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# ON SOME MORITA INVARIANT RADICALS OF SEMIRINGS

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**Abstract**

In this paper we prove that if $R$ and $S$ are Morita equivalent semirings via Morita context $(R, S, P, Q, \theta, \phi)$, then there exists a one-to-one inclusion preserving correspondence between the set of all prime ((right) strongly prime, uniformly strongly prime) ideals of $R$ and the set of all prime (resp. (right) strongly prime, uniformly strongly prime) subsemimodules of $P$. We also show that prime radicals, (right) strongly prime radicals, uniformly strongly prime radicals are preserved under Morita equivalence of semirings.

**Keywords:** Morita context, Morita equivalence, semiring, semimodule, radical, prime subsemimodule, strongly prime subsemimodule, uniformly strongly prime subsemimodule.

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## 1. Introduction

In 1975, Handelman and Lawrence [7] introduced the notion of (right) strongly prime ring motivated by the notion of primitive group ring and characterized them. A ring $R$ is said to be (right) strongly prime if for each non-zero element $r$ of $R$, there is a finite subset $S(r)$ (right insulator for $r$) of $R$ such that for $t \in R$, $\{rst \mid s \in S(r)\} = \{0\}$ implies $t = 0$. Later in the year 1987, Olson [13] introduced the notion of uniformly strongly prime ring and uniformly strongly prime ideals of a ring. A ring $R$ is called uniformly strongly prime if the same insulator may be chosen for each non-zero element of $R$. In order to investigate

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the validity of these concepts of ring theory in the settings of semiring, Dutta and Das generalized the notion of (right) strongly prime rings and uniformly strongly prime rings to (right) strongly prime semirings [3] and uniformly strongly prime semirings [4] respectively.

In 1958 Morita [12] established the Morita equivalence theory for rings as one of the most important and fundamental tools in studying the structure of rings. In 2011, Katsov and Nam [10] transferred the ring theoretic approach of Morita equivalence to semirings, which was later connected with Morita context for semirings by Sardar *et al.* [14]. In [6, 15] Sardar and Gupta listed some properties that remain invariant under the Morita equivalence of semirings via Morita context $(R, S, P, Q, \theta, \phi)$. As a continuation of [6, 15] using the nice interplay between various components of a Morita context, in this paper we introduce the notion of (right) strongly prime subsemimodules and uniformly strongly prime subsemimodules of a semimodule related to a Morita context for semirings and prove that under Morita equivalence of semirings, structures like prime radical, (right) strongly prime radical, uniformly strongly prime radical etc. are preserved. We organize the paper as follows. In Section 2, we recall some necessary preliminaries on semirings, semimodules and Morita equivalence of semirings. In Section 3, we define (right) strongly prime subsemimodules and uniformly strongly prime subsemimodules of a semimodule related to a Morita context for semirings and prove that if $R$ and $S$ are Morita equivalent semirings via Morita context $(R, S, P, Q, \theta, \phi)$, then there exists a one-to-one inclusion preserving correspondence between the set of all prime ((right) strongly prime, uniformly strongly prime) ideals of $R$ and the set of all prime (resp. (right) strongly prime, uniformly strongly prime) subsemimodules of $P$. Similar correspondences can be established between $R$ and $Q$, $S$ and $P$, $S$ and $Q$, which in turn results in a one-to-one inclusion preserving correspondence between the set of all prime ((right) strongly prime, uniformly strongly prime) ideals of $R$ and $S$. Lastly we have shown that structures like prime radical, (right) strongly prime radical, uniformly strongly prime radical of semirings are preserved under Morita equivalence.

## 2. Preliminaries

A *semiring* [5] is a nonempty set $R$ on which operations of addition and multiplication have been defined such that (1) $(R, +)$ is a commutative monoid with identity element 0, (2) $(R, \cdot)$ is a monoid with identity element $1_R$, (3) multiplication distributes over addition from either side, (4) $0r = 0 = r0$ for all $r \in R$. A left *$R$-semimodule* over a semiring $R$ is a commutative monoid $(P, +, 0_P)$ together with a scalar multiplication from $R \times P$ to $P$, denoted by $(r, p) \mapsto rp$, which satisfies the following identities: (1) $(r + r')p = rp + r'p$, (2) $r(p + p') = rp + rp'$, (3)

$(rr')p = r(r'p)$, (4) $1_R p = p$, (5) $r0_P = 0_P = 0p$ for all $r, r' \in R$ and $p, p' \in P$. Right $R$-semimodules are defined analogously. For given semirings $R$ and $S$, an $R$-$S$-bisemimodule $P$ (in symbols, ${}_RP_S$) is a commutative monoid which is both a left $R$-semimodule and a right $S$-semimodule, with $(rp)s = r(ps)$ for all $r \in R,\ s \in S$ and $p \in P$. A nonempty subset $I$ of a semiring $R$ is called an ideal [5] of $R$ if $i + i' \in I$ and $ri, ir \in I$ for any $i, i' \in I$ and $r \in R$. A nonempty subset $M$ of an $R$-$S$-bisemimodule ${}_RP_S$ is called a subsemimodule of $P$ if $p + p' \in M$ and $rp, ps \in M$ for any $r \in R,\ s \in S$ and $p, p' \in M$. A proper ideal $I$ of a semiring $R$ is called prime ideal [5] if for ideals $H, K \subseteq R$, $HK \subseteq I$ implies either $H \subseteq I$ or $K \subseteq I$. The prime radical (also called lower nil radical in [5]) of the semiring $R$ is defined to be the intersection of all prime ideals of $R$. An ideal $I$ of a semiring $R$ is called a *$k$-ideal* [8] (also called subtractive ideal in [5]) of $R$ if for $x \in I,\ y \in R,\ x + y \in I$ implies $y \in I$. A subsemimodule $N$ of a semimodule $P$ is called a *$k$-subsemimodule* [2] (called subtractive subsemimodule in [5]) of $P$ if for $x \in N,\ y \in P,\ x + y \in N$ implies $y \in N$.

If $R$ and $S$ are two semirings, ${}_RP_S$ and ${}_SQ_R$ are $R$-$S$-bisemimodule and $S$-$R$-bisemimodule respectively, and $\theta : P \otimes Q \to R$ and $\phi : Q \otimes P \to S$ are respectively $R$-$R$-bisemimodule homomorphism and $S$-$S$-bisemimodule homomorphism such that $\theta(p \otimes q)p' = p\phi(q \otimes p')$ and $\phi(q \otimes p)q' = q\theta(p \otimes q')$ for all $p, p' \in P$ and $q, q' \in Q$ then the sixtuple $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$ is called a Morita context for semirings. Two semirings $R$, $S$ are Morita equivalent if and only if there exists a Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$ with $\theta$ and $\phi$ surjective [14]. Readers are referred to [5] and [10] for more notions of semirings, semimodules and Morita equivalence of semirings. We also refer to [1] and [11] for notions of category theory.

Let $R, S$ be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then for subsets $X \subseteq P$ and $Y \subseteq Q$ we write

$$\theta(X \otimes Y) = \left\{ \sum_{k=1}^{n} \theta(p_k \otimes q_k) \mid p_k \in X,\ q_k \in Y \text{ for all } k \text{ and } n \in \mathbb{Z}^+ \right\} \text{ and}$$

$$\phi(Y \otimes X) = \left\{ \sum_{k=1}^{n} \phi(q_k \otimes p_k) \mid q_k \in Y,\ p_k \in X \text{ for all } k \text{ and } n \in \mathbb{Z}^+ \right\}.$$

Also for subsets $U \subseteq R,\ X \subseteq P$ we write, $UX = \left\{ \sum_{k=1}^{n} r_k p_k \mid r_k \in U,\ p_k \in X \text{ for all } k \text{ and } n \in \mathbb{Z}^+ \right\}$, similarly for $V \subseteq S,\ Y \subseteq Q$, we define $XV,\ YU,\ VY$.

Let $R$ and $S$ be Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then in [6, Theorem 2.2] we see that the lattice of ideals of $R$ and the lattice of subsemimodules of $P$ are isomorphic. The isomorphisms are given below.

[2] In the present article subtractiveness is replaced by $k$.

$f_1 : Id(R) \to Sub(P)$ and $g_1 : Sub(P) \to Id(R)$ are defined by

$$f_1(I) := \left\{ \sum_{k=1}^{n} i_k p_k \mid p_k \in P,\ i_k \in I \text{ for all } k, n \in \mathbb{Z}^+ \right\} = IP, \quad \text{and}$$

$$g_1(M) := \left\{ \sum_{k=1}^{n} \theta(p_k \otimes q_k) \mid p_k \in M,\ q_k \in Q \text{ for all } k, n \in \mathbb{Z}^+ \right\} = \theta(M \otimes Q).$$

Similar isomorphisms can be defined for other pairs of the Morita context as follows.

$f_2 : Id(R) \to Sub(Q)$ and $g_2 : Sub(Q) \to Id(R)$ are defined by

$$f_2(I) := \left\{ \sum_{k=1}^{n} q_k i_k \mid q_k \in Q,\ i_k \in I \text{ for all } k,\ n \in \mathbb{Z}^+ \right\} = QI, \quad \text{and}$$

$$g_2(N) := \left\{ \sum_{k=1}^{n} \theta(p_k \otimes q_k) \mid p_k \in P,\ q_k \in N \text{ for all } k,\ n \in \mathbb{Z}^+ \right\} = \theta(P \otimes N)\,.$$

We can also define $f_3 : Id(S) \to Sub(P)$, $g_3 : Sub(P) \to Id(S)$, $f_4 : Id(S) \to Sub(Q)$, $g_4 : Sub(Q) \to Id(S)$ in a similar way. Again in [15, Theorem 2.2] we see that the lattice of ideals of $R$ and the lattice of ideals of $S$ are isomorphic via the following lattice isomorphisms. Moreover these isomorphisms preserve $k$-ideals [15, proof of Theorem 2.4].

$\Theta : Id(S) \to Id(R)$ and $\Phi : Id(R) \to Id(S)$ are defined by

$$\begin{aligned} \Theta(J) &:= \left\{ \sum_{k=1}^{n} \theta(p_k j_k \otimes q_k) \mid p_k \in P,\ q_k \in Q,\ j_k \in J \text{ for all } k \text{ and, } n \in \mathbb{Z}^+ \right\} \\ &= \theta(PJ \otimes Q) \\ \Phi(I) &:= \left\{ \sum_{k=1}^{n} \phi(q_k i_k \otimes p_k) \mid p_k \in P,\ q_k \in Q,\ i_k \in I \text{ for all } k \text{ and, } n \in \mathbb{Z}^+ \right\} \\ &= \phi(QI \otimes P)\,. \end{aligned}$$

Throughout this paper unless stated otherwise $1_R$ and $1_S$ denote respectively the identity elements of the Morita equivalent semirings $R$ and $S$ of the Morita context $(R, S, {}_R P_S, {}_S Q_R, \theta, \phi)$ and also we take $1_R = \sum_{v=1}^{n'} \theta(\bar{p_v} \otimes \bar{q_v})$, $1_S = \sum_{u=1}^{m'} \phi(\tilde{q_u} \otimes \tilde{p_u})$ (existence of such $\bar{p_v},\ \bar{q_v},\ \tilde{q_u},\ \tilde{p_u}$ is guaranteed since $\theta$ and $\phi$ are surjective).

## 3. Main results

For the Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$, unless otherwise stated $f_i$'s and $g_i$'s $(i = 1, 2, 3, 4)$ are mappings as explained above.

**Definition 3.1.** Let $R, S$ be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. A subsemimodule $M$ of $P$ is said to be a prime subsemimodule [2] if for subsemimodules $A, B$ of $P$, $\theta(A \otimes Q)B \subseteq M$ implies either $A \subseteq M$ or $B \subseteq M$. Analogously using $\phi$ we can define prime subsemimodule of $Q$.

**Proposition 3.2.** *Let $R, S$ be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then the mapping $f_1 : Id(R) \to Sub(P)$ defines a one-to-one inclusion preserving correspondence between the set of all prime ideals of $R$ and the set of all prime subsemimodules of $P$.*

***Proof.*** Let $I$ be a prime ideal of $R$ and $A$ and $B$ be subsemimodules of $P$ such that $\theta(A \otimes Q)B \subseteq f_1(I)$. Then using the fact that $f_1$ and $g_1$ are mutually inverse lattice isomorphisms and $I$ is a prime ideal, we have,

$$\begin{aligned}
&\theta(\theta(A \otimes Q)B \otimes Q) \subseteq \theta(f_1(I) \otimes Q)\\
&\text{i.e.,}\quad \theta(A \otimes Q)\,\theta(B \otimes Q) \subseteq g_1(f_1(I)) = I\\
&\text{i.e.,}\quad \theta(A \otimes Q) \subseteq I \quad\text{or}\quad \theta(B \otimes Q) \subseteq I\\
&\text{i.e.,}\quad g_1(A) \subseteq I \quad\text{or}\quad g_1(B) \subseteq I\\
&\text{i.e.,}\quad A = f_1(g_1(A)) \subseteq f_1(I) \quad\text{or}\quad B = f_1(g_1(B)) \subseteq f_1(I).
\end{aligned}$$

Hence $f_1(I)$ is a prime subsemimodule of $P$.

Conversely, let $M$ be a prime subsemimodule of $P$ and $I$ and $J$ be ideals of $R$ such that $IJ \subseteq g_1(M)$. Then using the fact that $\theta$ is surjective, i.e., $\theta(P \otimes Q) = R$ and $M$ is a prime subsemimodule, we have,

$$\begin{aligned}
&I\theta(P \otimes Q)J = IRJ \subseteq IJ \subseteq g_1(M)\\
&\text{i.e.,}\quad \theta(IP \otimes Q)J \subseteq g_1(M)\\
&\text{i.e.,}\quad \theta(IP \otimes Q)JP \subseteq g_1(M)P = f_1(g_1(M)) = M\\
&\text{i.e.,}\quad IP \subseteq M \quad\text{or}\quad JP \subseteq M\\
&\text{i.e.,}\quad f_1(I) \subseteq M \quad\text{or}\quad f_1(J) \subseteq M\\
&\text{i.e.,}\quad I = g_1(f_1(I)) \subseteq g_1(M) \quad\text{or}\quad J = g_1(f_1(J)) \subseteq g_1(M).
\end{aligned}$$

Therefore $g_1(M)$ is a prime ideal of $R$. Since $f_1$ and $g_1$ are mutually inverse maps, the proof follows. ■

Analogously we obtain the following result.

**Proposition 3.3.** *Let $R, S$ be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then the mapping $f_4 : Id(S) \to Sub(Q)$ defines a one-to-one inclusion preserving correspondence between the set of all prime ideals of $S$ and the set of all prime subsemimodules of $Q$.*

**Proposition 3.4.** *Let $R, S$ be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then the mapping $f_2 : Id(R) \to Sub(Q)$ defines a one-to-one inclusion preserving correspondence between the set of all prime ideals of $R$ and the set of all prime subsemimodules of $Q$.*

***Proof.*** Let $I$ be a prime ideal of $R$ and $C$ and $D$ be subsemimodules of $Q$ such that $\phi(C \otimes P)D \subseteq f_2(I)$. Then we have,

$$\begin{aligned} & g_2(C)g_2(D) = \theta(P \otimes C)\theta(P \otimes D) \\ & \subseteq \theta(\theta(P \otimes C)P \otimes D) \subseteq \theta(P\phi(C \otimes P) \otimes D) \\ & \subseteq \theta(P \otimes \phi(C \otimes P)D) \subseteq \theta(P \otimes f_2(I)) = g_2(f_2(I)) = I. \end{aligned}$$

Since $I$ is a prime ideal, we have,

$$\begin{aligned} & g_2(C) \subseteq I \quad \text{or} \quad g_2(D) \subseteq I \\ & \text{i.e.,} \quad C = f_2(g_2(C)) \subseteq f_2(I) \quad \text{or} \quad D = f_2(g_2(D)) \subseteq f_2(I). \end{aligned}$$

Hence $f_2(I)$ is a prime subsemimodule of $Q$.

Conversely, let $N$ be a prime subsemimodule of $Q$ and $I$ and $J$ be ideals of $R$ such that $IJ \subseteq g_2(N)$. Then using the fact that $f_2$ and $g_2$ are mutually inverse lattice isomorphisms and $N$ is a prime subsemimodule, we have,

$$\begin{aligned} & I\theta(P \otimes Q)J = IRJ \subseteq IJ \subseteq g_2(N) \\ & \text{i.e.,} \quad I\theta(P \otimes QJ) \subseteq g_2(N) \\ & \text{i.e.,} \quad QI\theta(P \otimes QJ) \subseteq Qg_2(N) = f_2(g_2(N)) = N \\ & \text{i.e.,} \quad QI \subseteq N \quad \text{or} \quad QJ \subseteq N \\ & \text{i.e.,} \quad f_2(I) \subseteq N \quad \text{or} \quad f_2(J) \subseteq N \\ & \text{i.e.,} \quad I = g_2(f_2(I)) \subseteq g_2(N) \quad \text{or} \quad J = g_2(f_2(J)) \subseteq g_2(N). \end{aligned}$$

Therefore $g_2(N)$ is a prime ideal of $R$. Since $f_2$ and $g_2$ are mutually inverse lattice isomorphisms, the proof follows. ■

Analogously we obtain the following result.

**Proposition 3.5.** *Let $R, S$ be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then the mapping $f_3 : Id(S) \to Sub(P)$ defines a one-to-one inclusion preserving correspondence between the set of all prime ideals of $S$ and the set of all prime subsemimodules of $P$.*

We recall below the following Proposition from [15, Prop. 2.7] for its use in proving some of the subsequent Theorems (cf. Theorem 3.8, Theorem 3.18 and Theorem 3.29).

**Proposition 3.6.** *If $\{A_i \mid i \in I\}$ is an arbitrary set of ideals of semiring $R$, then $\Phi\big(\bigcap_{i\in I} A_i\big) = \bigcap_{i\in I} \Phi(A_i)$. Similar results hold for the map $\Theta$.*

Although [15, Theorem 2.8] gives a direct proof of the following result we can prove it using Prop. 3.2 and Prop. 3.5.

**Theorem 3.7.** *Let $R, S$ be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then the mapping $\Theta : Id(S) \to Id(R)$ defines a one-to-one inclusion preserving correspondence between the set of all prime ideals of $S$ and the set of all prime ideals of $R$.*

***Proof.*** Let $J$ be a prime ideal of $S$. Then from Prop. 3.5, $f_3(J) = PJ$ is a prime subsemimodule of $P$ and therefore from the proof of Prop. 3.2 we see that, $g_1(PJ)$ is a prime ideal of $R$. Since $\Theta(J) = \theta(PJ \otimes Q) = g_1(PJ)$, therefore $\Theta(J)$ is a prime ideal of $R$. Analogously we can prove that for any prime ideal $I$ of $R$, $\Phi(I)$ is a prime ideal of $S$. Since $\Theta$ and $\Phi$ are mutually inverse lattice isomorphisms, the proof follows. ∎

**Theorem 3.8.** *Let $R, S$ be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then $\Theta : Id(S) \to Id(R)$ maps the prime radical $(\beta(S))$ of $S$, to the prime radical $(\beta(R))$ of $R$, i.e., $\Theta(\beta(S)) = \beta(R)$.*

***Proof.*** Let $\mathcal{C}_P(R)$ and $\mathcal{C}_P(S)$ be the collection of all prime ideals of $R$ and $S$ respectively. Then using Prop. 3.6, Theorem 3.7 and the fact that $\Theta$ preserves inclusion we have,

$$\Theta(\beta(S)) = \Theta\left(\bigcap_{J\in\mathcal{C}_P(S)} J\right) = \bigcap_{J\in\mathcal{C}_P(S)} \Theta(J) \supseteq \bigcap_{I\in\mathcal{C}_P(R)} I = \beta(R).$$

Similarly we have $\Phi(\beta(R)) \supseteq \beta(S)$. Since $\Theta$ and $\Phi$ are mutually inverse lattice isomorphisms, we have $\beta(R) \supseteq \Theta(\beta(S))$. Hence, $\Theta(\beta(S)) = \beta(R)$. ∎

**Definition 3.9** [3]**.** An ideal $I$ of a semiring $R$ is said to be a (right) strongly prime ideal of $R$ if for every $r$ in $R$ with $r \notin I$, there exists a finite subset $F \subseteq \langle r\rangle$ (ideal generated by $r$) such that for $r' \in R$, $Fr' \subseteq I$ implies that $r' \in I$.

**Definition 3.10.** Let $R, S$ be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. A subsemimodule $M$ of $P$ is said to be a (right) strongly prime subsemimodule if for every element $p$ of $P$ with $p \notin M$ there exist finite subsets $X \subseteq \langle p\rangle$ (subsemimodule generated by $p$) and $Y \subseteq Q$ such that for $p' \in P$, $\theta(X \otimes Y)p' \subseteq M$ implies that $p' \in M$.

**Definition 3.11.** Let $R, S$ be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. A subsemimodule $N$ of $Q$ is said to be a (right) strongly prime subsemimodule if for every element $q$ of $Q$ with $q \notin N$ there exist finite subsets $Y \subseteq \langle q \rangle$ (subsemimodule generated by $q$) and $X \subseteq P$ such that for $q' \in Q$, $\phi(Y \otimes X)q' \subseteq N$ implies that $q' \in N$.

**Proposition 3.12.** *Let $R, S$ be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then the mapping $f_1 : Id(R) \to Sub(P)$ defines a one-to-one inclusion preserving correspondence between the set of all (right) strongly prime ideals of $R$ and the set of all (right) strongly prime subsemimodules of $P$.*

***Proof.*** Let $I$ be a (right) strongly prime ideal of $R$ and $p \notin f_1(I) = IP$ for some $p \in P$. Then there exists $k \in \{1, 2, \ldots, m'\}$ such that $\theta(p \otimes \tilde{q_k}) \notin I$, otherwise $p = p1_S = p\sum_{u=1}^{m'} \phi(\tilde{q_u} \otimes \tilde{p_u}) = \sum_{u=1}^{m'} \theta(p \otimes \tilde{q_u})\tilde{p_u} \in IP$ - a contradiction. Since $\theta(p \otimes \tilde{q_k}) \notin I$, therefore by hypothesis there exists a finite subset $F \subseteq \langle \theta(p \otimes \tilde{q_k}) \rangle$ such that for $r' \in R$, $Fr' \subseteq I$ implies that $r' \in I$. Let $Y = \{\bar{q_v} \mid v = 1, 2, \ldots, n'\} \subseteq Q$ and $X = \{r\bar{p_v} \mid r \in F, v = 1, 2, \ldots, n'\}$. Then both $Y$ and $X$ are finite subsets of $Q$ and $P$ respectively. Since every element of $X$ is of the form $r\bar{p_v}$ for some $r \in F$, i.e., $r = \sum_{i=1}^{l} r_i\theta(p \otimes \tilde{q_k})r_i'$, for some $l \in \mathbb{Z}^+$, where $r_i, r_i' \in R$ for all $i = 1, 2, \ldots, l$, therefore $r\bar{p_v} = \sum_{i=1}^{l} r_i\theta(p \otimes \tilde{q_k})r_i'\bar{p_v} = \sum_{i=1}^{l} r_ip\phi(\tilde{q_k} \otimes r_i'\bar{p_v}) \in RpS = \langle p \rangle$, i.e., $X \subseteq \langle p \rangle$.

Suppose $p' \in P$ such that $\theta(X \otimes Y)p' \subseteq f_1(I) = IP$. Let $r \in F$ and $q \in Q$. Then using the fact that $f_1$ and $g_1$ are mutually inverse maps we have, $r\theta(p' \otimes q) = r1_R\theta(p' \otimes q) = r\sum_{v=1}^{n'} \theta(\bar{p_v} \otimes \bar{q_v})\theta(p' \otimes q) = \sum_{v=1}^{n'} \theta(r\bar{p_v} \otimes \bar{q_v})\theta(p' \otimes q) = \theta\Big(\sum_{v=1}^{n'} \theta(r\bar{p_v} \otimes \bar{q_v})p' \otimes q\Big) \in \theta(\theta(X \otimes Y)p' \otimes q) \subseteq \theta(f_1(I) \otimes Q) = g_1(f_1(I)) = I$.

Since every element of $F\theta(p' \otimes q)$ is a finite sum of elements of the form $r\theta(p' \otimes q)$ for some $r \in F$, therefore we see that $F\theta(p' \otimes q) \subseteq I$. Then by our hypothesis we have $\theta(p' \otimes q) \in I$, which is true for all $q \in Q$, in particular for all $\tilde{q_u}$, where $u = 1, 2, \ldots, m'$. Therefore $p' = p'1_S = p'\sum_{u=1}^{m'} \phi(\tilde{q_u} \otimes \tilde{p_u}) = \sum_{u=1}^{m'} \theta(p' \otimes \tilde{q_u})\tilde{p_u} \in IP = f_1(I)$. Hence $f_1(I)$ is a (right) strongly prime subsemimodule of $P$. Conversely, let $M$ be a (right) strongly prime subsemimodule of $P$ and $r \in R$ such that $r \notin g_1(M) = \theta(M \otimes Q)$. Then there exists $k \in \{1, 2, \ldots, n'\}$ such that $r\bar{p_k} \notin M$, otherwise $r = r1_R = r\sum_{v=1}^{n'} \theta(\bar{p_v} \otimes \bar{q_v}) = \sum_{v=1}^{n'} \theta(r\bar{p_v} \otimes \bar{q_v}) \in \theta(M \otimes Q) = g_1(M)$ — a contradiction. Since $r\bar{p_k} \notin M$, therefore there exist finite subsets $X \subseteq \langle r\bar{p_k} \rangle$ and $Y \subseteq Q$ such that for $p' \in P$, $\theta(X \otimes Y)p' \subseteq M$ implies that $p' \in M$. Let $F = \{\theta(x \otimes y) \mid x \in X,\ y \in Y\}$. Then clearly $F$ is a finite subset of $R$ and for any $\theta(x \otimes y) \in F$ we have, $\theta(x \otimes y) \in \theta(\langle r\bar{p_k} \rangle \otimes Q) = \theta(R(r\bar{p_k})S \otimes Q) \subseteq Rr\theta(\bar{p_k}S \otimes Q) \subseteq RrR = \langle r \rangle$, i.e., $F \subseteq \langle r \rangle$.

Suppose $r' \in R$ such that $Fr' \subseteq g_1(M) = \theta(M \otimes Q)$. Let $x \in X,\ y \in Y$ and $p \in P$. Then using the fact that $f_1$ and $g_1$ are mutually inverse maps we have, $\theta(x \otimes y)(r'p) \in F(r'p) = (Fr')p \subseteq g_1(M)P = f_1(g_1(M)) = M$. Since every element of the set $\theta(X \otimes Y)(r'p)$ is a finite sum of elements of the form $\theta(x \otimes y)r'p$ for some $x \in X,\ y \in Y$, therefore we see that $\theta(X \otimes Y)r'p \subseteq M$. Then by our hypothesis we have $r'p \in M$, which is true for all $p \in P$, in particular for all $\bar{p_v}$, where $v = 1, 2, \ldots, n'$. Therefore $r' = r'1_R = r' \sum_{v=1}^{n'} \theta(\bar{p_v} \otimes \bar{q_v}) = \sum_{v=1}^{n'} \theta(r'\bar{p_v} \otimes \bar{q_v}) \in \theta(M \otimes Q) = g_1(M)$. Thus $g_1(M)$ is a (right) strongly prime ideal of $R$. Since $f_1$ and $g_1$ are mutually inverse lattice isomorphisms, the proof follows. ■

Analogously we obtain the following result.

**Proposition 3.13.** *Let $R, S$ be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then the mapping $f_4 : Id(S) \to Sub(Q)$ defines a one-to-one inclusion preserving correspondence between the set of all (right) strongly prime ideals of $S$ and the set of all (right) strongly prime subsemimodules of $Q$.*

**Proposition 3.14.** *Let $R, S$ be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then the mapping $f_2 : Id(R) \to Sub(Q)$ defines a one-to-one inclusion preserving correspondence between the set of all (right) strongly prime ideals of $R$ and the set of all (right) strongly prime subsemimodules of $Q$.*

***Proof.*** Let $I$ be a (right) strongly prime ideal of $R$ and $q \notin f_2(I) = QI$ for some $q \in Q$. Then there exists $k \in \{1, 2, \ldots, m'\}$ such that $\theta(\tilde{p_k} \otimes q) \notin I$, otherwise $q = 1_Sq = \sum_{u=1}^{m'} \phi(\tilde{q_u} \otimes \tilde{p_u})q = \sum_{u=1}^{m'} \tilde{q_u}\theta(\tilde{p_u} \otimes q) \in QI$ — a contradiction. Since $\theta(\tilde{p_k} \otimes q) \notin I$, therefore by hypothesis there exists a finite subset $F \subseteq \langle \theta(\tilde{p_k} \otimes q) \rangle$ such that for $r' \in R$, $Fr' \subseteq I$ implies that $r' \in I$. Let $Y = \{\bar{q_v}r \mid r \in F,\ v = 1, 2, \ldots, n'\} \subseteq Q$ and $X = \{\tilde{p_u} \mid u = 1, 2, \ldots, m'\}$. Then both $Y$ and $X$ are finite subsets of $Q$ and $P$ respectively. Since every element of $Y$ is of the form $\bar{q_v}r$ for some $r \in F$, i.e., $r = \sum_{i=1}^{l} r_i\theta(\tilde{p_k} \otimes q)r'_i$ for some $l \in \mathbb{Z}^+$, where $r_i, r'_i \in R$ for all $i = 1, 2, \ldots, l$, therefore $\bar{q_v}r = \bar{q_v} \sum_{i=1}^{l} r_i\theta(\tilde{p_k} \otimes q)r'_i = \sum_{i=1}^{l} \phi(\bar{q_v} \otimes r_i\tilde{p_k})qr'_i \in SqR = \langle q \rangle$, i.e., $Y \subseteq \langle q \rangle$.

Suppose $q' \in Q$ such that $\phi(Y \otimes X)q' \subseteq f_2(I) = QI$. Let $r \in F$ and $u \in \{1, 2, \ldots, m'\}$. Then using the fact that $f_2$ and $g_2$ are mutually inverse maps we have, $r\theta(\tilde{p_u} \otimes q') = 1_Rr\theta(\tilde{p_u} \otimes q') = \sum_{v=1}^{n'} \theta(\bar{p_v} \otimes \bar{q_v})r\theta(\tilde{p_u} \otimes q') = \sum_{v=1}^{n'} \theta(\bar{p_v} \otimes \bar{q_v}r\theta(\tilde{p_u} \otimes q')) = \sum_{v=1}^{n'} \theta(\bar{p_v} \otimes \phi(\bar{q_v}r \otimes \tilde{p_u})q') \in \theta(P \otimes \phi(Y \otimes X)q') \subseteq \theta(P \otimes f_2(I)) = g_2(f_2(I)) = I$. Since every element of $F\theta(\tilde{p_u} \otimes q')$ is a finite sum of elements of the form $r\theta(\tilde{p_u} \otimes q')$ for some $r \in F$, therefore we see that $F\theta(\tilde{p_u} \otimes q') \subseteq I$. Then by our hypothesis we have $\theta(\tilde{p_u} \otimes q') \in I$, which is true for all $\tilde{p_u}$, where $u = 1, 2, \ldots, m'$. Therefore $q' = 1_Sq' = \sum_{u=1}^{m'} \phi(\tilde{q_u} \otimes \tilde{p_u})q' = \sum_{u=1}^{m'} \tilde{q_u}\theta(\tilde{p_u} \otimes q') \in QI = f_2(I)$. Hence $f_2(I)$ is a (right) strongly prime subsemimodule of $Q$.

Conversely, let $N$ be a (right) strongly prime subsemimodule of $Q$ and $r \in R$ such that $r \notin g_2(N) = \theta(P \otimes N)$. Then there exists $k \in \{1, 2, \ldots, n'\}$ such that

$\bar{q_k}r \notin N$, otherwise $r = 1_R r = \sum_{v=1}^{n'} \theta(\bar{p_v} \otimes \bar{q_v})r = \sum_{v=1}^{n'} \theta(\bar{p_v} \otimes \bar{q_v}r) \in \theta(P \otimes N) = g_2(N)$ — a contradiction. Since $\bar{q_k}r \notin N$, therefore there exist finite subsets $X \subseteq P$, $Y \subseteq \langle \bar{q_k}r \rangle$ such that for $q' \in Q$, $\phi(Y \otimes X)q' \subseteq N$ implies that $q' \in N$. Let $F = \{\theta(\tilde{p_u} \otimes y)\theta(x \otimes \bar{q_v}) \mid y \in Y,\ x \in X,\ u = 1, 2, \ldots, m',\ v = 1, 2, \ldots, n'\}$. Then clearly $F$ is a finite subset of $R$ and since $y \in \langle \bar{q_k}r \rangle$, $y = \sum_{i=1}^{l} s_i(\bar{q_k}r)r_i$ for some $l \in \mathbb{Z}^+$, where $s_i \in S$, $r_i \in R$ for all $i = 1, 2, \ldots, l$, therefore for any element of $F$, $\theta(\tilde{p_u} \otimes y)\theta(x \otimes \bar{q_v}) = \theta(\tilde{p_u} \otimes \sum_{i=1}^{l} s_i(\bar{q_k}r)r_i)\theta(x \otimes \bar{q_v}) = \sum_{i=1}^{l} \theta(\tilde{p_u} \otimes s_i\bar{q_k})rr_i\theta(x \otimes \bar{q_v}) \in \langle r \rangle$, i.e., $F \subseteq \langle r \rangle$.

Suppose $r' \in R$ such that $Fr' \subseteq g_2(N) = \theta(P \otimes N)$. Let $x \in X$, $y \in Y$ and $v \in \{1, 2, \ldots, n'\}$. Then using the fact that $f_2$ and $g_2$ are mutually inverse maps we have, $\phi(y \otimes x)\bar{q_v}r' = 1_S\phi(y \otimes x)\bar{q_v}r' = \sum_{u=1}^{m'} \phi(\tilde{q_u} \otimes \tilde{p_u})\phi(y \otimes x)\bar{q_v}r' = \sum_{u=1}^{m'} \phi(\tilde{q_u} \otimes \tilde{p_u})y\theta(x \otimes \bar{q_v})r' = \sum_{u=1}^{m'} \tilde{q_u}\theta(\tilde{p_u} \otimes y)\theta(x \otimes \bar{q_v})r' \in QFr' \subseteq Qg_2(N) = f_2(g_2(N)) = N$. Since every element of the set $\phi(Y \otimes X)(\bar{q_v}r')$ is a finite sum of elements of the form $\phi(y \otimes x)\bar{q_v}r'$ for some $x \in X$, $y \in Y$, therefore we see that $\phi(Y \otimes X)(\bar{q_v}r') \subseteq N$. Then by our hypothesis we have $\bar{q_v}r' \in N$, which is true for all $\bar{q_v}$, where $v = 1, 2, \ldots, n'$. Therefore $r' = 1_R r' = \sum_{v=1}^{n'} \theta(\bar{p_v} \otimes \bar{q_v})r' = \sum_{v=1}^{n'} \theta(\bar{p_v} \otimes \bar{q_v}r') \in \theta(P \otimes N) = g_2(N)$. Thus $g_2(N)$ is a (right) strongly prime ideal of $R$. Since $f_2$ and $g_2$ are mutually inverse lattice isomorphisms, the proof follows. ■

Analogously we obtain the following result.

**Proposition 3.15.** *Let $R, S$ be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then the mapping $f_3 : Id(S) \to Sub(P)$ defines a one-to-one inclusion preserving correspondence between the set of all (right) strongly prime ideals of $S$ and the set of all (right) strongly prime subsemimodules of $P$.*

**Theorem 3.16.** *Let $R, S$ be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then the mapping $\Theta : Id(S) \to Id(R)$ defines a one-to-one inclusion preserving correspondence between the set of all (right) strongly prime ideals of $S$ and the set of all (right) strongly prime ideals of $R$.*

***Proof.*** Let $J$ be a (right) strongly prime ideal of $S$. Then from Prop. 3.15, $f_3(J) = PJ$ is a (right) strongly prime subsemimodule of $P$ and therefore from the proof of Prop. 3.12 we see that, $g_1(PJ)$ is a (right) strongly prime ideal of $R$. Since $\Theta(J) = \theta(PJ \otimes Q) = g_1(PJ)$, therefore $\Theta(J)$ is a (right) strongly prime ideal of $R$. Analogously we can prove that for any (right) strongly prime ideal $I$ of $R$, $\Phi(I)$ is a (right) strongly prime ideal of $S$. Hence the proof follows in view of the fact that $\Theta$ and $\Phi$ are mutually inverse lattice isomorphisms. ■

**Definition 3.17** [9]**.** For a semiring $R$, the (right) strongly prime radical is defined to be the intersection of all (right) strongly prime $k$-ideals of $R$.

**Theorem 3.18.** *Let $R, S$ be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then $\Theta : Id(S) \to Id(R)$ maps the (right) strongly prime radical $(SP(S))$ of $S$ to the (right) strongly prime radical $(SP(R))$ of $R$, i.e., $\Theta(SP(S)) = SP(R)$.*

***Proof.*** Let $\mathcal{C}_{SP}(R)$ and $\mathcal{C}_{SP}(S)$ be the collection of all (right) strongly prime $k$-ideals of $R$ and $S$ respectively. Then using Theorem 3.16 and Prop. 3.6 and the fact that $\Theta$ preserves inclusion and $k$-ideals we have

$$\Theta(SP(S)) = \Theta\left(\bigcap_{J\in\mathcal{C}_{SP}(S)} J\right) = \bigcap_{J\in\mathcal{C}_{SP}(S)} \Theta(J) \supseteq \bigcap_{\mathcal{C}_{SP}(R)} I = SP(R).$$

Similarly we have $\Phi(SP(R)) \supseteq SP(S)$. Since $\Theta$ and $\Phi$ are mutually inverse lattice isomorphisms, we have $SP(R) \supseteq \Theta(SP(S))$. Hence, $\Theta(SP(S)) = SP(R)$. ■

**Definition 3.19** [4]**.** An ideal $I$ of a semiring $R$ is said to be a uniformly strongly prime ideal of $R$ if and only if there exists a finite subset $F$ of $R$ such that for $r', r'' \in R$, $r'Fr'' \subseteq I$ implies that $r' \in I$ or $r'' \in I$.

**Definition 3.20.** Let $R, S$ be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. A subsemimodule $M$ of $P$ is said to be a uniformly strongly prime subsemimodule if there exist finite subsets $X$ and $Y$ of $P$ and $Q$ respectively such that for $p', p'' \in P$, $\theta(p'\otimes Y)\theta(X\otimes Y)p'' \subseteq M$ implies that $p' \in M$ or $p'' \in M$.

**Definition 3.21.** Let $R, S$ be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. A subsemimodule $N$ of $Q$ is said to be a uniformly strongly prime subsemimodule if there exist finite subsets $Y$ and $X$ of $Q$ and $P$ respectively such that for $q', q'' \in Q$, $\phi(q'\otimes X)\phi(Y\otimes X)q'' \subseteq N$ implies that $q' \in N$ or $q'' \in N$.

**Lemma 3.22.** *Let $R, S$ be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then the following statements are equivalent for a subsemimodule $M \subseteq P$.*

(a) *$M$ is a uniformly strongly prime subsemimodule of $P$.*

(b) *There exist finite subsets $X$ of $P$ and $Y'$, $Y''$ of $Q$ such that for $p', p'' \in P$, $\theta(p' \otimes Y')\theta(X \otimes Y'')p'' \subseteq M$ implies that $p' \in M$ or $p'' \in M$.*

***Proof.*** Clearly (a) $\Rightarrow$ (b).

(b) $\Rightarrow$ (a) Suppose $Y = Y' \cup Y''$, then clearly $Y$ is a finite subset of $Q$. Let $p'$, $p'' \in P$ such that $\theta(p' \otimes Y)\theta(X \otimes Y)p'' \subseteq M$. Then $\theta(p' \otimes Y')\theta(X \otimes Y'')p'' \subseteq \theta(p' \otimes Y)\theta(X \otimes Y)p'' \subseteq M$ and hence from (b) we get $p' \in M$ or $p'' \in M$. Consequently, $M$ is a uniformly strongly prime subsemimodule of $P$. ■

**Proposition 3.23.** *Let $R, S$ be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then the mapping $f_1 : Id(R) \to Sub(P)$ defines a one-to-one inclusion preserving correspondence between the set of all uniformly strongly prime ideals of $R$ and the set of all uniformly strongly prime subsemimodules of $P$.*

***Proof.*** Let $I$ be a uniformly strongly prime ideal of $R$. Then there exists a finite subset $F \subseteq R$ such that for $r', r'' \in R$, $r'Fr'' \subseteq I$ implies that $r' \in I$ or $r'' \in I$. Suppose $X = \{r\bar{p_v} \mid r \in F,\ v = 1, 2, \dots, n'\}$, $Y' = \{\tilde{q_u} \mid u = 1, 2, \dots, m'\}$, $Y'' = \{\bar{q_v} \mid v = 1, 2, \dots, n'\}$. Since $F$ is finite, clearly $X$ is a finite subset of $P$, also both $Y'$, $Y''$ are finite subsets of $Q$.

Let $p', p'' \in P$ such that $\theta(p' \otimes Y')\theta(X \otimes Y'')p'' \subseteq f_1(I) = IP$ and $p' \notin IP$. Then there exists $k \in \{1, 2, \dots, m'\}$ such that $\theta(p' \otimes \tilde{q_k}) \notin I$, otherwise $p' = p'1_S = p' \sum_{u=1}^{m'} \phi(\tilde{q_u} \otimes \tilde{p_u}) = \sum_{u=1}^{m'} \theta(p' \otimes \tilde{q_u})\tilde{p_u} \in IP$ — a contradiction. Now for any $r \in F$, $q \in Q$ we have,

$$
\begin{aligned}
&\theta(p' \otimes \tilde{q_k}) r \theta(p'' \otimes q)\\
&= \theta(p' \otimes \tilde{q_k}) r 1_R \theta(p'' \otimes q) = \theta(p' \otimes \tilde{q_k}) r \sum_{v=1}^{n'} \theta(\bar{p_v} \otimes \bar{q_v})\theta(p'' \otimes q)\\
&= \theta(p' \otimes \tilde{q_k}) \sum_{v=1}^{n'} \theta(r\bar{p_v} \otimes \bar{q_v})\theta(p'' \otimes q) = \sum_{v=1}^{n'} \theta(\theta(p' \otimes \tilde{q_k})\theta(r\bar{p_v} \otimes \bar{q_v})p'' \otimes q)\\
&\in \theta(\theta(p' \otimes Y')\theta(X \otimes Y'')p'' \otimes q) \subseteq \theta(f_1(I) \otimes Q) = g_1(f_1(I)) = I.
\end{aligned}
$$

This is true for all $r \in F$. Therefore $\theta(p' \otimes \tilde{q_k}) F \theta(p'' \otimes q) \subseteq I$. Now since $\theta(p' \otimes \tilde{q_k}) \notin I$, therefore by our hypothesis $\theta(p'' \otimes q) \in I$, which is true for all $q \in Q$, in particular for all $\tilde{q_u}$, $u = 1, 2, \dots, m'$. So we get $p'' = p''1_S = p'' \sum_{u=1}^{m'} \phi(\tilde{q_u} \otimes \tilde{p_u}) = \sum_{u=1}^{m'} \theta(p'' \otimes \tilde{q_u})\tilde{p_u} \in IP$. Hence by Lemma 3.22, $f_1(I)$ is a uniformly strongly prime subsemimodule of $P$.

Conversely, let $M$ be a uniformly strongly prime subsemimodule of $P$. Then there exist finite subsets $X \subseteq P$ and $Y \subseteq Q$ such that for $p', p'' \in P$, $\theta(p' \otimes Y)\theta(X \otimes Y)p'' \subseteq M$ implies that $p' \in M$ or $p'' \in M$. Let $F = \{\theta(\bar{p_v} \otimes y')\theta(x \otimes y'') \mid x \in X,\ y', y'' \in Y,\ v = 1, 2 \dots, n'\}$. Then clearly $F$ is a finite subset of $R$.

Suppose $r', r'' \in R$ such that $r'Fr'' \subseteq g_1(M) = \theta(M \otimes Q)$ and $r' \notin \theta(M \otimes Q)$, then there exists $k \in \{1, 2, \dots, n'\}$ such that $r'\bar{p_k} \notin M$, otherwise $r' = r'1_R = r' \sum_{v=1}^{n'} \theta(\bar{p_v} \otimes \bar{q_v}) = \sum_{v=1}^{n'} \theta(r'\bar{p_v} \otimes \bar{q_v}) \in \theta(M \otimes Q)$ — a contradiction. Now for any $y', y'' \in Y$, $x \in X$ and $p \in P$, using the fact that $f_1$ and $g_1$ are mutually inverse maps we have, $\theta(r'\bar{p_k} \otimes y')\theta(x \otimes y'')r''p = r'\theta(\bar{p_k} \otimes y')\theta(x \otimes y'')r''p \in r'Fr''p \subseteq g_1(M)P = f_1(g_1(M)) = M$. Since every element of $\theta(r'\bar{p_k} \otimes Y)\theta(X \otimes Y)r''p$ is finite sum of elements of the form $\theta(r'\bar{p_k} \otimes y')\theta(x \otimes y'')r''p$ for some $x \in X$, $y', y'' \in Y$, therefore $\theta(r'\bar{p_k} \otimes Y)\theta(X \otimes Y)r''p \subseteq M$. As $r'\bar{p_k} \notin M$, by our hypothesis $r''p \in M$, which is true for all $p \in P$, in particular for all $\bar{p_v}$, where

$v = 1, 2, \ldots, n'$. Therefore $r'' = r''1_R = r'' \sum_{v=1}^{n'} \theta(\bar{p_v} \otimes \bar{q_v}) = \sum_{v=1}^{n'} \theta(r''\bar{p_v} \otimes \bar{q_v}) \in \theta(M \otimes Q) = g_1(M)$. Thus $g_1(M)$ is a uniformly strongly prime ideal of $R$. Since $f_1$ and $g_1$ are mutually inverse lattice isomorphisms, the proof follows. ■

Analogously we obtain the following result.

**Proposition 3.24.** *Let $R, S$ be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then the mapping $f_4 : Id(S) \to Sub(Q)$ defines a one-to-one inclusion preserving correspondence between the set of all uniformly strongly prime ideals of $S$ and the set of all uniformly strongly prime subsemimodules of $Q$.*

**Proposition 3.25.** *Let $R, S$ be two Morita equivalent semirings via Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then the mapping $f_2 : Id(R) \to Sub(Q)$ defines a one-to-one inclusion preserving correspondence between the set of all uniformly strongly prime ideals of $R$ and the set of all uniformly strongly prime subsemimodules of $Q$.*

***Proof.*** Let $I$ be a uniformly strongly prime ideal of $R$. Then there exists a finite subset $F \subseteq R$ such that for $r', r'' \in R$, $r'Fr'' \subseteq I$ implies that $r' \in I$ or $r'' \in I$. Suppose $X' = \{r\bar{p_v} \mid r \in F,\ v = 1, 2, \ldots, n'\}$, $X'' = \{\tilde{p_u} \mid u = 1, 2, \ldots, m'\}$, $Y = \{\bar{q_v} \mid v = 1, 2, \ldots, n'\}$. Since $F$ is finite, $X'$ is a finite subset of $P$, also both $X''$ and $Y$ are finite subsets of $P$ and $Q$ respectively.

Let $q', q'' \in Q$ such that $\phi(q' \otimes X')\phi(Y \otimes X'')q'' \subseteq f_2(I) = QI$ and $q' \notin QI$. Then there exists $k \in \{1, 2, \ldots, m'\}$ such that $\theta(\tilde{p_k} \otimes q') \notin I$, otherwise $q' = 1_Sq' = \sum_{u=1}^{m'} \phi(\tilde{q_u} \otimes \tilde{p_u})q' = \sum_{u=1}^{m'} \tilde{q_u}\theta(\tilde{p_u} \otimes q') \in QI$ — a contradiction. Now for any $r \in F$, $u \in \{1, 2, \ldots, m'\}$ we have

$$\begin{aligned}
&\theta(\tilde{p_k} \otimes q')r\theta(\tilde{p_u} \otimes q'') \\
&= \theta(\tilde{p_k} \otimes q')r1_R\theta(\tilde{p_u} \otimes q'') = \theta(\tilde{p_k} \otimes q')r\sum_{v=1}^{n'}\theta(\bar{p_v} \otimes \bar{q_v})\theta(\tilde{p_u} \otimes q'') \\
&= \theta(\tilde{p_k} \otimes q')\sum_{v=1}^{n'}\theta(r\bar{p_v} \otimes \bar{q_v})\theta(\tilde{p_u} \otimes q'') = \sum_{v=1}^{n'}\theta(\tilde{p_k} \otimes q'\theta(r\bar{p_v} \otimes \bar{q_v})\theta(\tilde{p_u} \otimes q'')) \\
&= \sum_{v=1}^{n'}\theta(\tilde{p_k} \otimes \phi(q' \otimes r\bar{p_v})\bar{q_v}\theta(\tilde{p_u} \otimes q'')) = \sum_{v=1}^{n'}\theta(\tilde{p_k} \otimes \phi(q' \otimes r\bar{p_v})\phi(\bar{q_v} \otimes \tilde{p_u})q'') \\
&\in \theta(P \otimes \phi(q' \otimes X')\phi(Y \otimes X'')q'') \subseteq \theta(P \otimes f_2(I)) = g_2(f_2(I)) = I.
\end{aligned}$$

This is true for all $r \in F$. Therefore $\theta(\tilde{p_k} \otimes q')F\theta(\tilde{p_u} \otimes q'') \subseteq I$. Now since $\theta(\tilde{p_k} \otimes q') \notin I$, therefore by our hypothesis $\theta(\tilde{p_u} \otimes q'') \in I$, which is true for all $\tilde{p_u}$, $u = 1, 2, \ldots, m'$. So we get $q'' = 1_Sq'' = \sum_{u=1}^{m'} \phi(\tilde{q_u} \otimes \tilde{p_u})q'' = \sum_{u=1}^{m'} \tilde{q_u}\theta(\tilde{p_u} \otimes q'') \in$

$QI$. Hence by $Q$ analogue of Lemma 3.22, $f_2(I)$ is a uniformly strongly prime subsemimodule of $Q$.

Conversely, let $N$ be a uniformly strongly prime subsemimodule of $Q$. Then there exist finite subsets $X \subseteq P$ and $Y \subseteq Q$ such that for $q', q'' \in Q$, $\phi(q' \otimes X)\phi(Y \otimes X)q'' \subseteq N$ implies that $q' \in N$ or $q'' \in N$. Let $F = \{\theta(x' \otimes y)\theta(x'' \otimes \bar{q_v}) \mid x', x'' \in X,\ y \in Y,\ v = 1, 2 \dots, n'\}$. Then clearly $F$ is a finite subset of $R$.

Suppose $r', r'' \in R$ such that $r'Fr'' \subseteq g_2(N) = \theta(P \otimes N)$ and $r' \notin \theta(P \otimes N)$, then there exists $k \in \{1, 2, \dots, n'\}$ such that $\bar{q_k}r' \notin N$, otherwise $r' = 1_R r' = \sum_{v=1}^{n'} \theta(\bar{p_v} \otimes \bar{q_v})r' = \sum_{v=1}^{n'} \theta(\bar{p_v} \otimes \bar{q_v}r') \in \theta(P \otimes N)$ — a contradiction. Now for any $x', x'' \in X$, $y \in Y$ and $v \in \{1, 2 \dots, n'\}$, using the fact that $f_2$ and $g_2$ are mutually inverse maps we have, $\phi(\bar{q_k}r' \otimes x')\phi(y \otimes x'')\bar{q_v}r'' = \phi(\bar{q_k}r' \otimes x'\phi(y \otimes x''))\bar{q_v}r'' = \phi(\bar{q_k}r' \otimes \theta(x' \otimes y)x'')\bar{q_v}r'' = \bar{q_k}r'\theta(\theta(x' \otimes y)x'' \otimes \bar{q_v})r'' = \bar{q_k}r'\theta(x' \otimes y)\theta(x'' \otimes \bar{q_v})r'' \in \bar{q_k}r'Fr'' \subseteq Qg_2(N) = f_2(g_2(N)) = N$. Since every element of $\phi(\bar{q_k}r' \otimes X)\phi(Y \otimes X)\bar{q_v}r''$ is finite sum of elements of the form $\phi(\bar{q_k}r' \otimes x')\phi(y \otimes x'')\bar{q_v}r''$ for some $x', x'' \in X$, $y \in Y$, therefore $\phi(\bar{q_k}r' \otimes X)\phi(Y \otimes X)\bar{q_v}r'' \subseteq N$. As $\bar{q_k}r' \notin N$, by our hypothesis $\bar{q_v}r'' \in N$, which is true for all $v = 1, 2, \dots, n'$. Therefore $r'' = 1_R r'' = \sum_{v=1}^{n'} \theta(\bar{p_v} \otimes \bar{q_v})r'' = \sum_{v=1}^{n'} \theta(\bar{p_v} \otimes \bar{q_v}r'') \in \theta(P \otimes N) = g_2(N)$. Thus $g_2(N)$ is a uniformly strongly prime ideal of $R$. This completes the proof as $f_2$ and $g_2$ are mutually inverse lattice isomorphisms. ■

Analogously we obtain the following result.

**Proposition 3.26.** *Let $R, S$ be two Morita equivalent semirings via Morita context $(R, S, {}_R P_S, {}_S Q_R, \theta, \phi)$. Then the mapping $f_3 : Id(S) \to Sub(P)$ defines a one-to-one inclusion preserving correspondence between the set of all uniformly strongly prime ideals of $S$ and the set of all uniformly strongly prime subsemimodules of $P$.*

**Theorem 3.27.** *Let $R, S$ be two Morita equivalent semirings via Morita context $(R, S, {}_R P_S, {}_S Q_R, \theta, \phi)$. Then the mapping $\Theta : Id(S) \to Id(R)$ defines a one-to-one inclusion preserving correspondence between the set of all uniformly strongly prime ideals of $S$ and the set of all uniformly strongly prime ideals of $R$.*

***Proof.*** Let $J$ be a uniformly strongly prime ideal of $S$. Then from Prop. 3.26, $f_3(J) = PJ$ is a uniformly strongly prime subsemimodule of $P$ and therefore from the proof of Prop. 3.23 we see that, $g_1(PJ)$ is a uniformly strongly prime ideal of $R$. Since $\Theta(J) = \theta(PJ \otimes Q) = g_1(PJ)$, therefore $\Theta(J)$ is a uniformly strongly prime ideal of $R$. Analogously we can prove that for any uniformly strongly prime ideal $I$ of $R$, $\Phi(I)$ is a uniformly strongly prime ideal of $S$. In view of the fact that $\Theta$ and $\Phi$ are mutually inverse lattice isomorphisms, the proof follows. ■

**Definition 3.28** [9]**.** For a semiring $R$, the uniformly strongly prime radical is defined to be the intersection of all uniformly strongly prime $k$-ideals of $R$.

**Theorem 3.29.** *Let $R, S$ be two Morita equivalent semirings via Morita context $(R, S,_R P_S,_S Q_R, \theta, \phi)$. Then $\Theta : Id(S) \to Id(R)$ maps the uniformly strongly prime radical $(USP(S))$ of $S$ to the uniformly strongly prime radical $(USP(R))$ of $R$, i.e., $\Theta(USP(S)) = USP(R)$.*

***Proof.*** Let $\mathcal{C}_{USP}(R)$ and $\mathcal{C}_{USP}(S)$ be the collection of all uniformly strongly prime $k$-ideals of $R$ and $S$ respectively. Then using Theorem 3.27 and Prop. 3.6 and the fact that $\Theta$ preserves inclusion and $k$-ideals we have

$$\Theta(USP(S)) = \Theta\left(\bigcap_{J \in \mathcal{C}_{USP}(S)} J\right) = \bigcap_{J \in \mathcal{C}_{USP}(S)} \Theta(J) \supseteq \bigcap_{I \in \mathcal{C}_{USP}(R)} I = USP(R).$$

Similarly we have $\Phi(USP(R)) \supseteq USP(S)$. Since $\Theta$ and $\Phi$ are mutually inverse lattice isomorphisms, we have $USP(R) \supseteq \Theta(USP(S))$. Hence, $\Theta(USP(S)) = USP(R)$. ∎

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## References

[1] J. Adamek, H. Herrlich and G. Strecker, Abstract and Concrete Categories (John Wiley & Sons, Inc., New York, 1990).

[2] K. Dey, S. Gupta and S.K. Sardar, *Morita invariants of semirings related to a Morita context*, Asian-European J. Math. **12** (2) (2019) 1950023 (15 pages). https://doi.org/10.1142/S1793557119500232

[3] T.K. Dutta and M.L. Das, *On strongly prime semiring*, Bull. Malaysian Math. Sci. Soc. **30** (2) (2007) 135–141.

[4] T.K. Dutta and M.L. Das, *On uniformly strongly prime semiring*, Int. J. Math. Anal. **2** (1–3) (2006) 73–82.

[5] J.S. Golan, Semirings and Their Applications (Kluwer Academic Publishers, Dordrecht, 1999).

[6] S. Gupta and S.K. Sardar, *Morita invariants of semirings*-II, Asian-European J. Math. **11** (1) (2018) 1850014. https://doi.org/10.1142/S1793557118500146

[7] D. Handelman and J. Lawrence, *Strongly prime rings*, Trans. Amer. Math. Soc. **211** (1975) 209–223. https://doi.org/10.1090/s0002-9947-1975-0387332-0

[8] U. Hebisch and H.J. Weinert, *Semirings and semifields*, Handbook Alg. **1** (1996) 425–462. https://doi.org/10.1016/s1570-7954(96)80016-7

[9] U. Hebisch and H.J. Weinert, *Radical theory for semirings*, Quaest. Math. **20** (4) (1997) 647–661.
https://doi.org/10.1080/16073606.1997.9632232

[10] Y. Katsov and T.G. Nam, *Morita equivalence and homological characterization of semirings*, J. Alg. Its Appl. **10** (3) (2011) 445–473.
https://doi.org/10.1142/S0219498811004793

[11] S. Mac Lane, Categories for the Working Mathematician (Springer, New York, 1971).

[12] K. Morita, *Duality of modules and its applications to the theory of rings with minimum condition*, Sci. Rep. Tokyo Kyoiku Daigaku, Section A **6** (150) (1958) 83–142.

[13] D.M. Olson, *A uniformly strongly prime radical*, J. Austral. Math. Soc., Ser. A **43** (1) (1987) 95–102.
https://doi.org/10.1017/s1446788700029013

[14] S.K. Sardar, S. Gupta and B.C. Saha, *Morita equivalence of semirings and its connection with Nobusawa* Γ*-semirings with unities*, Alg. Colloq. **22** (Spec 1) (2015) 985–1000.
https://doi.org/10.1142/S1005386715000826

[15] S.K. Sardar and S. Gupta, *Morita invariants of semirings*, J. Algebra Appl. **15** (2) (2016) 14pp. 1650023.
https://doi.org/10.1142/S0219498816500237

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