Discussiones Mathematicae
General Algebra and Applications 43 (2023) 161–171
https://doi.org/10.7151/dmgaa.1407

ON PLEASANT EULERIAN POSETS

R. Subbarayan

Department of Mathematics
Ramakrishna Mission Vivekananda College
Chennai, Tamilnadu, India 600004

e-mail: rsubbarayan2010@gmail.com

A. Vethamanickam

and

V. Gladys Mano Amirtha[1]

PG and Research Department of Mathematics
Rani Anna Government College for Women
Tirunelveli, Tamilnadu, India, 627008

e-mail: dr_vethamanickam@yahoo.co.in

Abstract

In this paper, we classify Eulerian posets of small rank which are pleasant and further show that there is no pleasant Eulerian poset of rank less than are equal to 7.

Keywords: posets, lattices, Eulerian lattices, pleasant poset.

2020 Mathematics Subject Classification: 06A06, 06A07, 06A11.

1. Introduction

The concept of pleasant poset was introduced by Stanley [3]. Some specific pleasant posets were given by Stanley. The study of pleasant posets which are also Eulerian was done by Vethamanickam [5].

Are initial intervals of a pleasant poset pleasant? The question of finding all the Eulerian posets which are pleasant will be completely known, if we can answer the above question in the affirmative.

[1]Research Scholar (Reg. No. 20111172092013) in Manonmaniam Sundaranar University, Tirunelveli.

We provide the basic definitions and examples of Eulerian lattices that are needed to study pleasant Eulerian posets.

Definition 1.1. Let P be a finite poset with a unique minimum and a unique maximum element. The poset P is said to be graded if all the maximal chains in P have the same length.

Definition 1.2. A function $r : P \to \{0, 1, \ldots, n\}$ is said to be the rank function on P if $r(x) = 0$ if x is a minimal element of P and $r(y) = r(x) + 1$ if y covers x in P. If $r(x) = i$ then we say that x has rank i.

Definition 1.3. The *Möbius function* μ on a poset P is an integer-valued function $\mu : P \times P \to Z$ satisfying the following conditions:

$$\mu(x, y) = \begin{cases} 1 & \text{if } x = y \\ -\sum\limits_{x \le z < y} \mu(x, z) & \text{if } x \le y \\ 0 & \text{if } x \not< y. \end{cases}$$

Definition 1.4. A finite graded poset P is said to be Eulerian if its Möbius function assumes the value $\mu(x, y) = (-1)^{l(x,y)}$ for all $x \le y$ in P, where $l(x, y) = r(y) - r(x)$ and r is the rank function on P.

Equivalently, a finite graded poset P is Eulerian if and only if all intervals $[x, y]$ of length $l \ge 1$ in P contain an equal number of elements of odd and even rank.

Lemma 1.5 [2]. *If P is an Eulerian poset of rank $d+1$, then $\sum_{i=1}^{d}(-1)^{i-1}a_i = 1 + (-1)^{d+1}$, where a_i is the number of elements of rank i.*

Every Boolean algebra of rank n is Eulerian. We observe that any element of rank 2 of an Eulerian poset contains exactly two atoms. For the concept of Eulerian poset refer to [3, 4, 6].

Definition 1.6. Let P be a poset. An order ideal of P is a subset of P, say I such that, if $x \in I$ and $y < x$ then $y \in I$.

The collection of all order ideals of P including empty set forms a lattice under the usual ordering of inclusion. We will denote it by $J(P)$. $J(P)$ is a ranked poset.

Definition 1.7. Let P be a finite graded poset with rank function r. The rank generating function $F(P, q)$ of P is defined by $F(P, q) = \sum_{i=0}^{rankP} a_i q^i$, where a_i is the number of elements of rank i in P.

Example. For a Boolean algebra of rank n, the rank generating function is $\sum_{i=0}^{n} \binom{n}{i} q^i$.

Definition 1.8. Let $F(J(P),q)$ be the rank generating function of $J(P)$. A finite graded poset P is said to be pleasant if $F(J(P),q) = \Pi_{x\in P}\left[\frac{1-q^{r(x)+2}}{1-q^{r(x)+1}}\right]$.

2. PLEASANT EULERIAN POSETS

Some examples of pleasant posets are:

(1) Boolean algebras of rank ≤ 2 are pleasant.

(2) Any finite chain is pleasant. Since the only rank 2 Eulerian poset is the Boolean algebra of rank 2, all Eulerian posets of rank 2 are pleasant.

(3) The diamond lattice M_3 is not pleasant.

For, $F(J(M_3),q) = 1 + q + 3q^2 + 3q^3 + q^4 + q^5$ but,

$$\begin{aligned}
\Pi_{x\in P}\left[\frac{1-q^{r(x)+2}}{1-q^{r(x)+1}}\right] &= \left[\frac{1-q^2}{1-q}\right]\left[\frac{1-q^3}{1-q^2}\right]^3\left[\frac{1-q^4}{1-q^3}\right] \\
&= \frac{(1-q^3)^2(1-q^4)}{(1-q^2)^2(1-q)} \\
&= \frac{(1+q+q^2)(1+q^2)(1-q^3)}{1-q^2} \\
&= \frac{(1+q+q^2)^2(1+q^2)}{1+q} \neq F(J(M_3),q)
\end{aligned} \tag{1}$$

in fact, it is not even a polynomial.

In fact, any poset of rank 2 can be pleasant if and only if the number of elements of rank 1 is ≤ 2, that is, it is either chain or a Boolean algebra.

Now,we consider Eulerian posets of rank 3.

Lemma 2.1. *The Boolean algebra of rank* 3*, that is,* B_3 *is pleasant.*

Proof. In B_3, the number of order ideals containing $1,2,3,4,5,6,7$ and 8 elements are $1,3,3,4,3,3,1$ and 1 respectively. Hence,

$$F(J(B_3),q) = 1 + q + 3q^2 + 3q^3 + 4q^4 + 3q^5 + 3q^6 + q^7 + q^8.$$

Furthermore,

$$\Pi_{x\in B_3}\left(\frac{1-q^{r(x)+2}}{1-q^{r(x)+1}}\right) = \left[\frac{1-q^2}{1-q}\right]\left[\frac{1-q^3}{1-q^2}\right]^3\left[\frac{1-q^4}{1-q^3}\right]^3\left[\frac{1-q^5}{1-q^4}\right].$$

This clearly equals $F(J(B_3),q)$. Hence the Lemma.

Lemma 2.2. *The only pleasant Eulerian poset of rank* 3 *is* B_3.

Proof. Since the number of elements of rank 1 and rank 2 are equal, let n be the number of elements of rank 1 and rank 2 in an Eulerian poset P of rank 3.

Clearly, the number of order ideals of P containing $1, 2$ and 3 elements are respectively 1, n and $\binom{n}{2}$.

So, $F(J(P), q) = 1 + q + nq^2 + \binom{n}{2}q^3$ + terms containing higher powers. Also,

$$\begin{aligned}\Pi_{x\in P}\left(\frac{1-q^{r(x)+2}}{1-q^{r(x)+1}}\right) &= \left[\frac{1-q^2}{1-q}\right]\left[\frac{1-q^3}{1-q^2}\right]^n\left[\frac{1-q^4}{1-q^3}\right]^n\left[\frac{1-q^5}{1-q^4}\right]\\ &= \frac{(1-q^2)^{1-n}(1-q^4)^{n-1}(1-q^5)}{(1-q)}\\ &= (1+q^2)^{n-1}(1+q+q^2+q^3+q^4)\\ &= 1+q+nq^2+[(n-1)+1]q^3+\cdots .\end{aligned}$$

If this P is pleasant as $F(J(P), q) = \Pi_{x\in P}\left[\frac{1-q^{r(x)+2}}{1-q^{r(x)+1}}\right]$, $\binom{n}{2} = n$ and so $n = 3$. Hence $P = B_3$. So, the Lemma is proved.

Now, we consider the case of rank 4. In this case, simple calculation shows that even B_4 is not pleasant. Hence, one could come to the conclusion that no Eulerian poset of rank 4 can be pleasant. This is exactly the content of the next theorem.

Theorem 2.3. *No Eulerian poset of rank* 4 *is pleasant.*

Proof. Let P be an Eulerian poset of rank 4 and let a_1, a_2 and a_3 be the number of elements of rank $1, 2$ and 3, respectively. So,

$$\Pi_{x\in P}\left(\frac{1-q^{r(x)+2}}{1-q^{r(x)+1}}\right) = \left[\frac{1-q^2}{1-q}\right]\left[\frac{1-q^3}{1-q^2}\right]^{a_1}\left[\frac{1-q^4}{1-q^3}\right]^{a_2}\left[\frac{1-q^5}{1-q^4}\right]^{a_3}\left[\frac{1-q^6}{1-q^5}\right].$$

Since, by Lemma 1.5, $a_1 - a_2 + a_3 = 2$, the above is equal to

$$\begin{aligned}\Pi_{x\in P}\left(\frac{1-q^{r(x)+2}}{1-q^{r(x)+1}}\right) &= \frac{(1-q^3)^{2-a_3}(1-q^4)^{a_1-2}(1-q^5)^{a_3-1}(1-q^6)}{(1-q)(1-q^2)^{a_1-1}}\\ &= \frac{(1+q^2)^{a_1-2}(1-q^5)^{a_3-1}(1+q^3)}{(1-q^3)^{a_3-3}(1-q)(1-q^2)}.\end{aligned}$$

Cancel the factor $(1-q)^{a_3-1}$, we get,

$$\begin{aligned}\Pi_{x\in P}\left(\frac{1-q^{r(x)+2}}{1-q^{r(x)+1}}\right) &= \frac{(1+q^2)^{a_1-2}(1+q+q^2+q^3+q^4)^{a_3-1}(1+q^3)}{(1+q+q^2)^{a_3-3}(1+q)}\\ &= \frac{(1+q^2)^{a_1-2}(1+q+q^2+q^3+q^4)^{a_2-a_1+1}(1-q+q^2)}{(1+q+q^2)^{a_3-3}}.\end{aligned}$$

All the factors that appear are irreducible. If this is to be a polynomial equal to $F(J(P), q)$, then $a_3 \leq 3$, $a_1 \geq 2$, $a_2 - a_1 + 1 \geq 0$. But, $a_3 > 0$ and $a_3 \neq 1$, as P is Eulerian. Hence $a_3 = 2$ or 3.

Case (i): If $a_3 = 2, a_1 = a_2 = n$ (say). Then,

$$
\begin{aligned}
\Pi_{x\in P}\left(\frac{1-q^{r(x)+2}}{1-q^{r(x)+1}}\right) &= (1+q+q^2)(1+q^2)^{n-2}(1+q+q^2+q^3+q^4)(1-q+q^2)\\
&= (1+q^2)^{n-2}(1+q+2q^2+2q^3+3q^4+2q^5+2q^6+q^7+q^8)\\
&= 1+q+nq^2+nq^3+\left[(2n-1)+\binom{n-2}{2}\right]q^4 \qquad (2)\\
&\quad + \text{higher powers of } q.
\end{aligned}
$$

By considering the order ideals of P having 1, 2, 3 and 4 elements, we find, the number of such ideals are respectively 1, n, $\binom{n}{2}$ and $\binom{n}{3} + n$. So,

$$
F(J(P), q) = 1 + q + nq^2 + \binom{n}{2}q^3 + \left[\binom{n}{3} + n\right]q^4 + \text{higher powers of } q. \tag{3}
$$

For P to be pleasant, the equations (2) and (3) must be the same.

Comparing the coefficients q^3 and q^4 from the above equations we get, $\binom{n}{2} = n$ and $\binom{n}{3} + n = 2n - 1 + \binom{n-2}{2}$ this implies $n = 3$ from the first equation, but this cannot be satisfied by the second. Hence a_3 cannot be 2.

Case (ii): $a_3 = 3$. Since by Lemma 1.5, $a_1 - a_2 + a_3 = 2$, $a_1 - a_2 = -1$. Hence $a_1 = n$ and $a_2 = n + 1$. Then,

$$
\begin{aligned}
\Pi_{x\in P}\left(\frac{1-q^{r(x)+2}}{1-q^{r(x)+1}}\right) &= (1+q^2)^{n-2}(1-q+q^2)(1+q+q^2+q^3+q^4)^2\\
&= (1+q^2)^{n-2}(1+q+2q^2+3q^3+4q^4+3q^5+4q^6 \qquad (4)\\
&\quad +3q^7+2q^8+q^9+q^{10})\\
&= 1+q+nq^2+(n+1)q^3+\text{higher powers of } q.
\end{aligned}
$$

For P to be pleasant, the equations (4) and (3) must be the same.

Comparing the coefficient of q^3, $n+1 = \binom{n}{2}$, this implies, $2(n+1) = n(n-1)$, so, $n^2 - 3n - 2 = 0$. As n is an integer, this is impossible. Hence no Eulerian poset of rank 4 is pleasant.

Theorem 2.4. *No Eulerian poset of rank* 5 *is pleasant.*

Proof. Let P be an Eulerian poset of rank 5 with a_i elements of rank i, $1 \leq i \leq 4$. Since by Lemma 1.5, $a_1 - a_2 + a_3 - a_4 = 0$. Since $F(J(P), q) = 1 + q + a_1q^2 +$

$\binom{a_1}{2}q^3 + \left[\binom{a_1}{3} + a_2\right]q^4 + \cdots$ is a polynomial, and P is pleasant, this must be equal to

$$\left[\frac{1-q^2}{1-q}\right]\left[\frac{1-q^3}{1-q^2}\right]^{a_1}\left[\frac{1-q^4}{1-q^3}\right]^{a_2}\left[\frac{1-q^5}{1-q^4}\right]^{a_3}\left[\frac{1-q^6}{1-q^5}\right]^{a_4}\left[\frac{1-q^7}{1-q^6}\right]$$

$= (1+q+q^2)^{a_1-a_2+a_4-1}(1+q^2)^{a_1-a_4}(1-q+q^2)^{a_4-1}(1+q+q^2$

$+ q^3+q^4)^{a_2-a_1}(1+q+\cdots+q^6)$. We observe that the power of $q+1$ becomes zero.

$= (1+q+q^2)^{n_1}(1+q^2)^{n_2}(1+q^2+q^4)^{n_3}(1+q+q^2+q^3+q^4)^{n_4}(1+q+\cdots+q^6)$

where,

$$\begin{aligned} n_1 &= a_1 - a_2 \\ n_2 &= a_1 - a_4 \\ n_3 &= a_4 - 1 \\ n_4 &= a_2 - a_1. \end{aligned}$$

Here, the coefficient of q^3 is

$$\begin{aligned} &1 + 2n_1 + n_2 + n_3 + 3n_4 + n_1n_2 + 3n_1n_4 + n_1n_3 + n_3n_4 + n_2n_4 \\ &+ \binom{n_1}{2}[3+n_4] + \binom{n_1}{3} + \binom{n_4}{2}[3+n_1] + \binom{n_4}{3} = a_2. \end{aligned}$$

If the above is to be a polynomial then, $a_1 - a_2 + a_4 - 1 \geq 0$, $a_1 \geq a_4$ and $a_2 \geq a_1$. Assuming these and comparing the coefficient of q^3 in $F(J(P), q)$, we get, $a_2 = \binom{a_1}{2}$. Since $a_1 - a_2 + a_4 - 1 \geq 0$ and $a_1 \geq a_4$, we get $2a_1 - a_2 - 1 \geq 0$. Therefore $2a_1 \geq a_2 + 1$. Since $a_2 = \binom{a_1}{2}$, $2a_1 \geq \binom{a_1}{2} + 1$.

This implies $2a_1 \geq \frac{a_1(a_1-1)}{2} + 1$. That is, $4a_1 \geq {a_1}^2 - a_1 + 2$.

This implies $a_1^2 - 5a_1 + 2 \leq 0$. Therefore $a_1 \leq 4$.

(i) If $a_1 = 2$, then $a_2 = 1$, in which case , P cannot be Eulerian.

(ii) If $a_1 = 3$, then $a_2 = \binom{3}{2} = 3$ $a_1 = 3 = a_2$ implies $a_3 = a_4$.

Since $a_4 \leq a_1 = 3$ implies $a_4 \leq 3$. Hence, either $a_4 = 2$ or $a_4 = 3$. If $a_4 = a_3 = a_2 = a_1 = 3$, the polynomial becomes,

$$(1+q^2+q^4)^2(1+q+\cdots+q^6) = \left[(1+2(q^2+q^4)+(q^2+q^4)^2\right](1+q+\cdots+q^6).$$

The Coefficient of q^4 is $6 \neq 4 = \binom{a_1}{3} + a_2$. If $a_4 = a_3 = 2$, then the polynomial becomes,

$$(1+q^2+q^4)(1+q^2)(1+q+\cdots+q^6) = (1+2q^2+2q^4+q^6)(1+q+\cdots+q^6).$$

The Coefficient of q^4 is $5 \neq 4 = \binom{a_1}{3} + a_2$. So, there is no such Eulerian poset.

(iii) If $a_1 = 4$, then $a_2 = 6, a_3 - a_4 = 2, a_3 = a_4 + 2$.

The possibilities are:

$$\begin{aligned}
&a_1 = 4,\ a_2 = 6,\ a_3 = 4,\ a_4 = 2,\\
&a_1 = 4,\ a_2 = 6,\ a_3 = 5,\ a_4 = 3,\\
&a_1 = 4,\ a_2 = 6,\ a_3 = 6,\ a_4 = 4,\\
&a_4 \neq 2,\ \text{since } a_1 - a_2 + a_4 - 1 \geq 0.
\end{aligned}$$

When $a_4 = 3$, the polynomial becomes,

$$\begin{aligned}
&(1-q+q^2)^2(1+q^2)(1+q+q^2+q^3+q^4)^2(1+q+q^2+\cdots+q^6)\\
&= (1-2q+3q^2-2q^3+q^4)(1+q^2)(1+2q+3q^2+4q^3\\
&+5q^4+\cdots)(1+q+q^2+\cdots+q^6).
\end{aligned}$$

The coefficient of q^4 is $11 \neq \binom{a_1}{3} + a_2 = 10$. Similarly, when $a_4 = 4$, we get the coefficient of q^4 of the polynomial is $12 \neq \binom{a_1}{3} + a_2$.

Hence, there is no pleasant poset which is Eulerian of rank 5.

Theorem 2.5. *No Eulerian poset of rank* 6 *is pleasant.*

Proof. Let P be an Eulerian poset of rank 6 with a_i elements of rank i, $1 \leq i \leq 5$. Since by Lemma 1.5, $a_1 - a_2 + a_3 - a_4 + a_5 = 2$. Since

$$F(J(P), q) = 1 + q + a_1 q^2 + \binom{a_1}{2} q^3 + [\binom{a_1}{3} + a_2] q^4 + \cdots \tag{5}$$

is a polynomial and P is pleasant, this must be equal to

(6)
$$\begin{aligned}
&\left(\frac{1-q^2}{1-q}\right)\left(\frac{1-q^3}{1-q^2}\right)^{a_1}\left(\frac{1-q^4}{1-q^3}\right)^{a_2}\left(\frac{1-q^5}{1-q^4}\right)^{a_3}\left(\frac{1-q^6}{1-q^5}\right)^{a_4}\left(\frac{1-q^7}{1-q^6}\right)^{a_5}\left(\frac{1-q^8}{1-q^7}\right)\\
&= \left(1+q\right)^{1-a_1+a_2-a_3+a_4-a_5}\left(1+q+q^2\right)^{n_1}\left(1+q^2\right)^{n_2}\left(1+q+q^2+q^3+q^4\right)^{n_3}\\
&(1-q+q^2\Big)^{n_4}\left(1+q+q^2+\cdots+q^6\right)^{n_5}\left(1+q+q^2+\cdots+q^7\right).
\end{aligned}$$

where,

$$\begin{aligned}
n_1 &= a_1 - a_2 + a_4 - a_5\\
n_2 &= a_2 - a_3 + 1\\
n_3 &= a_3 - a_4\\
n_4 &= a_4 - a_5\\
n_5 &= a_5 - 1.
\end{aligned}$$

In (6), $\big(1+q\big)^{1-a_1+a_2-a_3+a_4-a_5} = \big(1+q\big)^{-1}$. Since $\big(1+q\big)^{-1}$ is not a polynomial, we can not compare the right hand side of (6) with that of (5).

Therefore, we conclude that an Eulerian poset P of rank 6 is not pleasant.

Remark 2.6. Suppose, the rank of P is even, say $2m$, where m is a positive integer. Then, the power of $1+q$ in the expression $\Pi_{x\in P}\left(\frac{1-q^{r(x)+2}}{1-q^{r(x)+1}}\right)$ is $1-a_1+a_2-a_3+a_4-a_5+\cdots+a_{2m-2}-a_{2m-1}=-1$, as $a_1+a_2-a_3+a_4-a_5+\cdots+a_{2m-2}-a_{2m-1}=2$ which shows that it is not a polynomial.

So, there is no pleasant Eulerian poset of even rank ≥ 6. With this observation, it is enough to look for pleasant Eulerian posets of odd ranks.

Theorem 2.7. *No Eulerian poset of rank* 7 *is pleasant.*

Proof. Let P be an Eulerian poset of rank 7 with a_i elements of rank i, $1\leq i\leq 6$. Now, $a_1-a_2+a_3-a_4+a_5-a_6=0$, by Lemma 1.5. Since

$$\begin{aligned} F\big(J(P),q\big) &= 1+q+a_1q^2+\tbinom{a_1}{2}q^3 \\ &\quad + \left[\binom{a_1}{3}+a_2\right]q^4+\left[\binom{a_1}{4}+a_2(a_1-2)\right]q^5+\cdots . \end{aligned} \tag{7}$$

is a polynomial and P is pleasant, this must be equal to

$$\begin{aligned} &\left(\frac{1-q^2}{1-q}\right)\left(\frac{1-q^3}{1-q^2}\right)^{a_1}\left(\frac{1-q^4}{1-q^3}\right)^{a_2}\left(\frac{1-q^5}{1-q^4}\right)^{a_3} \\ &\left(\frac{1-q^6}{1-q^5}\right)^{a_4}\left(\frac{1-q^7}{1-q^6}\right)^{a_5}\left(\frac{1-q^8}{1-q^7}\right)^{a_6}\left(\frac{1-q^9}{1-q^8}\right) \\ &= \big(1+q+q^2\big)^{n_1}\big(1+q^2\big)^{n_2}\big(1+q+q^2+q^3+q^4\big)^{n_3}\big(1+q^2+q^4\big)^{n_4} \\ &\quad \big(1+q+q^2+q^3+q^4+q^5+q^6\big)^{n_5}\big(1+q^4\big)^{n_6}\big(1+q^3+q^6\big) \end{aligned} \tag{8}$$

where,

$$\begin{aligned} n_1 &= a_1-a_2+1 \\ n_2 &= a_2-a_3+a_6-1 \\ n_3 &= a_3-a_4 \\ n_4 &= a_4-a_5 \\ n_5 &= a_5-a_6 \\ n_6 &= a_6-1. \end{aligned}$$

If the above is to be a polynomial, then

$$\begin{aligned} &a_1-a_2+1\geq 0,\ a_2-a_3+a_6-1\geq 0,\ a_3-a_4\geq 0, \\ &a_4-a_5\geq 0,\ a_5-a_6\geq 0 \ \text{ and } \ a_6-1\geq 0. \end{aligned} \tag{9}$$

This implies that $a_1-a_2\geq -1$, $a_2-a_3+a_6\geq +1$ and $a_3\geq a_4\geq a_5\geq a_6\geq 1$.

Assuming these and computing the coefficient of q and q^2 in the right hand side of (8), we get, $n_1+n_3+n_5=a_1-a_2+a_3-a_4+a_5-a_6+1=1$, as the coefficient of q and

$$n_1+n_2+n_3+n_4+n_5+\binom{n_1}{2}+\binom{n_3}{2}+\binom{n_5}{2}+\ n_1n_5+n_1n_3+n_3n_5=a_1,$$

as the coefficient of q^2, which are same as the coefficients of q and q^2 respectively in the right hand side of (7).

Since P is Eulerian, $a_1-a_2+a_3-a_4+a_5-a_6=0$, which implies $a_1-a_2\le 0$, as $a_3-a_4\ge 0$ and $a_5-a_6\ge 0$, by (9). This together with the inequality $a_1-a_2\ge -1$, imply that, either $a_1-a_2=0$ or $a_1-a_2=-1$. If $a_1-a_2=0$, then $(a_3-a_4)+(a_5-a_6)=0$. This implies $a_3-a_4=0$ and $a_5-a_6=0$, as $a_3-a_4\ge 0$ and $a_5-a_6\ge 0$. If $a_1-a_2=-1$, then $(a_3-a_4)+(a_5-a_6)=1$, which implies that $(a_3-a_4)=1-(a_5-a_6)$, implies $0\le(a_3-a_4)\le 1$ as $a_5-a_6\ge 0$. Hence, $(a_3-a_4)=0$ or 1 according as $(a_5-a_6)=1$ or 0.

Hence, we have three possibilities:

(i) $a_1-a_2=0, a_3-a_4=0$ and $a_5-a_6=0$.

(ii) $a_1-a_2=-1, a_3-a_4=1$ and $a_5-a_6=0$.

(iii) $a_1-a_2=-1, a_3-a_4=0$ and $a_5-a_6=1$.

In Case (i), when $a_1-a_2=0, a_3-a_4=0$ and $a_5-a_6=0$, the polynomial becomes,

(10)
$$\begin{aligned}&\left(1+q+q^2\right)\left(1+q^2\right)^{n_2}\left(1+q^2+q^4\right)^{n_4}\left(1+q^4\right)^{n_6}\left(1+q^3+q^6\right)\\&=\left[1+q+q^2\right]\Big[1+(n_2+n_4)q^2+q^3+\Big(n_4+n_6+\binom{n_2}{2}+\binom{n_4}{2}+n_2n_4\Big)q^4\\&+(n_2+n_4)q^5+\cdots\Big].\end{aligned}$$

The coefficient of q^3 is $1+n_2+n_4=a_1$.

Comparing this with the coefficient of q^3 in $F(J(P),q)$, we get, $a_1=\binom{a_1}{2}$. On solving we get, $a_1=3$. The coefficient of q^4 in (10) is

$$\begin{aligned}1+n_2+2n_4+n_6+\binom{n_2}{2}+\binom{n_4}{2}+n_2n_4&=1+n_2+2n_4+n_6+1\\&=a_3+3,\ \text{by substituting } a_1=3.\end{aligned}$$

Comparing this with the coefficient of q^4 in $F(J(P),q)$, we get, $a_3+3=\binom{a_1}{3}+a_2=1+3=4$, then $a_3=1$, in which case, P cannot be Eulerian.

In Case (ii), when $a_1-a_2=-1, a_3-a_4=1$ and $a_5-a_6=0$, the polynomial becomes,

$$
\begin{aligned}
&(1+q^2)^{n_2}(1+q+q^2+q^3+q^4)(1+q^2+q^4)^{n_4}(1+q^4)^{n_6}(1+q^3+q^6)\\
(11)\qquad &= (1+q^2)^{n_2}\Big[1+q+(1+n_4)q^2+(1+n_4)q^3\\
&+ \big(2+2n_4+\binom{n_4}{2}+n_6\big)q^4+\cdots\Big]
\end{aligned}
$$

The coefficient of q^3 is $1+n_2+n_4=a_1$, which is the same as that of Case (i). Therefore, again we have $a_1=3$. The coefficient of q^4 in (11) is

$$
\begin{aligned}
2+n_2+2n_4+n_6+\binom{n_2}{2}+\binom{n_4}{2}+n_2n_4 &= 2+n_2+2n_4+n_6+1\\
&= a_4+4, \text{ by substituting } a_1=3.
\end{aligned}
$$

On comparing this with the coefficient of q^4 in $F(J(P),q)$, we get, $a_4+4=\binom{a_1}{3}+a_2=1+4=5$ then $a_4=1$, in which case P cannot be Eulerian. In Case (iii), when $a_1-a_2=-1, a_3-a_4=0$ and $a_5-a_6=1$, the polynomial becomes,

$$
\begin{aligned}
&(1+q^2)^{n_2}(1+q+\cdots+q^6)(1+q^2+q^4)^{n_4}(1+q^4)^{n_6}(1+q^3+q^6)\\
(12)\qquad &= (1+q^2)^{n_2}\Big[1+q+(1+n_4)q^2+(2+n_4)q^3+\big(2+2n_4+\binom{n_4}{2}+n_6\big)q^4\\
&+ \big(3n_4+\binom{n_4}{2}+n_6\big)q^5+\cdots\Big].
\end{aligned}
$$

Here, the coefficient of q^3 is $2+n_2+n_4=a_1+1$. Comparing this with the coefficient of q^3 in $F(J(P),q)$, we get, $a_1+1=\binom{a_1}{2}=\frac{(a_1)(a_1-1)}{2}$. Which implies ${a_1}^2-3a_1-2=0$, whose solution a_1 is not even an integer. So, we conclude that there exist no such Eulerian poset P in this case. Hence, there is no pleasant poset which is Eulerian of rank 7.

3. Conclusion

The road ahead is not so clear. In this paper, the Eulerian posets up to rank 7 have been investigated for pleasantness. It is a challenging task to find out coefficients for higher ranks. But we strongly believe that there could be no pleasant Eulerian posets of ranks greater than 7 which is still open.

Acknowledgment

We are thankful to the referee for his critical and helpful comments and suggestions while revising this paper.

References

[1] G. Grätzer, General Lattice Theory (Birkhauser Verlag, Basel, 1978). https://doi.org/10.1007/978-3-0348-7633-9

[2] V.K. Santhi, Topics in Commutative Algebra, Ph.D. Thesis (Madurai Kamaraj University, 1992).

[3] R.P. Stanley, Enumerative Combinatorics **1** (Wordsworth & Brooks/Cole, 1986). https://doi.org/10.1007/978-1-4615-9763-6_1

[4] R. Subbarayan and A. Vethamanickam, *Some properties of Eulerian lattices*, Comment. Math. Univ. Carolin. **55 (4)** (2014) 499–507.

[5] A. Vethamanickam, Topics in Universal Algebra, Ph.D. Thesis (Madurai Kamaraj University, 1994).

[6] A. Vethamanickam and R. Subbarayan, *Some simple extensions of Eulerian lattices*, Acta Math. Univ. Comenianae **79 (1)** (2010) 47–54.

Received 8 July 2021
First Revised 26 October 2021
Second Revised 6 April 2022
Third Revised 22 June 2022
Accepted 1 August 2022