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THE AUTOMORPHISMS HAVING THE EXTENSION PROPERTY IN A CATEGORY OF A FINITE DIRECT SUM OF CYCLIC MODULES

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Abstract

It is well known that the problem of characterizing the automorphisms, in the category of abelian groups, with the extension property is resolved [1]. But in other categories, it is a very difficult problem. This paper extends the result in [1] to a category of modules. Let A be a unique factorization integral domain (UFD). Consider M a direct finite sum of cyclic modules over A where $Ann_A(M) = \{0\}$ and α an automorphism of M. We give a necessary and sufficient condition such that α satisfies the extension property.

Keywords: integral domain, factorization, module, automorphism, torsion and torsion-free.

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1. Introduction

Let A be a unique factorization integral domain (UFD) and M be a module over A. We say that α, an automorphism of M, satisfies the extension property if for all monomorphisms $\lambda : M \longrightarrow N$ there exists $\widetilde{\alpha}$ such that:

$$\begin{array}{ccc} M & \stackrel{\lambda}{\longrightarrow} & N \\ \alpha\downarrow & & \downarrow\widetilde{\alpha} \\ M & \stackrel{\lambda}{\longrightarrow} & N \end{array}$$

is commutative, i.e., $\lambda \circ \alpha = \widetilde{\alpha} \circ \lambda$.

It is known that all automorphisms of a vector space satisfy the extension property. However, it is not always true for other categories. But there are some very important results in the category of groups. Schupp [7] proved that the automorphisms having the property of extension in the category of groups characterize the inner automorphisms. Pettet [6] provided a simpler proof of Schupp's result. Then, Ben Yacoub [4] proved that this result is not true in the algebra category. In order to generalize the result of Schupp, Abdelalim *et al.* [1] gave the characterization of the automorphisms having the property of extension in the category of abelian groups.

In another work [2], the authors considered the category that contains abelian groups. They proved that if a module M is a direct sum of cyclic torsion-free modules over a BFD, the automorphisms of M that satisfy the property of the extension are none other than the homotheties of invertible ratio.

Certainly, one of the generalizations of [1] other than [2] is the case where all or some of the co-generators of M are not torsion-free. To do this, consider the following example which illustrates where the problem of the extension's ownership lies. We know that $\mathbb{Z}[i]$ is a UFD. Let $n \in \mathbb{N} : n \geqslant 2$. For all $s \in \{2, 3, \ldots, n\}$, we will denote by p_s the $(s-1)^e$ prime number. $(p_2 = 2 < p_3 = 3 < p_4 = 5 < \cdots < p_n)$.

Consider, in the $\mathbb{Z}[i]$-module $N = \mathbb{C}/\mathbb{Z}[i]$, the sub-module $M = \bigoplus_{s=1}^{s=n} \mathbb{Z}[i]\overline{x_s}$ where $\overline{x_1} \in \mathbb{C}/\mathbb{Z}[i] - \mathbb{Q}[i]/\mathbb{Z}[i]$ any torsion-free element and the torsion element $\overline{x_s} = \overline{1 + i \cdot \frac{1}{p_s}}$ for all $s \in \{2, \ldots, n\}$. We can prove that, for all $s \in \{2, \ldots, n\}$, there exists a_s in $\mathbb{Z}[i]$ such that $\alpha(\overline{x_s}) = a_s \cdot \overline{x_s}$. Without using that α satisfies the extension property, we cannot conclude that $a_2, \ldots, a_n$ are units in $\mathbb{Z}[i]$ neither $(**)\, \alpha(\overline{x_1}) \in \mathbb{Z}[i]\overline{x_1}$. The key idea is to prove $(**)$.

In this paper, we are interested in the extension property of a special category of modules over A which is a unique factorization integral domain (not a field).

Let n be a non-zero natural integer and $M = Ax_1 \oplus Ax_2 \oplus \cdots \oplus Ax_n$ a direct finite sum of cyclic modules over A, such that $Ann_A(M) = \{0_A\}$.

Our paper is organized as follows. In the first section, we will briefly give necessarly importants results. In the second part, all the generators of M are torsion-free elements. We will prove that an automorphism $\alpha : M \longrightarrow M$ satisfies the extension property, if and only if there exists a unit k in A such that $\alpha = k \cdot 1_M$. And in the last part, M is no longer considered as a torsion-free module. Again with much more advanced techniques, we will prove that the homotheties of invertible ratio are the only automorphisms of M satisfying the extension property.

2. Preliminaries and notations

Definition. Let A be an integral domain, M an A-module x an element of M. The annihilator of x is the ideal $Ann_A(x) = \{a \in A / a \cdot x = 0_M\}$. The annihilator of M, is the ideal $Ann_A(M) = \{a \in A / \forall y \in M : a \cdot y = 0_M\}$.

Definition [5]**.**

- Let x be an element of M. x is a torsion-free element if $Ann_A(x) = \{0_A\}$ and x is a torsion element if $Ann_A(x) \neq \{0_A\}$.
- We say that M is a torsion-free module if all its elements are torsion free.
- We say that M is a torsion module if all its elements are torsion.

Lemma 1. *Let $M = Ax_1 \oplus Ax_2 \oplus \cdots \oplus Ax_n$ be a direct finite sum of cyclic modules over A.*

M is a torsion module if and only if $Ann_A(M) \neq \{0_A\}$.

Proof. The proof is easy. ■

Lemma 2. *For an irreducible element p in A, the set $I_p = \{p^n / n \in \mathbb{N}\}$ is infinite.*

Proof. Suppose that I_p is finite, then necessarily $\exists\,(n, m) \in \mathbb{N}^2$ such that $n < m$ and $p^n = p^m$. So $p^n(1 - p^{m-n}) = 1_A \Longrightarrow p^{m-n} = 1_A$ (integral domain). Therefore p is a unit in A, which is not true. ■

Notations. Let $E_p(M)$ denote the set of all A-automorphisms of M satisfying the extension property. For $i \in \{1, 2, \ldots, n\}$, E_i will denote an injective envelope of Ax_i, $\mu_i : Ax_i \longrightarrow E_i$ a monomorphism of A-modules and

$$M_i = E_i + \sum_{k=1, k\neq i}^{k=n} A \cdot x_k.$$

Lemma 3. *If α satisfies the extension property then α^{-1} also satisfies the extension property.*

Proof. Let $M \overset{\lambda}{\to} N$ a monomorphism of A-modules, then there exists $N \overset{\widetilde{\alpha}}{\to} N$ an automorphism of A-modules such that $\lambda o \alpha = \widetilde{\alpha} o \lambda$. We define $\widetilde{\alpha^{-1}} = \widetilde{\alpha}^{-1}$, it's clear that $\widetilde{\alpha^{-1}} \in Aut\,(N)$. Let $x \in M$, so $\exists\, y \in M : x = \alpha(y)$ then $\lambda\left[\alpha^{-1}(x)\right] = \lambda\,(y)$. On the other hand, if we put $\widetilde{\alpha}^{-1}\,[\lambda\,(x)] = t$. We will have $\lambda\,(x) = \widetilde{\alpha}\,(t) \Longrightarrow \widetilde{\alpha}\,(t) = \lambda\,[\alpha\,(y)]$, then $\widetilde{\alpha}\,(t) = \widetilde{\alpha}\,[\lambda\,(y)] \Longrightarrow t = \lambda\,(y)$. From where, $\lambda o \alpha^{-1} = \widetilde{\alpha^{-1}} o \lambda$ hence the lemma. ■

3. Finite sum of torsion free cyclic A-modules

In this section, we suppose that x_i is a torsion-free element for all i in $\{1, 2, \dots, n\}$. Let $\mu_i : Ax_i \longrightarrow E_i$ be a monomorphism of A-modules.

Lemma 4. *For all $i \in \{1, 2, \dots, n\}$, we have*

$$M_i = E_i \bigoplus \bigoplus_{k=1, k\neq i}^{k=n} A \cdot x_k$$

and

$$E_i = \bigcap_{a \in A^*} a \cdot M_i.$$

Proof. We know that $E_1 \oplus E_2 \oplus \cdots \oplus E_n$ is an injective envelope of $Ax_1 + Ax_2 + \cdots + Ax_n$ (see [3]). Then

$$M_i = E_i + \sum_{k=1, k\neq i}^{k=n} A \cdot x_k = E_i \bigoplus \bigoplus_{k=1, k\neq i}^{k=n} Ax_k.$$

As E_i is an injective envelope of Ax_i then a divisible module. For all $a \in A^*$,

$$E_i = a \cdot E_i \subset a \cdot M_i \Longrightarrow E_i \subset \bigcap_{a \in A^*} a \cdot M_i.$$

Reciprocally:

Let

$$t = e + \sum_{j=1, j\neq i}^{j=n} m_j \cdot x_j \in \bigcap_{a \in A^*} a \cdot M_i \subset \bigcap_{k \in \mathbb{N}} p^k \cdot M_i \text{ where } e \in E_i \text{ and } m_j \in A.$$

Then $(\forall\, k \in \mathbb{N})$ $(\exists\, e_k \in E_i)$ $(\exists\, (m_{j,k})_{j=1, j\neq i, j=n} \in A^{n-1})$ such that

$$t = p^k \left(e_k + \sum_{j=1, j\neq i}^{j=n} m_{j,k} \cdot x_j \right).$$

Moreover, as M_i is a direct sum of A-modules, by identification, we have

$$e = p^k e_k \text{ and } \sum_{j=1, j\neq i}^{j=n} m_j \cdot x_j = p^k \cdot \left(\sum_{j=1, j\neq i}^{j=n} m_{j,k} \cdot x_j \right).$$

If $\sum_{j=1, j\neq i}^{j=n} m_j \cdot x_j \neq 0_M$ then there exists $j \neq i \in \{1, 2, \dots, n\}$ such that $m_j \neq 0_A$ and $m_j = p^k \cdot m_{j,k}$. So, $p^k \mid m_j$ for all integer k, which is false in a factorial integral domain by Lemma 2.

Hence $t = p^k e_k \in E_1$, that concludes the proof. ∎

Under the previous notations, we have the following lemma.

Lemma 5. *If $\alpha \in E_p(M)$, then $\widetilde{\alpha}\,(E_i) = E_i$.*

Proof. We have

$$E_i = \bigcap_{a\in A^*} aM_i \text{ then } \widetilde{\alpha}(E_i) = \bigcap_{a\in A^*} a\widetilde{\alpha}(M_i) \subset E_i.$$

Let now y in the divisible module E_i, then $\forall a \in A^*, \exists\, y_a \in M_i$ such that $y = a\cdot y_a$. As $\widetilde{\alpha} \in Aut\,(M_i)$ then there exists $x_a \in M_i$: $y_a = \widetilde{\alpha}(x_a)$, so

$$y = \widetilde{\alpha}\,(a\cdot x_a) \Longrightarrow y \in \widetilde{\alpha}(\bigcap_{a\in A^*} a\cdot M_i) = \widetilde{\alpha}\,(E_i)$$

Therefore $\widetilde{\alpha}\,(E_i) = E_i$. ■

Lemma 6. *For all i and j in $\{1, 2, \ldots, n\}$. If $i \neq j$, then $M = A\,(x_i + x_j) \bigoplus \bigoplus_{k=1,k\neq i}^{k=n} A\cdot x_k$.*

Proof. Let i and j in $\{1, 2, \ldots, n\}$ such that $i \neq j$. Consider $(a_1, a_2, \ldots, a_n) \in A^n$ and suppose that $a_i(x_i + x_j) + \sum_{k=1,k\neq i}^{k=n} a_k x_k = 0_M$.

Then $a_i x_i + (a_i + a_j)x_j + \sum_{k=1,k\neq i,k\neq j}^{k=n} a_k x_k = 0_M$. As $\{x_1, x_2, \ldots, x_n\}$ is A free, then $a_i = a_i + a_j = a_k = 0_A$ for all $k \in \{1, 2, \ldots, n\}/\{i, j\}$. So, $a_1 = a_2 = \cdots = a_n = 0_A$. It's clear that $A\,(x_i + x_j) \oplus Ax_j \subset Ax_i \oplus Ax_j$. Let $x \in Ax_i \oplus Ax_j$ then $x = ax_i + bx_j$ where a and b are in A. So, $x = b(x_i + x_j) + (a - b)x_i \in A(x_i + x_j) \oplus Ax_i$. Then the lemma follows. ■

Theorem 7. *If $\alpha \in E_p(M)$ then $(\forall i \in \{1, 2, \ldots, n\})$ $(\exists\, k_i \in A^*)$ $(\alpha\,(x_i) = k_i\cdot x_i)$.*

Proof. Let $\lambda : M \longrightarrow M_i$ defined by If $x = \sum_{j=1}^{j=n} m_j\cdot x_j$, where $m_j \in A$ for all $j \in \{1, 2, \ldots, n\}, \lambda\,(x) = m_i\cdot\mu_i\,(x_i) + \sum_{j=1,j\neq i}^{j=n} m_j\cdot x_j$.

As defined λ is a morphism of A-modules and we have

$$\lambda(x) = m_i\cdot\mu_i\,(x_i) + \sum_{j=1,j\neq i}^{j=n} m_j\cdot x_j = 0_M \Longrightarrow m_i\cdot\mu_i\,(x_i) = 0_M \text{ and } \sum_{j=1,j\neq i}^{j=n} m_j\cdot x_j = 0_M.$$

However μ_i is injective then $m_i = 0_A$. As $\{x_1, x_2, \ldots, x_n\}\backslash\{x_i\}$ is A-free, then $m_j = 0_A$ for all $j \in \{1, 2, \ldots, n\}/\,\{i\}$. Then, $x = 0_M$. Therefore λ is a monomorphism of A-modules. We know that

$$\alpha\,(x_i) = k_i\cdot x_i + \sum_{j=1,j\neq i}^{j=n} k_j\cdot x_j \text{ where } k_1, k_2, \ldots, k_{n-1} \text{ and } k_n \text{ are in } A.$$

As $\lambda o\alpha = \widetilde{\alpha} o\lambda$ and by Lemma 5 we have

$$\lambda\left[\alpha\left(x_i\right)\right] = k_i\cdot\mu\left(x_i\right) + \sum_{j=1, j\neq i}^{j=n} k_j\cdot\lambda\left(x_j\right) = k_i\cdot\mu_i\left(x_i\right) + \sum_{j=1, j\neq i}^{j=n} k_j\cdot x_j = \widetilde{\alpha}\left[\lambda\left(x_i\right)\right] \in E_i.$$

And since E_i is a direct factor in M_i, $\lambda(\alpha(x_i)) = k_i x_i \Longrightarrow \sum_{j=1,j\neq i}^{j=n} k_j x_j = 0_M$. So $\alpha(x_i) = k_i \cdot x_i$. ∎

Corollary 8. *We have $E_p(M) = \{kid_M / \, k \in A^\times\}$, where $A^\times$ is the group of units in A.*

Proof. Applying the Theorem 7 for all $i \in \{1, 2, \ldots, n\}$ there exists $k_i \in A^*$ such that $\alpha\left(x_i\right) = k_i \cdot x_i$. We must find $k_i = k_j$, for all i and j in $\{1, 2, \ldots, n\}$ such that $i \neq j$. Applying now the Theorem 7, to M as asserted in Lemma 6, there exists $k_{i,j} \in A^*$ such that $\alpha\left(x_i + x_j\right) = k_{i,j}.\left(x_i + x_j\right) = k_{i,j}\cdot x_i + k_{i,j}\cdot x_j = k_i \cdot x_i + k_j \cdot x_j \Longrightarrow k_{i,j} = k_i = k_j$ since $\{x_i, x_j\}$ is A-free. Then, there is $k \in A^*$ such that for all $i \in \{1, 2, \ldots, n\}$, $\alpha\left(x_i\right) = k \cdot x_i$. Therefore $\alpha = k \cdot id_M$.

We must proved that k is a unit in A. As α satisfies the extension property then α^{-1} also satisfies the extension property by Lemma 3. Then there exists $r \in A^*$ such that $\alpha^{-1} = r \cdot id_M$. As $\alpha \circ \alpha^{-1} = \alpha^{-1} \circ \alpha = k \cdot r \cdot id_M$ then $r \cdot k = 1_A$ which proved that k is a unit in A. ∎

4. Direct finite sum of torsion free and torsion cyclics A-modules

Let $n' \in \mathbb{N}^*$ such that $n' \leqslant n - 1$. In this section, we assume that $x_1, x_2, \ldots, x_{n'}$ are the torsion-free co-generators and $x_{n'+1}, x_{n'+2}, \ldots, x_n$ are the torsions co-generators.

For all $i \in \{1, 2, \ldots, n'\}$, E_i will denote an injective envelope of $A \cdot x_i$ and $\mu_i : A \cdot x_i \longrightarrow E_i$ a monomorphism of A-modules. Same as the Theorem 7 and corollary 8, we have the same result, but with a demonstration using the notion of a torsion element.

Lemma 9. *For all $i \in \{1, \ldots, n'\}$,we have, $E_i = \bigcap_{a\in A^*} a \cdot M_i$.*

Proof. It's clear that E_1 is a divisible A-module, so $E_1 \subset a \cdot E_1 \subset a \cdot M_1$ for all a in $A^* \Longrightarrow E_1 \subset \bigcap_{a\in A^*} a \cdot M_1$. Reciprocally, let $x \in M_1$ then $(\exists\,(f, c_2, c_3, \ldots, c_n) \in E_1 \times (A^*)^{n-1})$ $(x = f + c_2 \cdot x_2 + c_3 \cdot x_3 + \cdots + c_n \cdot x_n)$.

We know that $(\forall i \in \{n' + 1, n' + 2, \ldots, n\})$ $(\exists\, b_i \in A^* : b_i \cdot x_i = 0_M)$. Let $b = b_{n'+1} b_{n'+2} \cdots b_n \in A^*$. Therefore, for all $i \in \{n' + 1, n' + 2, \ldots, n\}, b \cdot x_i = 0$. Suppose that for all $a \in A^*$, $x \in a \cdot M_1$, then for all m a nonzero naturel integer $x \in b^m \cdot M_1$, consequently.

If $n' = 1$, then $x \in E_1$. If $n' \neq 1$, then

$$\exists\,(e_m, a_{2,m}, a_{3,m}, \ldots, a_{n,m}) \in E_1 \times (A^*)^{n-1} :$$
$$x = b^m e_m + \sum_{j=2}^{j=n} a_{j,m} b^m x_j = f + \sum_{j=2}^{j=n'} c_j x_j.$$

Let is suppose that $c_2x_2+c_3x_3+\cdots+c_{n'}x_{n'} \neq 0$, then there exists $j \in \{2,3,\ldots,n'\}$ such that $c_j \neq 0_A$ and $c_j = a_{j,m}b^m$. So, there is an infinity divisors of c_j, which is false (because A is UFD). Therefore, $c_2\cdot x_2+c_3\cdot x_3+\cdots+c_{n'}\cdot x_{n'} = 0 \Longrightarrow x \in E_1$. For any $i \in \{2,3,\ldots,n'\}$ with the simple transposition $(i,1)$, we find the first case. Then the lemma follows. ∎

Lemma 10. *If* $\alpha \in E_p(M)$, *then* $(\forall\, i \in \{1,2,\ldots,n'\})$ $(\exists\, k_i \in A^* : \alpha(x_i) = k_i x_i)$.

Proof. Let $\lambda : M \longrightarrow M_1$ be the morphism defined by $\lambda(a_1x_1 + a_2x_2 + \cdots + a_nx_n) = a_1\mu_1(x_1) + a_2x_2 + \cdots + a_nx_n$. It's clear that λ is a monomorphism of A modules.

As α satisfies the extension property, there exists $\widetilde{\alpha} \in Aut(M_1)$ such that $\lambda \circ \alpha = \widetilde{\alpha} \circ \lambda$. As $\alpha(x_1) \in M$ then $\alpha(x_1) = d_1\cdot x_1 + d_2\cdot x_2 + \cdots + d_n\cdot x_n$, where $(d_1, d_2, \ldots, d_n) \in A^n$. So $\lambda\,[\alpha\,(x_1)] = d_1\cdot\mu_1(x_1) + \sum_{j=2}^{j=n} d_j\cdot x_j = \widetilde{\alpha}\,[\lambda\,(x_1)]$. And as we know that $\widetilde{\alpha}\,[\lambda\,(x_1)] \in E_1$, which is a direct factor in M_1, then $\sum_{j=2}^{j=n} d_j\cdot x_j = 0_M \Longrightarrow \alpha(x_1) = d_1\cdot x_1$. Same proof for all i in $\{2,3,\ldots,n'\}$, hence $\alpha(x_i) = d_i\cdot x_i$ $(k_i = d_i)$. ∎

Lemma 11. *If* $n' \neq 1$, *let* $i_0 \in \{2,3,\ldots,n'\}$. *We have*

- $x_1 + x_{i_0}$ *is a torsion-free element.*
- $M = A(x_1 + x_{i_0}) \bigoplus \bigoplus_{i=1, i\neq i_0}^{i=n'} Ax_i \bigoplus \bigoplus_{j=n'+1}^{i=n} Ax_j$ $(*)$.

Proof.

- We have $x_{i_0} = x_1 + x_{i_0} - x_1 \in A(x_1 + x_{i_0}) \bigoplus \bigoplus_{i=1,i\neq i_0}^{i=n'} Ax_i \bigoplus \bigoplus_{j=n'+1}^{i=n} Ax_j$. Let $a \in A$ such that $a(x_1 + x_{i_0}) = 0_M$ then $ax_1 + ax_{i_0} = 0_M \Longrightarrow a = 0_A$. ($\{x_1, x_{i_0}\}$ is A free). Therefore, $\{x_1 + x_{i_0}\}$ is a A-free.
- Let $(a_1, a_2, \ldots, a_n) \in A^n$ and suppose that

  $$a_{i_0}(x_1 + x_{i_0}) + \sum_{i=1, i\neq i_0}^{i=n} a_i x_i = 0_M.$$

  Then

  $$(a_{i_0} + a_1)x_1 + \sum_{i=2}^{i=n} a_i x_i = 0_M \Longrightarrow a_1 = a_2 = \cdots = a_{n'} = 0_A$$

and $a_{n'+1}x_{n'+1} = a_{n'+2}x_{n'+2} = \cdots = a_nx_n = 0_M$ ($M = \bigoplus_{i=1}^{i=n} Ax_i$ and $\{x_1, \dots, x_{n'}\}$ is A free). So $M = A(x_1 + x_{i_0}) \bigoplus \bigoplus_{i=1, i\neq i_0}^{i=n'} Ax_i \bigoplus \bigoplus_{j=n'+1}^{i=n} Ax_j$ $(*)$.

Then the lemma follows. ∎

Corollary 12. *For all $i \in \{1, 2, \dots, n'\}$, $k_i = k_1$.*

Proof. By Lemma 10 we have for all i in $\{1, 2, \dots, n'\}$, there exists an $k_i \in A^*$ such that $\alpha(x_i) = k_i \cdot x_i$.

Let M be defined as in Lemma 11, then by Lemma 10, there exists an $r_{i_0} \in A^*$ such that $\alpha(x_1 + x_{i_0}) = r_{i_0}(x_1 + x_{i_0})$. So $r_{i_0} = k_{i_0} = k_1$, and this for all i_0 a fixed element in $\{2, \dots, n'\}$. Hence the corollary. ∎

Lemma 13. *For all $i \in \{1, 2, \dots, n'\}$, $x_i + x_{n'+1}$ is a torsion-free element and*

$$M = \bigoplus_{i=1}^{i=n'} A(x_i + x_{n'+1}) \bigoplus \bigoplus_{j=n'+1}^{j=n} Ax_j.$$

Proof. We have $x_i = x_i + x_{n'+1} - x_{n'+1}$ for all $i \in \{1, 2, \dots, n'\}$, then $M = A(x_1 + x_{n'+1}) + A(x_2 + x_{n'+1}) + \cdots + A(x_{n'} + x_{n'+1}) + Ax_{n'+1} + Ax_{n'+2} + \cdots + Ax_n$. Let $(a_1, a_2, \dots, a_n) \in A^n$ such that:

$$x = \sum_{i=1}^{i=n'} a_i(x_i + x_{n'+1}) + \sum_{j=n'+1}^{i=n} a_jx_j = 0_M.$$

Then

$$\sum_{i=1}^{i=n'} a_i \cdot x_i + \left(\left(\sum_{i=1}^{i=n'} a_i\right) + a_{n'+1}\right) \cdot x_{n'+1} + \sum_{j=n'+2}^{i=n} a_j \cdot x_j = 0_M.$$

So,

$$\sum_{i=1}^{i=n'} a_i \cdot x_i = 0_M, \left(\left(\sum_{i=1}^{i=n'} a_i\right) + a_{n'+1}\right) \cdot x_{n'+1} = 0_M \text{ and } \sum_{j=n'+2}^{i=n} a_j \cdot x_j = 0_M.$$

And as $x_1, x_2, \dots, x_{n'}$ are a torsion-free elements, then $a_i = 0_A$ for all $i \in \{1, 2, \dots, n'\}$ and $a_j \cdot x_j = 0_M$ for all $j \in \{n'+1, n'+2, \dots, n\}$. Let $i \in \{1, 2, \dots, n'\}$ and suppose that $a_i \cdot (x_i + x_{n'+1}) = 0_M$. Then $a_i \cdot x_i + a_i \cdot x_{n'+1} = 0_M \Longrightarrow a_i = 0_A \Longrightarrow x_i + x_{n'+1}$ is a torsion-free element. ∎

Theorem 14. *We have $E_p(M) = \{kid_M / \, k \in A^\times\}$.*

Proof. Let $b \in Ann_A(x_{n'+1})\backslash\{0_A\}$, so $b \cdot x_{n+1} = 0_M$. As $\alpha(x_{n'+1}) \in M$, then $\alpha(x_{n'+1}) = \sum_{j=1}^{j=n} a_j x_j$, where $(a_1, a_2, \ldots, a_n) \in A^n$. Then

$$\begin{aligned}
&\alpha(bx_{n'+1}) = ba_1x_1 + ba_2x_2 + \cdots + ba_nx_n = 0_M \\
&\Longrightarrow ba_1x_1 = ba_2x_2 = \cdots = ba_{n'}x_{n'} = 0_M \Longrightarrow a_1 = a_2 = \cdots = a_{n'} = 0_A \\
&\Longrightarrow \alpha(x_{n'+1}) = a_{n'+1}x_{n'+1} + a_{n'+2}x_{n'+2} + \cdots + a_nx_n.
\end{aligned}$$

By applying the result of Lemma 10 to $M = \bigoplus_{i=1}^{i=n'} A(x_i + x_{n'+1}) \bigoplus \bigoplus_{j=n'+1}^{j=n} Ax_j$ there exists $k_{n'+1} \in A^*$ such that $\alpha(x_i + x_{n'+1}) = k_{n'+1}x_i + k_{n'+1}x_{n'+1} = k_1x_i + \alpha(x_{n'+1})$. Then $(k_{n'+1} - k_1)x_i + k_{n'+1}x_{n'+1} = a_{n'+1}x_{n'+1} + a_{n'+2}x_{n'+2} + \cdots + a_nx_n$.

Also, $M = \bigoplus_{k=1}^{k=n} Ax_k$ and x_i is a torsion free element, then $k_{n'+1} = k_1$ and $\sum_{j=n'+2}^{j=n} a_jx_j = 0_M \Longrightarrow \alpha(x_{n'+1}) = k_1x_{n'+1}$. The same proof for $\alpha(x_{n'+j}) = k_1x_{n'+j}$ for $j \in \{2, 3, \ldots, n - n'\}$, so $\alpha = k_1 id_M$. As we know α^{-1} satisfies the extension property, then there exists $r \in A^*$such that $\alpha^{-1} = rid_M$. Consequently, $r \cdot k_1 = 1_A$. Which completes the proof of the theorem. ■

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