z_0^+ = \{0, 4, 8, 16, \dots\}$ of S. Then $\sqrt{I}$ is a prime ideal of S.

Result 2.4. A proper ideal I of a semiring S is a quasi-primary ideal of S if and only if whenever $a, b \in S$ and $ab \in I$, then $a \in \sqrt{I}$ or $b \in \sqrt{I}$.

Proof. Let I be a quasi-primary ideal of S and $ab \in I \subseteq \sqrt{I}$. Since $\sqrt{I}$ is a prime ideal of S, we have $a \in \sqrt{I}$ or $b \in \sqrt{I}$. Conversely, let I be a proper ideal of S and $ab \in I$ for some $a, b \in S$, then either $a \in \sqrt{I}$ or $b \in \sqrt{I}$. Suppose that $ab \in \sqrt{I}$ but $a \notin \sqrt{I}$. Since $ab \in \sqrt{I}$, therefore for some positive integer n, $a^n b^n \in I$. Since $a^n \notin \sqrt{I}$, we have $b^n \in \sqrt{I}$, that is , $b \in \sqrt{I}$. Hence I is a quasi-primary ideal of S. ■

Theorem 2.5. *Let $f : S \mapsto S'$ be a homomorphism of commutative semirings. Then, if I' is a quasi-primary ideal of S', then $f^{-1}(I')$ is a quasi-primary ideal of S.*

Proof. Let $ab \in f^{-1}(\sqrt{I'})$ for some $a, b \in S$. Then $f(ab) \in \sqrt{I'}$, that is, $f(a)f(b) \in \sqrt{I'}$. Since I' is a quasi-primary ideal of S', therefore $\sqrt{I'}$ is a prime ideal of S'. Therefore, $f(a) \in \sqrt{I'}$ or $f(b) \in \sqrt{I'}$. Hence, $a \in f^{-1}(\sqrt{I'})$ or $b \in f^{-1}(\sqrt{I'})$. Since $f^{-1}(\sqrt{I'}) = \sqrt{f^{-1}(I')}$, we have $f^{-1}(I')$ is a quasi-primary ideal of S. ■

Definition 2.6 ([3], Definition 1(i))**.** A proper ideal I of a semiring S is said to be strong ideal, if for each $a \in I$ there exists $b \in I$ such that $a + b = 0$.

Proposition 2.7. Let S and S' be semirings, $f : S \mapsto S'$ be an epimorphism such that $f(0) = 0$ and I be a subtractive and strong ideal of S. If I is a quasi-primary ideal of S such that $kerf \subseteq I$, then $f(I)$ is a quasi-primary ideal of S'.

Proof. Let $a, b \in S'$ be such that $ab \in f(I)$. Then there exists an element $m \in I \subseteq \sqrt{I}$ such that $ab = f(m)$. Since f is an epimorphism, therefore there exist $p, q \in S$ such that $f(p) = a$, $f(q) = b$. Also, since I is a strong ideal of S and $m \in I$, therefore there exists $n \in I$ such that $m + n = 0$. This implies $f(n + m) = 0$, that is, $f(pq + n) = 0$, implies $pq + n \in kerf \subseteq I$. Since, $n \in I$ and I is a subtractive ideal of S, we have $pq \in I$. Since $\sqrt{I}$ is a prime ideal of S, therefore either $p \in \sqrt{I}$ or $q \in \sqrt{I}$. This gives, $p^n \in I$ or $q^m \in I$ for some positive integers n, m. This gives, $f(p^n) \in f(I)$ or $f(q^m) \in f(I)$. Therefore, $f(p)^n \in f(I)$ or $f(q)^m \in f(I)$ for some $n, m \in Z^+$. Hence $a = f(p) \in \sqrt{f(I)}$ or $b = f(q) \in \sqrt{f(I)}$. Thus, $f(I)$ is a quasi-primary ideal of S' (by Result 2.4). ■

Theorem 2.8. *If I is a quasi-primary ideal of a semiring S, then the following holds:*

(i) $(\sqrt{I} : x)$ *is a quasi-primary ideal of S for all $x \in S \setminus \sqrt{I}$.*

(ii) $(\sqrt{I}:x)=(\sqrt{I}:x^2)$ *for all* $x\in S\setminus\sqrt{I}$.

Proof. (i) Let $a,b\in S$ be such that $ab\in(\sqrt{I}:x)$. Then $abx\in\sqrt{I}$. Since I is quasi-primary, therefore $\sqrt{I}$ is a prime ideal of S. Therefore, either $a\in\sqrt{I}$ or $b\in\sqrt{I}$, since $x\notin\sqrt{I}$. This implies, $ax\in\sqrt{I}$ or $bx\in\sqrt{I}$. Hence $a\in(\sqrt{I}:x)\subseteq\sqrt{(\sqrt{I}:x)}$ or $a\in(\sqrt{I}:x)\subseteq\sqrt{(\sqrt{I}:x)}$.

(ii) It is clear that $(\sqrt{I}:x)\subseteq(\sqrt{I}:x^2)$. Let $y\in(\sqrt{I}:x^2)$. Then $x^2y\in\sqrt{I}$. Therefore, $y\in\sqrt{I}$, since $x\notin\sqrt{I}$ and $\sqrt{I}$ is a prime ideal. Thus, $xy\in\sqrt{I}$. Hence $y\in(\sqrt{I}:x)$ and we are done. ■

Consider $S=S_1\times S_2$ where each S_i, $i=1,2$ is a commutative semiring with unity and $(a_1,a_2)(b_1,b_2)=(a_1b_1,a_2b_2)$ for all $a_1,b_1\in S_1$ and $a_2,b_2\in S_2$.

Proposition 2.9. Let I be a proper ideal of a semiring S_1. Then the following statements are equivalent:

(i) I is a quasi-primary ideal of S_1.

(ii) $I\times S_2$ is a quasi-primary ideal of $S=S_1\times S_2$.

Proof. (i)⇒(ii) Let $(a_1,a_2),(b_1,b_2)\in S$ be such that $(a_1,a_2)(b_1,b_2)\in I\times S_2$. Then $(a_1b_1,a_2b_2)\in I\times S_2$ implies $a_1b_1\in I\subseteq\sqrt{I}$. This gives, either $a_1\in\sqrt{I}$ or $b_1\in\sqrt{I}$, that is, either ${a_1}^l\in I$ or ${b_1}^m\in I$ for some positive integers l,m, since I is quasi-primary ideal of S_1. If ${a_1}^l\in I$ for some positive integer l, then $({a_1}^l,{a_2}^l)\in I\times S_2$. If ${b_1}^m\in I$ for some positive integer m, then $(b_1^m,{b_2}^m)\in I\times S_2$. Hence, $I\times S_2$ is a quasi-primary ideal of S (by Result 2.4).

(ii)⇒(i) Let $ab\in I$ for some $a,b\in S_1$. Then for each $r_1,r_2\in S_2$, we have $(a,r_1)(b,r_2)\in I\times S_2$. By Result 2.4, we have either $(a^l,{r_1}^l)\in I\times S_2$ or $(b^m,{r_2}^m)\in I\times S_2$ for some positive integers l,m, since $I\times S_2$ is quasi-primary ideal of S. That is, either $a^l\in I$ or $b^m\in I$ for some positive integers l,m. This shows that I is a quasi-primary ideal of S_1 (by Result 2.4). ■

Theorem 2.10. *Let S be a regular semiring. Then every irreducible ideal I of S is a quasi-primary ideal of S.*

Proof. Let S be a regular semiring and I be an irreducible ideal of S. Let $ab\in I$ for some $a,b\in S$. We need to show that either $a\in\sqrt{I}$ or $b\in\sqrt{I}$. On contrary, we assume that $a^m\notin I$ and $b^n\notin I$ for some positive integers m,n. Then, $H=(I+<a^m>)$ and $K=(I+<b^n>)$ are two ideals of S properly contained in I. Since I is irreducible, therefore $I\neq H\cap K$. Thus, there exists $p\in S$ such that $p\in(I+<a^m>)\cap(I+<b^n>)\setminus I$. Also, by regularity of S, we have $H\cap K=HK$, therefore $p\in(I+<a^m>)(I+<b^n>)\setminus I$. Then, there are $p_1,p_2\in I$ and $r_1,r_2\in S$ such that $p=(p_1+r_1a^m)(p_2+r_2b^n)=p_1p_2+r_1a^mp_2+p_1r_2b^n+r_1r_2a^mb^n$. This implies that $p\in I$, which is a contradiction. Hence I is a quasi-primary ideal of S. ■

Theorem 2.11. *Let S be a noetherian semiring and I be a proper ideal of S such that $\sqrt{I}$ is subtractive and irreducible. Then I is quasi-primary.*

Proof. Let I be a proper ideal of S such that $\sqrt{I}$ is subtractive and irreducible. Let $ab \in \sqrt{I}$ and $a \notin \sqrt{I}$. Consider the ascending chain $(\sqrt{I} : a) \subseteq (\sqrt{I} : a^n) \subseteq \cdots \subseteq \cdots$. Since S is noetherian, there exists an element n such that $(\sqrt{I} : a^{n+1}) = (\sqrt{I} : a^n)$. We show that $\sqrt{I} = (\sqrt{I} : a) \cap (\sqrt{I} + Sa^n)$. Let $r \in (\sqrt{I} : a) \cap (\sqrt{I} + Sa^n)$. This gives, $r \in (\sqrt{I} : a)$ and $r \in \sqrt{I} + Sa^n$. Therefore, $ar \in \sqrt{I}$ and $r = i + sa^n$ where $i \in \sqrt{I}$ and $s \in S$. This gives, $ra = ia + sa^{n+1}$ and hence $sa^{n+1} \in \sqrt{I}$, since $\sqrt{I}$ is subtractive. Thus, $s \in (\sqrt{I} : a^{n+1}) = (\sqrt{I} : a^n)$. Now, $r = i + sa^n$ and hence $r \in \sqrt{I}$. Thus, $\sqrt{I} = (\sqrt{I} : a) \cap (\sqrt{I} + Sa^n)$. Since $\sqrt{I}$ is irreducible and $a \notin \sqrt{I}$, we have $\sqrt{I} \neq \sqrt{I} + Sa^n$. This gives, $\sqrt{I} = (\sqrt{I} : a)$. Since $ab \in \sqrt{I}$, therefore, $b \in \sqrt{I}$. ■

Definition 2.12 ([1], Definition 4)**.** An ideal I of a semiring S is called a Q-ideal (partitioning ideal) if there exists a subset Q of S such that

(i) $S = \cup\{q + I : q \in Q\}$

(ii) If $q_1, q_2 \in Q$, then $(q_1 + I) \cap (q_2 + I) \neq \emptyset \Leftrightarrow q_1 = q_2$.

Let I be a Q-ideal of a semiring S. Then $S/I_Q = \{q + I : q \in Q\}$ forms a semiring under the following addition '$\oplus$' and multiplication '$\odot$', $(q_1 + I) \oplus (q_2 + I) = q_3 + I$, where $q_3 \in Q$ is unique such that $q_1 + q_2 + I \subseteq q_3 + I$ and $(q_1+I) \odot (q_2+I) = q_4+I$, where $q_4 \in Q$ is unique such that $q_1q_2+I \subseteq q_4+I$. This semiring S/I_Q is called the quotient semiring of S and denoted by $(S/I_Q, \oplus, \odot)$ or S/I_Q. By definition of Q-ideal, there exists a unique $q_0 \in Q$ such that $0 + I \subseteq q_0 + I$. Then $q_0 + I$ is a zero element of S/I_Q. Clearly, if S is commutative, then S/I_Q is commutative.

Theorem 2.13. *Let I be a Q-ideal of S and P a subtractive ideal of S such that $I \subseteq P$. Then P is a quasi-primary ideal of S if and only if $P/I_{Q\cap P}$ is a quasi-primary ideal of S/I_Q.*

Proof. Let P be a quasi-primary ideal of S. Then $\sqrt{P}$ is a prime ideal of S. Suppose that $q_1 + I, q_2 + I \in S/I_Q$ are such that $(q_1 + I) \odot (q_2 + I) \in P/I_{Q\cap P}$. Therefore, $q_1q_2 + I \subseteq q_3 + I \in P/I_{Q\cap P}$ where $q_3 \in Q \cap P$ is a unique element. So $q_1q_2 = q_3 + i$ for some $i \in I$. Since $\sqrt{P}$ is a prime ideal of S and $q_1q_2 \in P \subseteq \sqrt{P}$, therefore either $q_1 \in \sqrt{P}$ or $q_2 \in \sqrt{P}$. Thus, either ${q_1}^l \in P$ or ${q_2}^m \in P$ for some $l, m \in Z^+$. First suppose that ${q_1}^l \in P$. Suppose that ${q_1}^l + I = q + I$ with q such that ${q_1}^l + I \subseteq q + I$, with $q \in Q$. Therefore, ${q_1}^l = q + i \in P$ and P is subtractive, it gives $q \in P$. Thus, $Q \cap P$ is non-empty. This gives ${q_1}^l + I \in P/I_{Q\cap P}$. Similarly, ${q_2}^m + I \in P/I_{Q\cap P}$. Thus, $P/I_{Q\cap P}$ is a quasi-primary ideal of S/I_Q.

Conversely, let $P/I_{Q\cap P}$ be a quasi-primary ideal of S/I_Q. Let $ab \in P$ for some $a, b \in S$. Since I is a Q-ideal of S, therefore there exist $q_1, q_2, q_3 \in Q$ such that $a \in q_1+I$, $b \in q_2+I$. Now, $ab \in (q_1+I)\odot(q_2+I) = q_3+I$. So, $ab = q_3+i \in P$ for some $i \in I$. Since P is a subtractive ideal of S and $I \subseteq P$, we have $q_3 \in P$. So, $(q_1 + I) \odot (q_2 + I) = q_3 + I \in P/I_{Q\cap P}$. This implies, $q_1 + I \in \sqrt{P/I_{Q\cap P}}$ or $q_2+I \in \sqrt{P/I_{Q\cap P}}$, that is, $(q_1+I)^{m_1} \in P/I_{Q\cap P}$ or $(q_2+I)^{m_2} \in P/I_{Q\cap P}$ for some positive integer m_1, m_2. If $q_1{}^{m_1} + I \in P/I_{Q\cap P}$, then $a^{m_1} \in (q_1{}^{m_1} + I) \in P/I_{Q\cap P}$. Thus $a^{m_1} \in P$. Similarly, $b^{m_2} \in P$. Hence, P is a quasi-primary ideal of S. ∎

Theorem 2.14. *Let S be a semiring, I a Q-ideal of S and P a subtractive ideal of S such that $I \subseteq P$. If I and $P/I_{Q\cap P}$ are quasi-primary ideals of S and S/I_Q respectively, then P is a quasi-primary ideal of S.*

Proof. Let $a, b \in S$ be such that $ab \in P$. If $ab \in I \subseteq \sqrt{I}$, then either $a \in \sqrt{I} \subseteq \sqrt{P}$ or $b \in \sqrt{I} \subseteq \sqrt{P}$, since I is a quasi-primary ideal of S. So, assume that $ab \notin I$. Then there are elements $q_1, q_2 \in Q$ such that $a \in q_1 + I$, $b \in q_2 + I$. Therefore, for some $i_1, i_2 \in I$, $a = q_1 + i_1$, $b = q_2 + i_2$. As $ab = q_1q_2 + q_1i_2 + q_2i_1 + i_1i_2 \in P$ and since P is subtractive, we have $q_1q_2 \in P$. Consider, $(q_1 + I) \odot (q_2 + I) = q_3 + I$ where q_3 is the unique element such that $q_1q_2 + I \subseteq q_3 + I$. Since P is subtractive, we have $q_3 \in P \cap Q$, hence $q_1q_2 + I \subseteq q_3 + I \in P/I_{Q\cap P}$, that is, $(q_1 + I) \odot (q_2 + I) \in P/I_{Q\cap P}$. This gives, either $q_1 + I \in \sqrt{P/I_{Q\cap P}}$ or $q_2 + I \in \sqrt{P/I_{Q\cap P}}$, since $P/I_{Q\cap P}$ is quasi-primary ideal of S/I_Q. Thus, either $a^l \in q_1{}^l + I \in P/I_{Q\cap P}$ or $b^m \in q_2{}^m + I \in P/I_{Q\cap P}$ for some positive integers l, m. Thus, either $a^l \in P$ or $b^m \in P$ and hence P is a quasi-primary ideal of S (by Result 2.4). ∎

3. 2-absorbing quasi-primary ideals

In this section, we introduce the concept of 2-absorbing quasi-primary ideal of a semiring and prove some results related to the same.

Definition 3.1. Let S be a commutative semiring and I be a proper ideal of S. Then I is said to be a 2-absorbing quasi-primary ideal of S if $\sqrt{I}$ is a 2-absorbing ideal of S.

Taking $n = 10$ and $i = 7$ in Example 2.2, we get $S = B(10, 7) = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Let $I = \{0, 3, 6, 9\}$ be an ideal of S. Then it is easy to check that $\sqrt{I} = \{0, 3, 6, 9\}$ is a 2-absorbing ideal of S.

Proposition 3.2. A proper ideal I of S is a 2-absorbing quasi-primary ideal of S if and only if whenever $a, b, c \in S$ and $abc \in I$, then $ab \in \sqrt{I}$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$.

Proof. Let I be a proper ideal of S and let for some $a, b, c \in S$, $abc \in I$, we have $ab \in \sqrt{I}$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. Let $a, b, c \in S$ such that $abc \in \sqrt{I}$, $ac \notin \sqrt{I}$ and $bc \notin \sqrt{I}$. Since $abc \in \sqrt{I}$, there exists a positive integer l such that $(abc)^l = a^l b^l c^l \in I$. Since $a^l c^l \notin \sqrt{I}$ and $b^l c^l \notin \sqrt{I}$, we conclude that $a^l b^l = (ab)^l \in \sqrt{I}$ and so $ab \in \sqrt{I}$. Thus $\sqrt{I}$ is a 2-absorbing ideal of S. Hence I is a 2-absorbing quasi-primary ideal of S. Conversely, let I be a 2-absorbing quasi-primary ideal of S and $a, b, c \in S$ such that $abc \in I$. Since $I \subseteq \sqrt{I}$ and $\sqrt{I}$ is a 2-absorbing ideal of S, it is clear that $ab \in \sqrt{I}$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. ■

Proposition 3.3. If I is a quasi-primary ideal of S, then I is a 2-absorbing quasi primary ideal of S.

Proof. Since I is a quasi-primary ideal of S, therefore $\sqrt{I}$ is a prime ideal of S. Hence $\sqrt{I}$ is a 2-absorbing ideal of S. ■

Theorem 3.4. *Let I be a 2-absorbing quasi-primary ideal of S and let P, P_1, P_2 are prime ideals of S.*

(i) *If $\sqrt{I} = P$, then $(I : x)$ is a 2-absorbing quasi-primary ideal of S for all $x \notin P$.*

(ii) *If $\sqrt{I} = P_1 \cap P_2$, then $(I : x)$ is a 2-absorbing quasi-primary ideal of S for all $x \notin P_1 \cup P_2$.*

Proof. (i) Let $x \notin P$. If $\sqrt{I} = P$, then it is easy to see that $\sqrt{(I : x)} = P$. Hence $(I : x)$ is a quasi-primary ideal and so 2-absorbing quasi-primary ideal of S.

(ii) Let $x \notin P_1 \cup P_2$ and $a, b, c \in S$ such that $abc \in (I : x)$, then $(abc)x \in I \subseteq \sqrt{I} = P_1 \cap P_2$. Since $x \notin P_1 \cup P_2$ and P_1, P_2 are prime ideals of S, then $abc \in P_1 \cap P_2 = \sqrt{I}$. Since $\sqrt{I}$ is a 2-absorbing ideal of S, then we have $ab \in \sqrt{I}$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. This gives, $(ab)^l \in I$ or $(ac)^m \in I$ or $(bc)^n \in I$ for some positive integers l, m, n, thus $(ab)^l x \in I$ or $(ac)^m x \in I$ or $(bc)^n x \in I$. It gives, $(ab)^l \in (I : x)$ or $(ac)^m \in (I : x)$ or $(bc)^n \in (I : x)$. Hence $(I : x)$ is a 2-absorbing quasi-primary ideal of S. ■

Proposition 3.5. Let $\sqrt{I}$ be a subtractive ideal of S and I be a 2-absorbing quasi-primary ideal of S and suppose that $abJ \subseteq I$ for some elements $a, b \in S$ and some ideal J of S. If $aJ \nsubseteq \sqrt{I}$ and $bJ \nsubseteq \sqrt{I}$, then $ab \in \sqrt{I}$.

Proof. Suppose that $ab \notin \sqrt{I}$. Since $aJ \nsubseteq \sqrt{I}$ and $bJ \nsubseteq \sqrt{I}$, then $aj_1 \notin \sqrt{I}$ and $bj_2 \notin \sqrt{I}$ for some $j_1, j_2 \in J$. Since $abj_1 \in I$ and $ab \notin \sqrt{I}$ and $aj_1 \notin \sqrt{I}$, we have $bj_1 \in \sqrt{I}$. Since $abj_2 \in I$ and $ab \notin \sqrt{I}$ and $bj_2 \notin \sqrt{I}$, we have $aj_2 \in \sqrt{I}$. Since $ab(j_1 + j_2) \in I$ and $ab \notin \sqrt{I}$, then we have $a(j_1 + j_2) \in \sqrt{I}$ or $b(j_1 + j_2) \in \sqrt{I}$. Suppose that $a(j_1 + j_2) = aj_1 + aj_2 \in \sqrt{I}$. Since $aj_2 \in \sqrt{I}$, we have $aj_1 \in \sqrt{I}$, a contradiction. Suppose that $b(j_1 + j_2) = bj_1 + bj_2 \in \sqrt{I}$. Again, $bj_1 \in \sqrt{I}$, we have $bj_2 \in \sqrt{I}$, a contradiction again. Thus $ab \in \sqrt{I}$. ■

Theorem 3.6. *Let I be a proper subtractive ideal of S and suppose that $\sqrt{I}$ is a subtractive ideal of S. Then I is a 2-absorbing quasi-primary ideal of S if and only if whenever $I_1I_2I_3 \subseteq I$ for some ideals I_1 I_2, I_3 of S, then either $I_1I_2 \subseteq \sqrt{I}$ or $I_2I_3 \subseteq \sqrt{I}$ or $I_3I_1 \subseteq \sqrt{I}$.*

Proof. Proof is similar to the proof of (Theorem 3.4, [9]). ∎

Theorem 3.7. *Let I be a Q-ideal of S and P a subtractive ideal of S such that $I \subseteq P$. Then P is a 2-absorbing quasi-primary ideal of S if and only if $P/I_{Q\cap P}$ is a 2-absorbing quasi-primary ideal of S/I_Q.*

Proof. Let P be a 2-absorbing quasi-primary ideal of S. Then $\sqrt{P}$ is a 2-absorbing ideal of S. Suppose that $q_1 + I, q_2 + I, q_3 + I \in S/I_Q$ are such that $(q_1 + I) \odot (q_2 + I) \odot (q_3 + I) \in P/I_{Q\cap P}$ such that $q_1q_2q_3 + I \subseteq q_4 + I \in P/I_{Q\cap P}$ where $q_4 \in Q \cap P$ is a unique element. Let $q_1q_2q_3 = q_4 + i$ for some $i \in I$. Since $\sqrt{P}$ is a 2-absorbing ideal of S and $q_1q_2q_3 \in \sqrt{P}$, therefore either $q_1q_2 \in \sqrt{P}$ or $q_2q_3 \in \sqrt{P}$ or $q_1q_3 \in \sqrt{P}$. Thus, either ${q_1}^l{q_2}^l \in P$ or ${q_2}^m{q_3}^m \in P$ or ${q_3}^r{q_1}^r \in P$ for some $l, m, r \in Z^+$. If ${q_1}^l{q_2}^l \in P$, then $({q_1}^l + I) \odot ({q_2}^l + I) \in P/I_{Q\cap P}$ (as explained in Theorem 2.13). Similarly, $({q_2}^m + I) \odot ({q_3}^m + I) \in P/I_{Q\cap P}$ or $({q_3}^r + I) \odot ({q_1}^r + I) \in P/I_{Q\cap P}$. Thus, $P/I_{Q\cap P}$ is a 2-absorbing quasi-primary ideal of S/I_Q.

Conversely, let $P/I_{Q\cap P}$ be a 2-absorbing quasi-primary ideal of S/I_Q. Let $abc \in P$ for some $a, b, c \in S$. Since I is a Q-ideal of S, therefore there exist $q_1, q_2, q_3, q_4 \in Q$ such that $a \in q_1 + I$, $b \in q_2 + I$ and $c \in q_3 + I$. Now, $abc \in (q_1 + I) \odot (q_2 + I) \odot (q_3 + I) = q_4 + I$. So, $abc = q_4 + i \in P$ for some $i \in I$. Since P is a subtractive ideal of S and $I \subseteq P$, we have $q_4 \in P$. So, $(q_1 + I) \odot (q_2 + I) \odot (q_3 + I) = q_4 + I \in P/I_{Q\cap P}$. This implies, $(q_1 + I) \odot (q_2 + I) \in \sqrt{P/I_{Q\cap P}}$ or $(q_2 + I) \odot (q_3 + I) \in \sqrt{P/I_{Q\cap P}}$ or $(q_3 + I) \odot (q_1 + I) \in \sqrt{P/I_{Q\cap P}}$, that is, $(q_1 + I)^{m_1} \odot (q_2 + I)^{m_1} \in P/I_{Q\cap P}$ or $(q_2 + I)^{m_2} \odot (q_3 + I)^{m_2} \in P/I_{Q\cap P}$ or $(q_3 + I)^{m_3} \odot (q_1 + I)^{m_3} \in P/I_{Q\cap P}$ for some positive integers m_1, m_2, m_3. If $(q_1+I)^{m_1} \odot (q_2+I)^{m_1} \in P/I_{Q\cap P}$, then $a^{m_1}b^{m_1} \in ({q_1}^{m_1}+I) \odot ({q_2}^{m_1}+I) \in P/I_{Q\cap P}$. Thus $a^{m_1}b^{m_1} \in P$. Similarly, $b^{m_2}c^{m_2} \in P$ or $a^{m_3}c^{m_3} \in P$. Hence, P is a 2-absorbing quasi-primary ideal of S. ∎

Theorem 3.8. *Let S be a semiring, I a Q-ideal of S and P a subtractive ideal of S such that $I \subseteq P$. If I and $P/I_{Q\cap P}$ are 2-absorbing quasi-primary ideals of S and S/I_Q respectively, then P is a 2-absorbing quasi-primary ideal of S.*

Proof. Let $a, b, c \in S$ be such that $abc \in P$. If $abc \in I \subseteq \sqrt{I}$, then either $ab \in \sqrt{I} \subseteq \sqrt{P}$ or $bc \in \sqrt{I} \subseteq \sqrt{P}$ or $ac \in \sqrt{I} \subseteq \sqrt{P}$, since I is a 2-absorbing quasi-primary ideal of S. So, assume that $abc \notin I$. Then there are elements $q_1, q_2, q_3 \in Q$ such that $a \in q_1 + I$, $b \in q_2 + I$ and $c \in q_3 + I$. Therefore, for some $i_1, i_2, i_3 \in I$, $a = q_1 + i_1$, $b = q_2 + i_2$ and $c = q_3 + i_3$. As $abc = q_1q_2q_3 + q_1q_3i_2 +$

$q_2q_3i_1 + q_3i_1i_2 + q_1q_2i_3 + q_1i_2i_3 + q_2i_1i_3 + i_1i_2i_3 \in P$ and since P is subtractive, we have $q_1q_2q_3 \in P$. Consider, $(q_1 + I) \odot (q_2 + I) \odot (q_3 + I) = q_4 + I$ where q_4 is the unique element such that $q_1q_2q_3 + I \subseteq q_4 + I$. Since P is subtractive, we have $q_4 \in P \cap Q$, hence $q_1q_2q_3+I \subseteq q_4+I \in P/I_{Q\cap P}$, that is, $(q_1+I)\odot(q_2+I)\odot(q_3+I) \in P/I_{Q\cap P}$. This gives, either $(q_1 + I) \odot (q_2 + I) \in \sqrt{P/I_{Q\cap P}}$ or $(q_2 + I) \odot (q_3 + I) \in \sqrt{P/I_{Q\cap P}}$ or $(q_1 + I) \odot (q_3 + I) \in \sqrt{P/I_{Q\cap P}}$, since $P/I_{Q\cap P}$ is 2-absorbing quasi-primary ideal of S/I_Q. Thus, either $a^lb^l \in ({q_1}^l + I) \odot ({q_2}^l + I) \in P/I_{Q\cap P}$ or $b^mc^m \in ({q_2}^m + I) \odot ({q_3}^m + I) \in P/I_{Q\cap P}$ or $a^rc^r \in ({q_1}^r + I) \odot ({q_3}^r + I) \in P/I_{Q\cap P}$ for some positive integers l, m, r. Thus, either $a^lb^l \in P$ or $b^mc^m \in P$ or $a^rc^r \in P$ and hence P is a 2-absorbing quasi-primary ideal of S. ■

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