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NOTES ON PLANAR SEMIMODULAR LATTICES IX $\mathcal{C}_1$-DIAGRAMS

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Abstract

A planar semimodular lattice L is *slim* if M_3 is not a sublattice of L. In a recent paper, G. Czédli introduced a very powerful diagram type for slim, planar, semimodular lattices, the $\mathcal{C}_1$-diagrams. This short note proves the existence of such diagrams.

Keywords: $\mathcal{C}_1$-diagrams, slim planar semimodular lattice.

2020 Mathematics Subject Classification: 06C10.

Background

The basic concepts and notation not defined in this note are available in Part I of the book [10], see `arXiv:2104.06539`; it is freely available. We will reference it, for instance, as [CFL2, p. 4]. In particular, a planar semimodular lattice L is *slim* if M_3 is not a sublattice of L and a grid G is a direct product of two nontrivial chains. For the lattice S_7, see Figure 1 and [10, pages xxi, 34]. Following my paper [15] with Knapp, a semimodular lattice L is *rectangular* if the left and right boundary chains have exactly one doubly-irreducible element each and these elements are complementary.

In my paper [16] with Lakser and Schmidt, we prove that every finite distributive lattice D can be represented as the congruence lattice of a (planar) semimodulare lattice L. Since M_3 sublattices play a crucial role in the construction of L, it was natural to raise the question what can be said about congruence lattices of slim, planar, semimodular (SPS) lattices (see [CFL2, Problem 24.1], originally raised in my paper [11]). The papers in the References list some contributions to this topic. In particular, my presentation [13] gently reviews the background of this topic.

$\mathcal{C}_1$-diagrams

This research tool played an important role in some recent papers, see Czédli [3] and [4], Czédli and Grätzer [6] and Grätzer [13]; for the definition, see Czédli [3, Definition 5.3], Czédli [4, Definition 2.1], and Czédli and Grätzer [6, Definition 3.1].

In the diagram of an SPS lattice K, a *normal edge* (*line*) has a slope of 45° or 135°. If it is the first, we call the edge (line) *normal-up*, otherwise, *normal-down*. Any edge (line) of slope strictly between 45° and 135° is *steep*.

A *cover-preserving* S_7 of a lattice L is a sublattice isomorphic to S_7 such that the covers in the sublattice are covers in the lattice L.

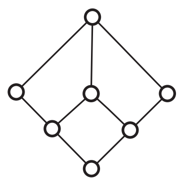

Figure 1. The lattice S_7.

Definition 1. A diagram of an SPS lattice L is a $\mathcal{C}_1$-*diagram* if the middle edge of any cover-preserving S_7 is steep and all other edges are normal.

Czédli [3, Definition 5.11] also defines the much smaller class of $\mathcal{C}_2$-diagrams.

This note presents a short and direct proof of the existence theorem of $\mathcal{C}_1$-diagrams, see Czédli [3, Theorem 5.5], utilizing only Theorem 3, the Structure Theorem of Slim Rectangular Lattices.

Theorem 2. *Every slim, planar, semimodular lattice L has a $\mathcal{C}_1$-diagram.*

For an SPS lattice K and 4-cell C in K, we denote the *fork extension* of K at C by $K[C]$, see Czédli and Schmidt [7] (see also [CFL2, Section 4.2]), illustrated by Figure 2.

Theorem 3 (Structure Theorem of Slim Rectangular Lattices)**.** *For every slim rectangular lattice K, there is a grid G and sequences*

$$G = K_1, K_2, \ldots, K_{n-1}, K_n = K \tag{1}$$

of slim rectangular lattices and

$$C_1 = \{o_1, c_1, d_1, i_1\}, C_2 = \{o_2, c_2, d_2, i_2\}, \ldots, C_{n-1} = \{o_{n-1}, c_{n-1}, d_{n-1}, i_{n-1}\} \tag{2}$$

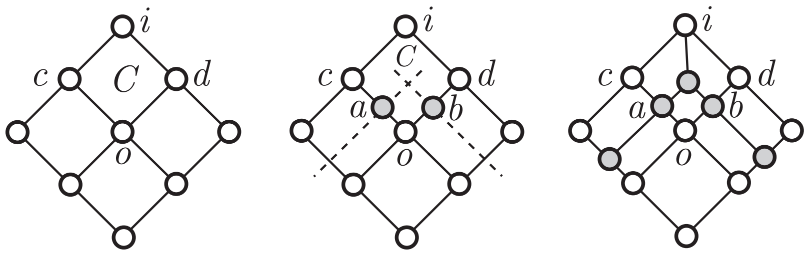

Figure 2. (i) The 4-cell with $0_C = o$ and $1_C = i$. (ii) Adding the elements a and b for the fork. (iii) Adding the fork.

of 4*-cells in the appropriate lattices such that*

$$G = K_1, K_1[C_1] = K_2, \ldots, K_{n-1}[C_{n-1}] = K_n = K. \tag{3}$$

Moreover, the principal ideals $\downarrow c_{n-1}$ *and* $\downarrow d_{n-1}$ *are distributive.*

Proof of Theorem 2 for rectangular lattices. Let the rectangular lattice K be represented as in (3). We prove the Theorem by induction on n. For $n = 1$, the statement is trivial. Let us assume that the statement holds for $n - 1$ and so K_{n-1} has $\mathcal{C}_1$-diagrams; we fix one. By the induction hypothesis, the 4-cell $C = C_{n-1}$ with $0_C = o$ and $1_C = i$ has (at least) two normal edges: $[o, c]$ and $[o, d]$, see Figure 2(i) and by the last clause of Theorem 3, the principal ideals $\downarrow c$ and $\downarrow d$ are distributive.

Utilizing that $\downarrow c$ is distributive, we place the element a inside the edge $[o, c]$ so that the area bounded by the (dotted) normal-up line through a and the normal-up line through o contains no element below a; we place the element b symmetrically on the other side, as in Figure 2(ii). The two dotted lines meet inside C since the two lower edges of C are normal and the upper edges are normal or steep. We place the third element of the fork at their intersection and connect it with a steep edge to the element i. We add more elements to the lower left and lower right of C as part of the fork construction, see Figure 2(iii). We can use normal edges for this because of the way a and b were placed. The diagram we obtain is a $\mathcal{C}_1$-diagram of K. ■

Now let K be an SPS lattice. Czédli and Schmidt define in [7] a *corner* element a of K as a doubly irreducible element on the boundary of K such that a_* is meet-reducible, a^* is join-reducible, and a^* has exactly two lower covers.

By Czédli and Schmidt [7], K is obtained from a slim rectangular lattice $\hat{K}$ with a fixed $\mathcal{C}_1$-diagram by removing corners. In a cover-preserving sublattice S_7 of K, there are only two doubly irreducible elements but neither is a corner (since the upper cover of a corner has at most two lower covers). Hence, when

S_7 is a cover-preserving sublattice (of $\hat{K}$ or any other SPS lattice), then this S_7 contains no corner of K. So the S_7-s remain S_7-s, the steep edges remain the "legitimately" steep edges of these remaining S_7-s. All other edges that are left after removing corners remain of normal slopes. Thus, K is a $\mathcal{C}_1$-diagram, as required.

References

[1] G. Czédli, *Patch extensions and trajectory colorings of slim rectangular lattices*, Algebra Universalis **72** (2014) 125–154.
https://doi.org/10.1007/s00012-014-0294-z

[2] G. Czédli, *A note on congruence lattices of slim semimodular lattices*, Algebra Universalis **72** (2014) 225–230.
https://doi.org/10.1007/s00012-014-0286-z

[3] G. Czédli, *Diagrams and rectangular extensions of planar semimodular lattices*, Algebra Universalis **77** (2017) 443–498.
https://doi.org/10.1007/s00012-017-0437-0

[4] G. Czédli, *Lamps in slim rectangular planar semimodular lattices*, Acta Sci. Math. (Szeged).
https://doi.org/10.14232/actasm-021-865-y0

[5] G. Czédli, *Non-finite axiomatizability of some finite structures.*
arXiv:2102.00526

[6] G. Czédli and G. Grätzer, *A new property of congruence lattices of slim, planar, semimodular lattices.*
arXiv:2103.04458

[7] G. Czédli and E.T. Schmidt, *Slim semimodular lattices* I. *A visual approach*, ORDER **29** (2012) 481-497.
https://doi.org/10.1007/s11083-011-9215-3

[8] G. Grätzer, *Congruences in slim, planar, semimodular lattices: The Swing Lemma*, Acta Sci. Math. (Szeged) **81** (2015) 381–397.
https://doi.org/10.1007/978-3-319-38798-_25

[9] G. Grätzer, *On a result of Gábor Czédli concerning congruence lattices of planar semimodular lattices*, Acta Sci. Math. (Szeged) **81** (2015) 25–32.
https://doi.org/10.14232/actasm-014-024-1

[10] G. Grätzer, The Congruences of a Finite Lattice, A Proof-by-Picture Approach, Second Edition (Birkhäuser, 2016). Part I is accessible at arXiv:2104.06539.
https://doi.org/10.1007/978-3-319-38798-7

[11] G. Grätzer, *Congruences of fork extensions of slim, planar, semimodular lattices*, Algebra Universalis **76** (2016) 139–154.
https://doi.org/10.1007/s00012-016-0394-z

[12] G. Grätzer, *Notes on planar semimodular lattices* VIII. *Congruence lattices of SPS lattices*, Algebra Universalis **81** (2020).
https://doi.org/10.1007/s00012-020-0641-1

[13] G. Grätzer, *Applying the Swing Lemma and Czédli diagrams to congruences of planar semimodular lattices.*
arXiv:214.13444

[14] G. Grätzer, *A gentle introduction to congruences of planar semimodular lattices*, Presentation at the meeting AAA 101 (Novi Sad, 2021).

[15] G. Grätzer and E. Knapp, *Notes on planar semimodular lattices* III. *Rectangular lattices*, Acta Sci. Math. (Szeged) **75** (2009) 29–48.

[16] G. Grätzer, H. Lakser and E.T. Schmidt, *Congruence lattices of finite semimodular lattices*, Canad. Math. Bull. **41** (1998) 290–297.
https://doi.org/10.4153/cmb-1998-041-7

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