

## NOTES ON PLANAR SEMIMODULAR LATTICES IX $\mathcal{C}_1$ -DIAGRAMS

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### Abstract

A planar semimodular lattice  $L$  is *slim* if  $M_3$  is not a sublattice of  $L$ . In a recent paper, G. Czédli introduced a very powerful diagram type for slim, planar, semimodular lattices, the  $\mathcal{C}_1$ -diagrams. This short note proves the existence of such diagrams.

**Keywords:**  $\mathcal{C}_1$ -diagrams, slim planar semimodular lattice.

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### Background

The basic concepts and notation not defined in this note are available in Part I of the book [10], see [arXiv:2104.06539](#); it is freely available. We will reference it, for instance, as [CFL2, p. 4]. In particular, a planar semimodular lattice  $L$  is *slim* if  $M_3$  is not a sublattice of  $L$  and a grid  $G$  is a direct product of two nontrivial chains. For the lattice  $S_7$ , see Figure 1 and [10, pages xxi, 34]. Following my paper [15] with Knapp, a semimodular lattice  $L$  is *rectangular* if the left and right boundary chains have exactly one doubly-irreducible element each and these elements are complementary.

In my paper [16] with Lakser and Schmidt, we prove that every finite distributive lattice  $D$  can be represented as the congruence lattice of a (planar) semimodular lattice  $L$ . Since  $M_3$  sublattices play a crucial role in the construction of  $L$ , it was natural to raise the question what can be said about congruence lattices of slim, planar, semimodular (SPS) lattices (see [CFL2, Problem 24.1], originally raised in my paper [11]). The papers in the References list some contributions to this topic. In particular, my presentation [13] gently reviews the background of this topic.

### $\mathcal{C}_1$ -diagrams

This research tool played an important role in some recent papers, see Czédli [3] and [4], Czédli and Grätzer [6] and Grätzer [13]; for the definition, see Czédli [3, Definition 5.3], Czédli [4, Definition 2.1], and Czédli and Grätzer [6, Definition 3.1].

In the diagram of an SPS lattice  $K$ , a *normal edge (line)* has a slope of  $45^\circ$  or  $135^\circ$ . If it is the first, we call the edge (line) *normal-up*, otherwise, *normal-down*. Any edge (line) of slope strictly between  $45^\circ$  and  $135^\circ$  is *steep*.

A *cover-preserving  $S_7$*  of a lattice  $L$  is a sublattice isomorphic to  $S_7$  such that the covers in the sublattice are covers in the lattice  $L$ .

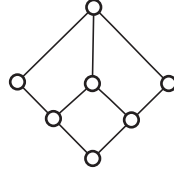


Figure 1. The lattice  $S_7$ .

**Definition 1.** A diagram of an SPS lattice  $L$  is a  $\mathcal{C}_1$ -*diagram* if the middle edge of any cover-preserving  $S_7$  is steep and all other edges are normal.

Czédli [3, Definition 5.11] also defines the much smaller class of  $\mathcal{C}_2$ -diagrams.

This note presents a short and direct proof of the existence theorem of  $\mathcal{C}_1$ -diagrams, see Czédli [3, Theorem 5.5], utilizing only Theorem 3, the Structure Theorem of Slim Rectangular Lattices.

**Theorem 2.** *Every slim, planar, semimodular lattice  $L$  has a  $\mathcal{C}_1$ -diagram.*

For an SPS lattice  $K$  and 4-cell  $C$  in  $K$ , we denote the *fork extension* of  $K$  at  $C$  by  $K[C]$ , see Czédli and Schmidt [7] (see also [CFL2, Section 4.2]), illustrated by Figure 2.

**Theorem 3** (Structure Theorem of Slim Rectangular Lattices). *For every slim rectangular lattice  $K$ , there is a grid  $G$  and sequences*

$$(1) \quad G = K_1, K_2, \dots, K_{n-1}, K_n = K$$

*of slim rectangular lattices and*

$$(2) \quad C_1 = \{o_1, c_1, d_1, i_1\}, C_2 = \{o_2, c_2, d_2, i_2\}, \dots, C_{n-1} = \{o_{n-1}, c_{n-1}, d_{n-1}, i_{n-1}\}$$

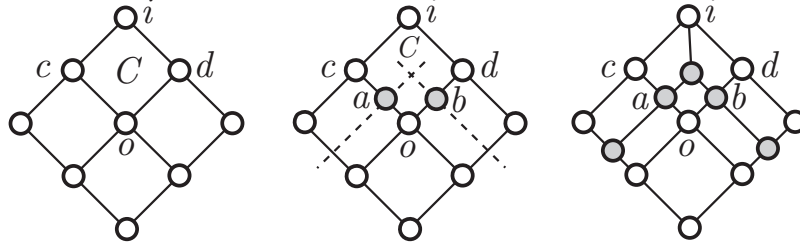


Figure 2. (i) The 4-cell with  $0_C = o$  and  $1_C = i$ . (ii) Adding the elements  $a$  and  $b$  for the fork. (iii) Adding the fork.

of 4-cells in the appropriate lattices such that

$$(3) \quad G = K_1, K_1[C_1] = K_2, \dots, K_{n-1}[C_{n-1}] = K_n = K.$$

Moreover, the principal ideals  $\downarrow c_{n-1}$  and  $\downarrow d_{n-1}$  are distributive.

**Proof of Theorem 2 for rectangular lattices.** Let the rectangular lattice  $K$  be represented as in (3). We prove the Theorem by induction on  $n$ . For  $n = 1$ , the statement is trivial. Let us assume that the statement holds for  $n - 1$  and so  $K_{n-1}$  has  $\mathcal{C}_1$ -diagrams; we fix one. By the induction hypothesis, the 4-cell  $C = C_{n-1}$  with  $0_C = o$  and  $1_C = i$  has (at least) two normal edges:  $[o, c]$  and  $[o, d]$ , see Figure 2(i) and by the last clause of Theorem 3, the principal ideals  $\downarrow c$  and  $\downarrow d$  are distributive.

Utilizing that  $\downarrow c$  is distributive, we place the element  $a$  inside the edge  $[o, c]$  so that the area bounded by the (dotted) normal-up line through  $a$  and the normal-up line through  $o$  contains no element below  $a$ ; we place the element  $b$  symmetrically on the other side, as in Figure 2(ii). The two dotted lines meet inside  $C$  since the two lower edges of  $C$  are normal and the upper edges are normal or steep. We place the third element of the fork at their intersection and connect it with a steep edge to the element  $i$ . We add more elements to the lower left and lower right of  $C$  as part of the fork construction, see Figure 2(iii). We can use normal edges for this because of the way  $a$  and  $b$  were placed. The diagram we obtain is a  $\mathcal{C}_1$ -diagram of  $K$ . ■

Now let  $K$  be an SPS lattice. Czédli and Schmidt define in [7] a *corner* element  $a$  of  $K$  as a doubly irreducible element on the boundary of  $K$  such that  $a_*$  is meet-reducible,  $a^*$  is join-reducible, and  $a^*$  has exactly two lower covers.

By Czédli and Schmidt [7],  $K$  is obtained from a slim rectangular lattice  $\hat{K}$  with a fixed  $\mathcal{C}_1$ -diagram by removing corners. In a cover-preserving sublattice  $S_7$  of  $K$ , there are only two doubly irreducible elements but neither is a corner (since the upper cover of a corner has at most two lower covers). Hence, when

$S_7$  is a cover-preserving sublattice (of  $\hat{K}$  or any other SPS lattice), then this  $S_7$  contains no corner of  $K$ . So the  $S_7$ -s remain  $S_7$ -s, the steep edges remain the “legitimately” steep edges of these remaining  $S_7$ -s. All other edges that are left after removing corners remain of normal slopes. Thus,  $K$  is a  $\mathcal{C}_1$ -diagram, as required.

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