# NOTES ON PLANAR SEMIMODULAR LATTICES IX $\mathcal{C}_{1}$-DIAGRAMS 

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#### Abstract

A planar semimodular lattice $L$ is slim if $\mathrm{M}_{3}$ is not a sublattice of $L$. In a recent paper, G. Czédli introduced a very powerful diagram type for slim, planar, semimodular lattices, the $\mathcal{C}_{1}$-diagrams. This short note proves the existence of such diagrams.


Keywords: $\mathcal{C}_{1}$-diagrams, slim planar semimodular lattice.
2020 Mathematics Subject Classification: 06C10.

## Background

The basic concepts and notation not defined in this note are available in Part I of the book [10], see arXiv:2104.06539; it is freely available. We will reference it, for instance, as [CFL2, p. 4]. In particular, a planar semimodular lattice $L$ is slim if $\mathrm{M}_{3}$ is not a sublattice of $L$ and a grid $G$ is a direct product of two nontrivial chains. For the lattice $S_{7}$, see Figure 1 and [10, pages xxi, 34]. Following my paper [15] with Knapp, a semimodular lattice $L$ is rectangular if the left and right boundary chains have exactly one doubly-irreducible element each and these elements are complementary.

In my paper [16] with Lakser and Schmidt, we prove that every finite distributive lattice $D$ can be represented as the congruence lattice of a (planar) semimodulare lattice $L$. Since $\mathrm{M}_{3}$ sublattices play a crucial role in the construction of $L$, it was natural to raise the question what can be said about congruence lattices of slim, planar, semimodular (SPS) lattices (see [CFL2, Problem 24.1], originally raised in my paper [11]). The papers in the References list some contributions to this topic. In particular, my presentation [13] gently reviews the background of this topic.

## $\mathcal{C}_{1}$-diagrams

This research tool played an important role in some recent papers, see Czédli [3] and [4], Czédli and Grätzer [6] and Grätzer [13]; for the definition, see Czédli [3, Definition 5.3], Czédli [4, Definition 2.1], and Czédli and Grätzer [6, Definition 3.1].

In the diagram of an SPS lattice $K$, a normal edge (line) has a slope of $45^{\circ}$ or $135^{\circ}$. If it is the first, we call the edge (line) normal-up, otherwise, normaldown. Any edge (line) of slope strictly between $45^{\circ}$ and $135^{\circ}$ is steep.

A cover-preserving $S_{7}$ of a lattice $L$ is a sublattice isomorphic to $S_{7}$ such that the covers in the sublattice are covers in the lattice $L$.


Figure 1. The lattice $\mathrm{S}_{7}$.

Definition 1. A diagram of an SPS lattice $L$ is a $\mathcal{C}_{1}$-diagram if the middle edge of any cover-preserving $S_{7}$ is steep and all other edges are normal.

Czédli [3, Definition 5.11] also defines the much smaller class of $\mathcal{C}_{2}$-diagrams.
This note presents a short and direct proof of the existence theorem of $\mathcal{C}_{1-}$ diagrams, see Czédli [3, Theorem 5.5], utilizing only Theorem 3, the Structure Theorem of Slim Rectangular Lattices.

Theorem 2. Every slim, planar, semimodular lattice L has a $\mathcal{C}_{1}$-diagram.
For an SPS lattice $K$ and 4-cell $C$ in $K$, we denote the fork extension of $K$ at $C$ by $K[C]$, see Czédli and Schmidt [7] (see also [CFL2, Section 4.2]), illustrated by Figure 2.

Theorem 3 (Structure Theorem of Slim Rectangular Lattices). For every slim rectangular lattice $K$, there is a grid $G$ and sequences

$$
\begin{equation*}
G=K_{1}, K_{2}, \ldots, K_{n-1}, K_{n}=K \tag{1}
\end{equation*}
$$

of slim rectangular lattices and

$$
\begin{equation*}
C_{1}=\left\{o_{1}, c_{1}, d_{1}, i_{1}\right\}, C_{2}=\left\{o_{2}, c_{2}, d_{2}, i_{2}\right\}, \ldots, C_{n-1}=\left\{o_{n-1}, c_{n-1}, d_{n-1}, i_{n-1}\right\} \tag{2}
\end{equation*}
$$



Figure 2. (i) The 4 -cell with $0_{C}=o$ and $1_{C}=i$. (ii) Adding the elements $a$ and $b$ for the fork. (iii) Adding the fork.
of 4-cells in the appropriate lattices such that

$$
\begin{equation*}
G=K_{1}, K_{1}\left[C_{1}\right]=K_{2}, \ldots, K_{n-1}\left[C_{n-1}\right]=K_{n}=K \tag{3}
\end{equation*}
$$

Moreover, the principal ideals $\downarrow c_{n-1}$ and $\downarrow d_{n-1}$ are distributive.

Proof of Theorem 2 for rectangular lattices. Let the rectangular lattice $K$ be represented as in (3). We prove the Theorem by induction on $n$. For $n=1$, the statement is trivial. Let us assume that the statement holds for $n-1$ and so $K_{n-1}$ has $\mathcal{C}_{1}$-diagrams; we fix one. By the induction hypothesis, the 4 -cell $C=C_{n-1}$ with $0_{C}=o$ and $1_{C}=i$ has (at least) two normal edges: $[o, c]$ and $[o, d]$, see Figure 2(i) and by the last clause of Theorem 3, the principal ideals $\downarrow c$ and $\downarrow d$ are distributive.

Utilizing that $\downarrow c$ is distributive, we place the element $a$ inside the edge $[o, c]$ so that the area bounded by the (dotted) normal-up line through $a$ and the normal-up line through $o$ contains no element below $a$; we place the element $b$ symmetrically on the other side, as in Figure 2(ii). The two dotted lines meet inside $C$ since the two lower edges of $C$ are normal and the upper edges are normal or steep. We place the third element of the fork at their intersection and connect it with a steep edge to the element $i$. We add more elements to the lower left and lower right of $C$ as part of the fork construction, see Figure 2(iii). We can use normal edges for this because of the way $a$ and $b$ were placed. The diagram we obtain is a $\mathcal{C}_{1}$-diagram of $K$.

Now let $K$ be an SPS lattice. Czédli and Schmidt define in [7] a corner element $a$ of $K$ as a doubly irreducible element on the boundary of $K$ such that $a_{*}$ is meet-reducible, $a^{*}$ is join-reducible, and $a^{*}$ has exactly two lower covers.

By Czédli and Schmidt [7], $K$ is obtained from a slim rectangular lattice $\hat{K}$ with a fixed $\mathcal{C}_{1}$-diagram by removing corners. In a cover-preserving sublattice $\mathrm{S}_{7}$ of $K$, there are only two doubly irreducible elements but neither is a corner (since the upper cover of a corner has at most two lower covers). Hence, when
$\mathrm{S}_{7}$ is a cover-preserving sublattice (of $\hat{K}$ or any other SPS lattice), then this $\mathrm{S}_{7}$ contains no corner of $K$. So the $\mathrm{S}_{7}$-s remain $\mathrm{S}_{7}$-S, the steep edges remain the "legitimately" steep edges of these remaining $\mathrm{S}_{7}$-s. All other edges that are left after removing corners remain of normal slopes. Thus, $K$ is a $\mathfrak{C}_{1}$-diagram, as required.

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Received 5 May 2021
Revised 30 May 2021
Accepted 30 May 2021

