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USING THE SWING LEMMA AND $\mathcal{C}_1$-DIAGRAMS FOR CONGRUENCES OF PLANAR SEMIMODULAR LATTICES

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Abstract

A planar semimodular lattice K is *slim* if M_3 is not a sublattice of K. In a recent paper, G. Czédli found four new properties of congruence lattices of slim, planar, semimodular lattices, including the *No Child Property*: *Let $\mathcal{P}$ be the ordered set of join-irreducible congruences of K. Let $x, y, z \in \mathcal{P}$ and let z be a maximal element of $\mathcal{P}$. If $x \neq y$ and $x, y \prec z$ in $\mathcal{P}$, then there is no element u of $\mathcal{P}$ such that $u \prec x, y$ in $\mathcal{P}$.*

The Swing Lemma and a standardized diagram type are used to give direct proofs of Czédli's four properties.

Keywords: rectangular lattice, slim planar semimodular lattice, congruence lattice.

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1. Introduction

Let K be a planar semimodular lattice. We call the lattice K *slim* if M_3 is not a sublattice of K. In the paper [19, Theorem 1.5], I found a property of congruence lattices of slim, planar, semimodular lattices. In the same paper (see also Problem 24.1 in Grätzer [18]), I proposed the following.

Problem. Characterize the congruence lattices of slim, planar, semimodular lattices.

Czédli [4, Corollaries 3.4, 3.5, Theorem 4.3] found four new properties of congruence lattices of slim, planar, semimodular lattices.

Theorem. *Let K be a slim, planar, semimodular lattice with at least three elements and let $\mathcal{P}$ be the ordered set of join-irreducible congruences of K.*

(i) Partition Property: *The set of maximal elements of $\mathcal{P}$ can be divided into a disjoint union of two nonempty subsets such that no two distinct elements in the same subset have a common lower cover.*

(ii) Maximal Cover Property: *If $v \in \mathcal{P}$ is covered by a maximal element u of $\mathcal{P}$, then u is not the only cover of v.*

(iii) No Child Property: *Let $x \neq y \in \mathcal{P}$ and let u be a maximal element of $\mathcal{P}$. If $x, y \prec u$ in $\mathcal{P}$, then there is no element $z \in \mathcal{P}$ such that $z \prec x, y$ in $\mathcal{P}$.*

(iv) Four-Cown Two-pendant Property: *There is no cover-preserving embedding of the ordered set $\mathcal{R}$ in Figure 1 into $\mathcal{P}$ satisfying the property*: *any maximal element of $\mathcal{R}$ is a maximal element of $\mathcal{P}$.*

In this note, we will provide a short and direct proof of this theorem using only the Swing Lemma and $\mathcal{C}_1$-diagrams, see Section 3.

Czédli [4] uses different names for three of the four properties of the Theorem.

Czédli [4]:	Our terminology:
Forbidden Marriage Property	No Child Property
Dioecious Maximal Elements Property	Maximal Cover Property
Bipartite Maximal Elements Property	Partition Property

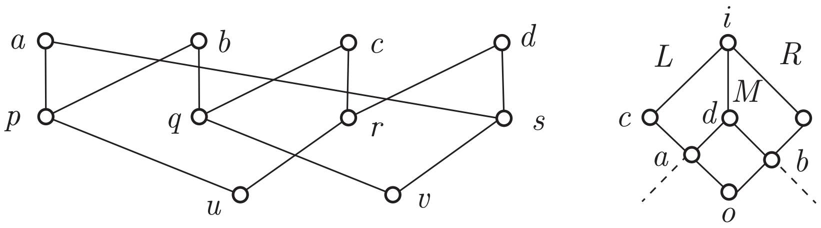

Figure 1. The Four-crown Two-pendant ordered set $\mathcal{R}$ with notation; the covering S_7 sublattice with edge and element notation.

Outline

Section 2 provides the motivation for Czédli's Theorem. Section 3 provides the tools we need: the Swing Lemma, $\mathcal{C}_1$-diagrams, and forks.

Section 4 proves the Partition Property, Section 5 does the Maximal Cover Property, while Section 6 verifies the No Child Property.

Finally, The Four-Crown Two-pendant Property is proved in Section 7.

2. Motivation

In my paper [29] with Lakser and Schmidt, we proved that every finite distributive lattice D can be represented as the congruence lattice of a semimodular lattice L. To our surprise, the semimodular lattice K we constructed was *planar*.

Grätzer and Knapp [24]–[28] started the study of planar semimodular lattices. I continued it with my "Notes on planar semimodular lattices" series (started with Knapp): [30] (with Wares), [6, 14] (with Czédli), [21, 23]. See also Czédli and Schmidt [10] and Czédli [1]–[5].

A major subchapter of the theory of planar semimodular lattices started with the observation that in the construction of the lattice K, as in the first paragraph of this section, M_3 sublattices play a crucial role. It was natural to raise the question what can be said about congruence lattices of slim, planar, semimodular (SPS) lattices (see [CFL2, Problem 24.1], originally raised in Grätzer [19]). In [19], I found the first necessary condition and Czédli [2] proved that this condition is not sufficient (see also my related papers [17] and [21]).

A number of papers developed tools to tackle this problem: the Swing Lemma (Grätzer [16]), trajectory coloring (Czédli [1]), special diagrams (Czédli [3]), lamps (Czédli [4]). Some of these results require long proofs. The proof of the trajectory coloring theorem is just shy of 20 pages, while the basic theory of lamps and its application to Theorem 1 is 23 pages.

There are a number of surveys of this field, see the book chapters Czédli and Grätzer [7] and Grätzer [13] in Grätzer and Wehrung, eds. [31]. My presentation [22] provides a gentle review for the background of this topic.

There is an approach to congruences of SPS lattices that is very different from what we are doing in this note. This started in McKenzie [33], Nation [34], Jónsson and Nation [33] and Day [11], viewing SPS lattices as upper bounded homomorphic images of free lattices. Thus the meet-prime elements of a rectangular SPS lattice are along the upper boundaries, so the edges on the upper boundary of a rectangular SPS lattice correspond to maximal join-irreducible congruences, providing a background for the Swing Lemma and its corollaries in Section 3.1. This topic is covered in detail in the book Freese, Ježek and Nation [12, Chapter II], see also the lecture notes of Nation [35, Chapter 10]. The referee informs me that the Maximal Cover Property holds for all finite upper bounded lattices.

3. The tools we need

Most basic concepts and notation not defined in this paper are available in Part I of the book [18], see

`https://www.researchgate.net/publication/299594715`
`arXiv:2104.06539`

It is available to the reader and will be referenced as [CFL2].

In particular, we use the notation $C \sim D$, $C \overset{\text{up}}{\sim} D$, and $C \overset{\text{dn}}{\sim} D$ for perspectivity, up-perspectivity, and down-perspectivity, respectively. As usual, for planar lattices, a prime interval (or covering interval) is called an *edge*. For a finite lattice K and a finite ordered set S, a *cover-preserving* embedding $\varepsilon\colon S \to K$ is an embedding ε mapping edges of S to edges of K. We define a *cover-preserving* sublattice similarly. For the lattice S_7 of Figure 1, we need a variant: an S_7 sublattice S (a sublattice isomorphic to S_7) is a *peak sublattice* if the three top edges (L, M, and R in Figure 1) are edges in K.

By Grätzer and Knapp [27], every slim, planar, semimodular lattice K has a congruence-preserving extension (see [CFL2, page 43]) $\hat{K}$ to a slim rectangular lattice. Any of the properties (i)–(iv) holds for K iff it holds for $\hat{K}$. Therefore, in the rest of this paper, we can assume that K is a slim rectangular lattice, simplifying the discussion.

3.1. Swing Lemma

For an edge E of an SPS lattice K, let $E = [0_E, 1_E]$ and define $\operatorname{col}(E)$, the *color of* E, as $\operatorname{con}(E)$, the (join-irreducible) congruence generated by collapsing E (see [CFL2, Section 3.2]). We write $\mathcal{P}$ for $\operatorname{Ji}\operatorname{Con} K$, the ordered set of join-irreducible congruences of K.

As in my paper [16], for the edges U, V of an SPS lattice K, we define a binary relation: U *swings* to V, in formula, $U \mathrel{\hookrightarrow} V$, if $1_U = 1_V$, the element $1_U = 1_V$ of K covers at least three elements, and 0_V is neither the left-most nor the right-most element covered by $1_U = 1_V$; if also 0_U is such, then the swing is *interior*, otherwise, it is *exterior*, denoted by $U \overset{\text{in}}{\hookrightarrow} V$ and $U \overset{\text{ex}}{\hookrightarrow} V$, respectively; see Figure 2.

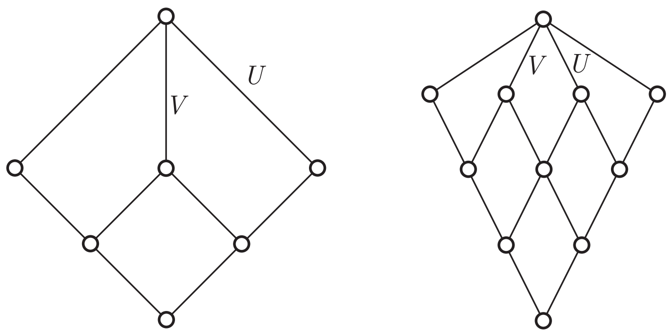

Figure 2. Two swings, $U \mathrel{\hookrightarrow} V$; the first $U \overset{\text{ex}}{\hookrightarrow} V$, the second $U \overset{\text{in}}{\hookrightarrow} V$.

Swing Lemma (Grätzer [16]). *Let K be an SPS lattice and let U and V be edges in K. Then* $\operatorname{col}(V) \leq \operatorname{col}(U)$ *iff there exists an edge R such that U is* *up-perspective to R and there exists a sequence of edges and a sequence of binary relations*

$$R = R_0 \,\varrho_1\, R_1 \,\varrho_2 \cdots \varrho_n\, R_n = V,$$

where each relation ϱ_i is $\overset{\mathrm{dn}}{\sim}$ (down-perspective) or $\curvearrowleft$ (swing). In addition, this sequence also satisfies

$$1_{R_0} \geq 1_{R_1} \geq \cdots \geq 1_{R_n}.$$

The following statements are immediate consequences of the Swing Lemma, see my papers [16] and [20].

Corollary 1. *We use the assumptions of the Swing Lemma.*

(i) *The equality* $\mathrm{col}(U) = \mathrm{col}(V)$ *holds in $\mathcal{P}$ iff there exist edges S and T in K, such that*

$$U \overset{\mathrm{up}}{\sim} S,\ S \overset{\mathrm{in}}{\curvearrowleft} T,\ T \overset{\mathrm{dn}}{\sim} V.$$

(ii) *Let us further assume that the element 0_U is meet-irreducible. Then the equality* $\mathrm{col}(U) = \mathrm{col}(V)$ *holds in $\mathcal{P}$ iff there exists an edge T such that $U \overset{\mathrm{in}}{\curvearrowleft} T \overset{\mathrm{dn}}{\sim} V$.*

(iii) *If the lattice K is rectangular and U is on the upper boundary of K, then the equality* $\mathrm{col}(U) = \mathrm{col}(V)$ *holds in $\mathcal{P}$ iff $U \overset{\mathrm{dn}}{\sim} V$.*

Note that in (i) the edges S, T, U, V need not be distinct, so we have as special cases $U = V$, $U \sim V$, $S = T$, and others.

Corollary 2. *We use the assumptions of the Swing Lemma.*

(i) *The covering* $\mathrm{col}(V) \prec \mathrm{col}(U)$ *holds in $\mathcal{P}$ iff there exist edges $R_1, \ldots, R_4$ in K, such that*

$$U \overset{\mathrm{up}}{\sim} R_1,\ R_1 \overset{\mathrm{in}}{\curvearrowleft} R_2,\ R_2 \overset{\mathrm{dn}}{\sim} R_3,\ R_3 \overset{\mathrm{ex}}{\curvearrowleft} R_4,\ R_4 \overset{\mathrm{dn}}{\sim} V.$$

(ii) *If the element 0_U is meet-irreducible, then the covering* $\mathrm{col}(V) \prec \mathrm{col}(U)$ *holds in $\mathcal{P}$ iff there exist edges S, T in K, so that*

$$U \overset{\mathrm{dn}}{\sim} S \overset{\mathrm{ex}}{\curvearrowleft} T \overset{\mathrm{dn}}{\sim} V.$$

Corollary 3. *Let K be a slim rectangular lattice, let U and V be edges in K, and let U be in the upper-left boundary of K.*

(i) *The covering* $\mathrm{col}(V) \prec \mathrm{col}(U)$ *holds in $\mathcal{P}$ iff there exist edges S, T in K, such that*

$$U \overset{\mathrm{dn}}{\sim} S \overset{\mathrm{ex}}{\curvearrowleft} T \overset{\mathrm{dn}}{\sim} V. \tag{1}$$

(ii) *Define the element $t = 1_S = 1_T \in K$ and let $S = E_1, E_2, \ldots, E_n = W$ enumerate, from left to right, all the edges E of K with $1_E = t$. Then*

$$\mathrm{col}(S) \neq \mathrm{col}(W), \tag{2}$$

$$\mathrm{col}(E_2) = \cdots = \mathrm{col}(E_{n-1}) = \mathrm{col}(T), \tag{3}$$

$$\mathrm{col}(T) \prec \mathrm{col}(S), \mathrm{col}(W). \tag{4}$$

Corollary 4. *Let the edge U be on the upper edge of the rectangular lattice K. Then* $\mathrm{col}(U)$ *is a maximal element of* $\mathcal{P}$.

The converse of this statement is stated in Corollary 8.

3.2. $\mathcal{C}_1$-diagrams

In the diagram of a planar lattice K, a *normal edge* (*line*) has a slope of 45° or 135°. If it is the first, we call it a *normal-up edge* (*line*), otherwise, a *normal-down edge* (*line*). Any edge of slope strictly between 45° and 135° is *steep* (Czédli [3] calls these edges *precipitous*). In Figure 3, for instance, the edges A and D are normal, while the edge S is steep.

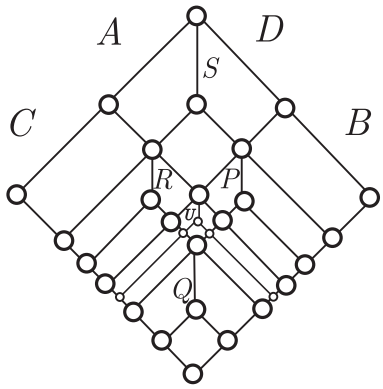

Figure 3. Illustrating the proof of The Four-Crown Two-pendant Property.

Definition 5. A diagram of a rectangular lattice K is a $\mathcal{C}_1$-*diagram* if the middle edge of any covering S_7 is steep and all other edges are normal.

This concept was introduced in G. Czédli [3, Definition 5.3], see also Czédli [4, Definition 2.1] and Czédli and Grätzer [8, Definition 3.1]. The following is the existence theorem of $\mathcal{C}_1$-diagrams in Czédli [3, Theorem 5.5].

Theorem 6. *Every rectangular lattice lattice K has a $\mathcal{C}_1$-diagram.*

See Figure 3 for a $\mathcal{C}_1$-diagram of a rectangular lattice. For a short and direct proof for the existence of $\mathcal{C}_1$-diagrams, see my paper [23].

In this paper, K denotes a slim rectangular lattice with a fixed $\mathcal{C}_1$-diagram and $\mathcal{P}$ is the ordered set of join-irreducible congruences of K.

Let C and D be maximal chains in an interval $[a, b]$ of K such that $C \cap D = \{a, b\}$. If there is no element of K between C and D, then we call $C \cup D$ a *cell*. A four-element cell is a 4-*cell*. Opposite edges of a 4-cell are called *consecutive*. Planar semimodular lattices are 4-cell lattices, that is, all of its cells are 4-cells, see Grätzer and Knapp [24, Lemmas 4, 5] and [CFL2, Section 4.1] for more detail.

The following statement illustrates the use of $\mathcal{C}_1$-diagrams.

Lemma 7. *Let K be a slim rectangular lattice K with a fixed $\mathcal{C}_1$-diagram and let X be a normal-up edge of K. Then X is up-perspective either to an edge in the upper-left boundary of K or to a steep edge.*

Proof. If X is not steep nor it is in the upper-left boundary of K, then there is a 4-cell C whose lower-right edge is X. If the upper-left edge is steep or it is in the upper-left boundary, then we are done. Otherwise, we proceed the same way until we reach a steep edge or an edge the upper-left boundary. ■

Corollary 8. *Let the edge U be on the upper boundary of K. Then* $\operatorname{col}(U)$ *is a maximal element of $\mathcal{P}$. Conversely, if u is a maximal element of $\mathcal{P}$, then there is an edge U on the upper boundary of K so that* $\operatorname{col}(U) = u$.

3.3. Trajectories

Czédli and Schmidt [9] introduced a *trajectory* in K as a maximal sequence of consecutive edges, see also [CFL2, Section 4.1]. The *top edge T* of a trajectory is either in the upper boundary of K or it is steep by Lemma 7. For such an edge T, we denote by $\operatorname{traj}(T)$ the trajectory with top edge T.

By Grätzer and Knapp [24, Lemma 8], an element a in an SPS lattice K has at most two covers. Therefore, a trajectory has at most one top edge and at most one steep edge. So we conclude the following statement.

Lemma 9. *Let K be a slim rectangular lattice K with a fixed $\mathcal{C}_1$-diagram. Let X and Y be distinct steep edges of K. Then* $\operatorname{traj}(X)$ *and* $\operatorname{traj}(Y)$ *are disjoint.*

4. THE PARTITION PROPERTY

First, we verify the Partition Property for the slim rectangular lattice K with a fixed $\mathcal{C}_1$-diagram. Let us start with a lemma.

Lemma 10. *Let X and Y be distinct edges on the upper-left boundary of K. Then there is no edge Z of K such that* $\operatorname{col}(Z) \prec \operatorname{col}(X), \operatorname{col}(Y)$.

Proof. By way of contradiction, let Z be an edge such that $\operatorname{col}(Z) \prec \operatorname{col}(X), \operatorname{col}(Y)$. Since X and Y are on the upper-left boundary, Corollary 3(i) applies. Therefore, there exist normal-up edges S_X, S_Y and steep edges T_X, T_Y such that

$$X \overset{\text{dn}}{\sim} S_X \overset{\text{ex}}{\hookrightarrow} T_X, \quad Y \overset{\text{dn}}{\sim} S_Y \overset{\text{ex}}{\hookrightarrow} T_Y, \quad Z \in \operatorname{traj}(T_X) \cap \operatorname{traj}(T_Y).$$

By Lemma 9, the third formula implies that $T_X = T_Y$. Since S_X, S_Y are normal-up, it follows that $X = Y$, contradicting the assumption that $X \neq Y$. ∎

By Corollary 8, the set of maximal elements of $\mathcal{P}$ is the same as the set of colors of edges in the upper boundaries, which we can partition into the set of edges $\mathcal{L}$ in the upper-left boundary and the set of edges $\mathcal{R}$ in the upper-right boundary. Let X and Y be distinct edges in $\mathcal{L}$. By Lemma 10, there is no edge Z of K such that $\operatorname{col}(Z) \prec \operatorname{col}(X), \operatorname{col}(Y)$. By symmetry, this verifies the Partition Property.

5. The Maximal Cover Property

Next, we verify the Maximal Cover Property for the slim rectangular lattice K and with a fixed $\mathcal{C}_1$-diagram.

Let $x \in \mathcal{P}$ be covered by a maximal element u of $\mathcal{P}$ in K. By Corollary 8, we can choose an edge U of color u on the upper boundary of K, by symmetry, on the upper-left boundary of K. By Corollary 3(ii), we can choose the edges S, T in K so that $U \overset{\text{dn}}{\sim} S \overset{\text{ex}}{\hookrightarrow} T$, $\operatorname{col}(S) = u$, and $\operatorname{col}(T) = x$. By Corollary 3(ii), specifically, by equations (2) and (4), we have $x \prec u, \operatorname{col}(W)$ and $u \neq \operatorname{col}(W)$, verifying the Maximal Cover Property.

6. The No Child Property

In this section, we verify the No Child Property for the slim rectangular lattice K and with a fixed $\mathcal{C}_1$-diagram.

Let $x, y, z, u \in \mathcal{P}$ with $x \neq y \in \mathcal{P}$, let u be a maximal element of $\mathcal{P}$, and let $x, y \prec u$ in $\mathcal{P}$. By way of contradiction, let us assume that there is an element $z \in \mathcal{P}$ such that $z \prec x, y$ in $\mathcal{P}$.

By Corollary 8, the element u colors an edge U on the upper boundary of K, say, in the upper-left boundary. By Corollary 2(i), for $z \prec x \in \mathcal{P}$, we get a peak sublattice S_7 in which the middle edge Z is colored by z and upper-left edge X is colored by x, or symmetrically. The upper-right edge Y must have color y.

Now we apply Corollary 3(ii) to the edge U and middle edge Z of the peak sublattice S_7, obtaining that $U \overset{\text{dn}}{\sim} Y \hookrightarrow Z$, in particular, $U \overset{\text{dn}}{\sim} Y$. This is a contradiction, since U is normal-up and Y is normal-down.

7. The Four-Crown Two-pendant Property

Finally, we verify the Four-Crown Two-pendant Property for the slim rectangular lattice K and with a fixed $\mathcal{C}_1$-diagram.

By way of contradiction, assume that the ordered set $\mathcal{R}$ of Figure 1 is a cover-preserving ordered subset of $\mathcal{P}$, where a, b, c, d are maximal elements of $\mathcal{P}$. By Corollary 8, there are edges A, B, C, D on the upper boundary of K, so that $\mathrm{col}(A) = a$, $\mathrm{col}(B) = b$, $\mathrm{col}(C) = c$, $\mathrm{col}(D) = d$. By left-right symmetry, we can assume that the edge A is on the upper-left boundary of K. Since $p \prec a, b$ in $\mathcal{P}$, it follows from Lemma 10 that the edge B is on the upper-right boundary of K, and so is D. Similarly, C is on the upper-left boundary of K.

Because of the automorphisms of the ordered set $\mathcal{R}$, it is sufficient to deal with one case only: C is below A and B is below D.

By Corollary 2(ii), there is a peak sublattice S_7 with middle edge P (as in the first diagram of Figure 3 so that A and B are down-perspective to the upper-left edge and the upper-right edge of this peak sublattice, respectively. We define, similarly, the edge Q for C and B, the edge S for A and D, the edge R for C and D, and the edge U for R and P. Finally, $v \prec q, s$ in $\mathcal{R}$, therefore, there is a peak sublattice S_7 with middle edge V with upper-left edge V_l and the upper-right edge V_r so that $S \overset{\mathrm{dn}}{\sim} V_l$ and $Q \overset{\mathrm{dn}}{\sim} V_r$, or symmetrically. Then V is in both $\mathrm{traj}(S)$ and $\mathrm{traj}(Q)$, contradicting Lemma 9. This concludes the proof of the Four-Crown Two-pendant Property and of Czédli's Theorem.

Of course, the diagram in Figure 3 is only an illustration. The grid could be much larger, the edges A, C and B, D may not be adjacent, and there maybe lots of other elements in K.

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