Discussiones Mathematicae
General Algebra and Applications 43 (2023) 41–52
https://doi.org/10.7151/dmgaa.1405

NOTE ON TRANJUGATE LATTICE MATRICES

Rajesh Gudepu

Department of Mathematics
IcfaiTech, FST IFHE Hyderabad-501203
Telangana, India

e-mail: rajesh.g@ifheindia.org

Abstract

In this paper, we extend the notion of tranjugate lattice matrices and we show that a square lattice matrix can be expressed as meet (or greatest lower bound or infimum) of symmetric and tranjugate lattice matrices and we discuss their uniqueness.

Keywords: complete and completely distributive lattice, lattice vector space, skew symmetric matix, tranjugate matrix.

2020 Mathematics Subject Classification: 06D99,15B99.

1. Introduction

The notion of lattice matrices appeared firstly in the work Lattice Matrices [4] by Give'on in 1964. A matrix is called a lattice matrix if its entries belong to a distributive lattice. All Boolean matrices and fuzzy matrices are lattice matrices. Lattice matrices in various special cases become useful tools in various domains like the theory of switching nets, automata theory, the theory of finite graphs [4] and cryptography [6]. In the classical theory of matrices, any square matrix can be written uniquely as a sum of symmetric and skew symmetric matrices [8]. As an analogue to this result in the classical theory of matrices, Luce has shown in [9], Theorem 2.1, that any square Boolean matrix can be uniquely decomposed as a disjoint sum of a symmetric and a skew symmetric matrix. Joy and Thomas [7] extended the notions of symmetric and skew-symmetric lattice matrices and they have generalized the theorem of Luce [9] to lattice matrices. But they did not discussed their uniqueness.

In the present work, we have discussed and given example of the uniqueness of symmetric and skew-symmetric lattice matrices in the representation of G. Joy and K.V. Thomas.

In the theory of matrices over Boolean algebra, Chen has introduced the tranjugate Boolean matrices ([3], Definition 1) and he has shown that any square Boolean matrix can be uniquely decomposed as a join intersection of a symmetric and a tranjugate matrix ([3], Theorem 3).

In the present work, we extend the notion of tranjugate Boolean matrices to lattice matrices and we generalize the theorem of Chen [3] to lattice Matrices. We discuss and give an example their uniqueness.

2. Preliminaries

Throughout this paper, $\mathbb{N}$ denotes the set of non zero natural numbers. We denote vectors as $\mathbf{u}, \mathbf{v}, \mathbf{w}$, etc. and scalars as a, b, c, etc. and also the zero vector as $\mathbf{0}$ and the vector $(1, 1, \ldots, 1)$ as $\mathbf{1}$.

We recall some basic definitions and results on lattice theory, lattice matrices, lattice vector spaces, pseudo-complements and their properties, that will be used in the sequel. For details see [1, 4, 5, 10, 11] and [12].

A partially ordered set $(L, \leq)$ is a lattice if for all $a, b \in L$, the least upper bound of a, b and the greatest lower bound of a, b exist in L. For any $a, b \in L$, the least upper bound and the greatest lower bound are denoted by $a \vee b$ and $a \wedge b$ (or ab), respectively. An element $a \in L$ is called the greatest element of L if $\alpha \leq a$, for all $\alpha \in L$. An element $b \in L$ is called the least element of L if $b \leq \alpha$, for all $\alpha \in L$. We use 1 and 0 to denote the greatest element and the least element of L, respectively.

If $a, b \in L$, the largest $x \in L$ satisfying the inequality $a \wedge x \leq b$ is called the relative pseudo-complement of a in b and is denoted by $a \to b$. If for any pair of elements $a, b \in L$, $a \to b$ exists, then L is said to be a Brouwerian lattice. Dually for $a, b \in L$, the least $x \in L$ satisfying $a \vee x \geq b$ is called the relative lower pseudo-complement of a in b and is denoted by $b - a$. If for any pair of elements $a, b \in L$, $b - a$ exists, then L is said to be a Dually Brouwerian lattice. We shall denote if exists $a \to 0$ by a^*.

A lattice L is a completely distributive lattice, if for any $x \in L$, and any set of elements $\{y_i \mid i \in I\}$, I being nonempty index set,

1. $x \wedge (\vee_{i \in I} y_i) = \wedge_{i \in I}(x \vee y_i)$;
2. $x \vee (\wedge_{i \in I} y_i) = \vee_{i \in I}(x \wedge y_i)$ and hold.

A complete lattice is a partially ordered set in which all subsets have both a supremum (join) and an infimum (meet).

It is known [11] that a complete lattice L is dually Brouwerian if and only if (2) is satisfied in L. Therefore, a complete and completely distributive lattice L is dually Brouwerian.

In this paper, L denotes a complete and completely distributive lattice with the greatest element 1 and the least element 0. Unless otherwise specified all matrices and vectors are of order n.

Lemma 1 ([12], Lemma 2.2)**.** *Let L be a complete and completely distributive lattice. Then for any $a, b, c \in L$*

1. $a - b \leq a$;
2. $b \leq c \Rightarrow a - b \geq a - c$ *and* $b - a \leq c - a$;
3. $a \leq b \Leftrightarrow a - b = 0$;
4. $a - (b \vee c) \leq (a - b)(a - c)$;
5. $a - (bc) = (a - b) \vee (a - c)$;
6. $(ab) - c \leq (a - b)(b - c)$;
7. $(a \vee b) - c = (a - c) \vee (b - c)$;
8. $(a - b) \vee (b - c) = (a \vee b) - (bc)$.

Definition [1]**.** For any element a of a distributive lattice L, the pseudo-complement a^* of a is an element satisfying the following property for all $x \in L$: $a \wedge x = 0 \Leftrightarrow a^* \wedge x = x \Leftrightarrow x \leq a^*$. A distributive lattice L in which every element has a pseudo-complement is called a pseudo-complemented distributive lattice.

Lemma 2 [5]**.** *For any two elements a, b of a pseudo-complemented distributive lattice, then*

1. $a^* = \max\{x \in L \mid a \wedge x = 0\}$;
2. $0^{**} = 0$;
3. $a \wedge a^* = 0$;
4. $a \leq b$ *implies* $b^* \leq a^*$;
5. $a \leq a^{**}$;
6. $a \wedge b = 0$ *if and only if* $a \leq b^*$ *if and only if* $b \leq a^*$;
7. $a \leq b$ *implies* $a \wedge b^* = 0$;
8. $a^{***} = a^*$;
9. $(a \vee b)^* = a^* \wedge b^*$;
10. $(a \wedge b)^{**} = a^{**} \wedge b^{**}$.

A pseudo-complemented distributive lattice with the greatest element 1 and the least element 0 such that for all $a \in L$, $a^* \vee a^{**} = 1$ is known as Stone lattice.

Every Stone lattice can be defined as a pseudo-complemented distributive lattice L in which any of the following equivalent conditions hold for all $x, y \in L$ [10].

1. $S(L)$ is a sublattice of L, where the set $S(L) = \{x^{**}|x \in L\}$;
2. $(x \wedge y)^* = x^* \vee y^*$;
3. $(x \vee y)^{**} = x^{**} \vee y^{**}$;
4. $x^* \vee x^{**} = 1$.

Let $M_n(L)$ be the set of $n \times n$ matrices over L, the elements of $M_n(L)$ denoted by capital letters and suppose $A \in M_n(L)$, then the $(i,j)^{th}$ entry of A is denoted by a_{ij}. Giveon [4] calls them as lattice matrices.

The following definitions are due to Giveon for the lattice matrices $A = [a_{ij}]$, $B = [b_{ij}]$, $C = [c_{ij}] \in M_n(L)$, where a_{ij}, b_{ij}, $c_{ij} \in L$, $1 \leq i, j \leq n$.

$A \leq B$ if and only if $a_{ij} \leq b_{ij}$;

$A + B(= A \vee B) = C$ if and only if $c_{ij} = a_{ij} \vee b_{ij}$;

$A \wedge B = C$ if and only if $c_{ij} = a_{ij} \wedge b_{ij} = a_{ij}b_{ij}$;

$A \cdot B = AB = C$ if and only if $c_{ij} = \bigvee_{k=1}^{n}(a_{ik} \wedge b_{kj})$;

$A^T = C$ if and only if $a_{ji} = c_{ij}$;

for $\alpha \in L, \alpha A = \alpha \cdot A = C$, if and only if $c_{ij} = \alpha a_{ij}$;

$A^{K+1} = A^K A$, $O = [0_{ij}], 0_{ij} = 0$, $E = [e_{ij}], e_{ij} = 1$, $1 \leq i, j \leq n$, where E is one matrix;

$A(BC) = (AB)C, AI = IA = A$, where I is an identity matrix, $AO = OA = O$, where O is zero matrix;

$A(B + C) = AB + AC$, $(A + B)C = AC + BC$, if $A \leq B$ and $C \leq D$, then $AC \leq BD$;

$(A + B)^T = A^T + B^T$, $(A \wedge B)^T = A^T \wedge B^T$, $(AB)^T = B^T A^T$, $((A)^T)^T = A$;

$M_n(L)$ is a distributive lattice with least element zero as O and greatest element one as E with respect to $\wedge$ and $+$ (or $\vee$) ([4], Statement (15)).

Let L be a complete and completely distributive lattice and $A = [a_{ij}]$, $B = [b_{ij}] \in M_{m \times n}(L)$, then the following definition is due to Tan [12].

$A - B = C$ if and only if $c_{ij} = a_{ij} - b_{ij}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. $\bar{A}(= [\bar{a_{ij}}]) = E - A = [1 - a_{ij}]$, where E is a square matrix in which all entries are 1.

3. Symmetric and skew-symmetric lattice matrices

In this section, we will discuss the uniqueness of symmetric and weakly skew-symmetric matrices in the representation [7], $A = C \vee D$, where C is symmetric and D is weakly skew-symmetric matrices.

Definition [7]**.** A matrix $A \in M_n(L)$ is said to be

1. symmetric if $A = A^T$,
2. skew-symmetric if $A - A^T = A$,
3. weakly skew-symmetric if $A \wedge A^T = O$.

Definition. The pseudo-complement of a matrix $A \in M_n(L)$ is given by $A^* = \max\{X \in M_n(L) \mid A \wedge X = O\}$.

Lemma 3. *Let $A \in M_n(L)$ and A is symmetric matrix, then*

1. $A^* \wedge A^T = 0$*;*
2. $\bar{A} \vee A^T = E$*.*

We observe that if A is a matrix over Stone lattice, then the concept of skew-symmetric and weakly skew-symmetric coincide.

Theorem 4 [Joy, (2018) Chapter 6, Theorem 6.3.8]**.** *If a matrix in $M_n(L)$ is weakly skew-symmetric, then it is skew-symmetric.*

Theorem 5 [Joy, (2018) Chapter 6, Theorem 6.3.14]**.** *Any matrix in $M_n(L)$ can be written as join* (*or least upper bound or supremum*) *of symmetric and skew-symmetric lattice matrices.*

From Theorem 3.5, consider the representation of A, that is, $A = C \vee D$, where $C = A \wedge A^T$ is symmetric and $D = A - A^T$ is skew-symmetric, here the symmetric and skew-symmetric may not be unique.

Now we will prove the uniqueness of the above representation if D is weakly skew-symmetric matrix.

Theorem 6. *Suppose $A = C \vee D$, where C is symmetric and D is weakly skew-symmetric, then C has unique representation and D is the greatest lower bound of set of weakly skew-symmetric matrices which satisfy the above representation of A.*

Proof. Let $A \in M_n(L)$ and let $A = C \vee D$, where $C = A \wedge A^T$ is symmetric and $D = A - A^T$ is weakly skew-symmetric. Let $A = H \vee F$ be any other representation where H is symmetric and F is weakly skew-symmetric. Then consider $C = A \wedge A^T = (H \vee F) \wedge (H \vee F)^T = (H \vee F) \wedge (H \vee F^T) = H \vee (F \wedge F^T) = H$. Therefore C has unique representation and consider $D = A - A^T = (H \vee F) - (H \vee F)^T = (H - (H \vee F^T)) \vee (F - (H \vee F^T)) = F - (H \vee F^T) \leq F$.

Let $\{F_i, i \in I\}$ be the set of weakly skew-symmetric matrices which satisfy the above representation. Clearly D is the lower bound of $\{F_i,\ i \in I\}$ which implies $D \leq \wedge_{i \in I} F_i$ but always $\wedge_{i \in I} F_i \leq D$. Therefore $D = \wedge_{i \in I} F_i$. So, D is the greatest lower bound of $\{F_i,\ i \in I\}$. Since, a greatest lower bound of a set is unique.

In this sense $D = A - A^T$ has unique representation. ∎

Example 7. Consider the lattice $L = \{0, a, b, c, d, e, f, 1\}$ where the Hasse diagram of L is shown below:

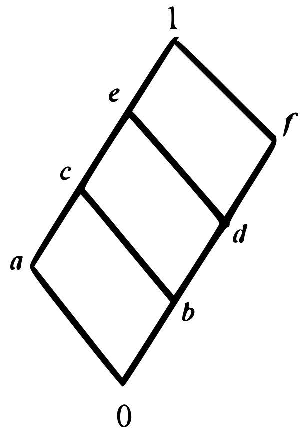

Figure 1.

Let $A = \begin{pmatrix} a & c & e \\ d & b & f \\ e & a & c \end{pmatrix}$, then $C = A \wedge A^T = \begin{pmatrix} a & b & e \\ b & b & 0 \\ e & 0 & c \end{pmatrix}$, $D = A - A^T = \begin{pmatrix} 0 & a & 0 \\ d & 0 & f \\ 0 & a & 0 \end{pmatrix}$. So, $A = C \vee D$. Clearly C is unique but we have other choices for D which are $F_1 = \begin{pmatrix} 0 & a & a \\ d & 0 & f \\ b & a & 0 \end{pmatrix}$, $F_2 = \begin{pmatrix} 0 & a & a \\ d & 0 & f \\ d & a & 0 \end{pmatrix}$, $F_3 = \begin{pmatrix} 0 & a & b \\ d & 0 & f \\ a & a & 0 \end{pmatrix}$, $F_4 = \begin{pmatrix} 0 & a & d \\ d & 0 & f \\ a & a & 0 \end{pmatrix}, \ldots$.

So, that each F_i is weakly skew-symmetric such that $A = C \vee F_1 = C \vee F_2 = C \vee F_3 = C \vee D = C \vee F_4 = \ldots$. So, we can obtain that $D = \bigwedge_{i \in I} F_i$.

4. Symmetric and tranjugate lattice matrices

In this section, we give some properties of complete and completely distributive lattice. We extend the concept of tranjugate Boolean matrices [3] to tranjugate lattice matrices and we prove that any square lattice matrix can be written as meet (or greatest lower bound or infimum) of symmetric and tranjugate lattice matrices. This is the dual of decomposition theorem (G. Joy (2018), Chapter 6, Theorem 6.3.14).

Some properties of a complete and completely distributive lattice L.

Lemma 8 [1, 10]**.** *For any elements a, b, c of a complete and completely distributive lattice L the following holds:*

(i) $a \leq b \to a$;

(ii) $a \to b = 1$ *if and only if* $a \leq b$;

(iii) $a \to 1 = 1$;

(iv) $1 \to a = a$;

(v) $a \wedge (a \to b) = a \wedge b$;

(vi) $(a \to b) \wedge (a \to c) = a \to (b \wedge c)$;

(vii) $(a \to c) \wedge (b \to c) = (a \vee b) \to c$;

(viii) $a \leq (b \to (a \wedge b))$;

(ix) *If* $a \leq b$, *then* $(c \to a) \leq (c \to b)$ *and* $(b \to c) \leq (a \to c)$;

(x) $a \to (b \to c) = (a \wedge b) \to c = (a \to b) \to (a \to c)$.

Lemma 9. *Let L be a complete and completely distributive lattice. Then for any $a, b \in L$, we have $(a \to b) \to (b \to a) = b \to a$.*

Proof. Let $a, b \in L$. According to the Claim (i) of Lemma 4.1. the inequality $b \to a \leq (a \to b) \to (b \to a)$ is valid. By Lemma 4.1 (x) and (i), we have

$$(a \to b) \to (b \to a) = [(a \to b) \wedge b] \to a \leq b \to a.$$

This proves the required equality. ∎

Definition. Let L be a complete and completely distributive lattice and $A = [a_{ij}]$, $B = [b_{ij}] \in M_{m\times n}(L)$, then $A \to B = C$ if and only if $c_{ij} = a_{ij} \to b_{ij}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Lemma 10. *Let L be a complete and completely distributive lattice with 1 and 0. Then for any $A, B \in M_n(L)$, we have*

(i) $(A \to B)^T = A^T \to B^T$;

(ii) $B \wedge (A \to B) = B$;

(iii) $E \to A = A$;

(iv) $(A \to C) \wedge (B \to C) = (A \vee B) \to C$;

(v) $B \wedge (B \to A) = A \wedge B = A \wedge (A \to B)$;

(vi) $(A \to B) \to (B \to A) = B \to A$;

(vii) *if* $(A \vee B) = E$, *then* $B \to A = A$.

Proof. By Lemma 4.1, 4.2 and Definition 4.3, we can obtain (i), (ii), (iii), (iv), (v) and (vi). For (vii), let $(A \vee B) = E$. Consider $A = E \to A = (A \vee B) \to A = (A \to A) \wedge (B \to A) = B \to A$. ■

Definition. A matrix $A \in M_n(L)$ is said to be tranjugate if $A^T \to A = A$.

Example 11. Consider the lattice $L = \{0, a, b, c, 1\}$ where the Hasse diagram of L is shown below:

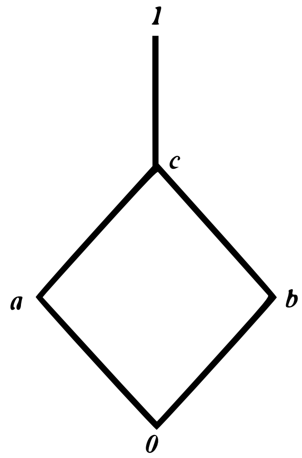

Figure 2.

Let $A = \begin{pmatrix} 1 & b \\ a & 1 \end{pmatrix}$ and $A^T = \begin{pmatrix} 1 & a \\ b & 1 \end{pmatrix}$, then $A^T \to A = \begin{pmatrix} 1 \to 1 & a \to b \\ b \to a & 1 \to 1 \end{pmatrix} = \begin{pmatrix} 1 & b \\ a & 1 \end{pmatrix} = A$ which implies A is a tranjugate matrix over L.

Theorem 12. *Let $A = [a_{ij}] \in M_n(L)$ is tranjugate. Then*

(i) *the diagonal elements must be* 1,

(ii) *a_{ij} and a_{ji} are incomparable.*

Proof. Let $A = [a_{ij}] \in M_n(L)$ be a tranjugate matrix. This means that $A^T \to A = A$.

(i) for $i = j$, $a_{ii} \to a_{ii} = a_{ii}$. So, $a_{ii} = 1$.

(ii) Suppose $a_{ij} \leq a_{ji}$.

By Lemma 4.1, $a_{ij} \to a_{ji} = 1$. A contradiction. Therefore a_{ij} and a_{ji} are incomparable. ■

Definition. A square lattice matrix A is said to be weakly tranjugate if $A \vee A^T = E$.

Example 13. Consider the lattice $L = \{0, a, b, c, d, 1\}$ where the Hasse diagram of L is shown below:

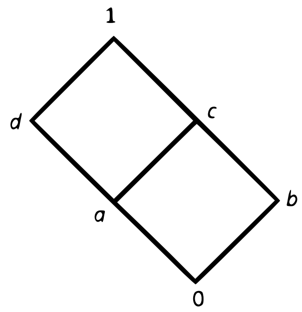

Figure 3.

Let $A = \begin{pmatrix} 1 & d \\ c & 1 \end{pmatrix}$ and $A^T = \begin{pmatrix} 1 & c \\ d & 1 \end{pmatrix}$, then $A \vee A^T = \begin{pmatrix} 1 & d \\ c & 1 \end{pmatrix} \vee \begin{pmatrix} 1 & c \\ d & 1 \end{pmatrix} = E$ which implies A is a weakly tranjugate matrix over L.

We can observe that if A is a matrix over a Stone lattice, then the concept of tranjugate and weakly tanjugate coincide. However, in the case of matrices over L, we prove that every weakly tranjugate lattice matrices are tranjugate and converse need not be true.

Theorem 14. *Let $A \in M_n(L)$ be a weakly tranjuagte matrix. Then A is tranjugate.*

Proof. Suppose $A \in M_n(L)$ is a weakly tranjuagte matrix, then $A \vee A^T = E$. By Lemma 4.4(vii), we can obtain $A^T \to A = A$. Therefore A is a tranjugate lattice matrix. ■

Example 15. Consider the lattice $L = \{0, a, b, c, 1\}$ where the Hasse diagram of L is shown in Figure 2. Let $A = \begin{pmatrix} 1 & b \\ a & 1 \end{pmatrix}$ Clearly A is tranjuagte. Consider $A \vee A^T = \begin{pmatrix} 1 & b \\ a & 1 \end{pmatrix} \vee \begin{pmatrix} 1 & a \\ b & 1 \end{pmatrix} = \begin{pmatrix} 1 & c \\ c & 1 \end{pmatrix} \neq E$. Therefore, A is not weakly tranjugate matrix over L.

Lemma 16. *For any $A, B \in M_n(L)$, then $A \vee B = E$ if and only if $\bar{A} \leq B$.*

Corollary 17. *A lattice matrix A is tranjugate whenever $\bar{A^T} \leq A$.*

Now we prove that any square matrix over L can be represented as meet (or greatest lower bound or infimum) of symmetric and tranjugate lattice matrices.

Theorem 18. *Any lattice matrix in $M_n(L)$ can be expressed as meet (or greatest lower bound or infimum) of symmetric and tranjugate lattice matrices.*

Proof. Suppose $A \in M_n(L)$ and let $S = A \vee A^T$, $T = A^T \to A$. Consider, $S^T = (A \vee A^T)^T = (A \vee A^T) = S$. Consider, $T^T \to T = (A^T \to A)^T \to (A^T \to A) = (A \to A^T) \to (A^T \to A) = A^T \to A = T$. Which shows that S is symmetric and T is tranjugate. Consider $S \wedge T = (A \vee A^T) \wedge (A^T \to A) = [A \wedge (A^T \to A)] \vee [A^T \wedge (A^T \to A)] = A \vee [A^T \wedge A] = A$. ∎

Remark 19. In the Theorem 4.14, $S \vee T$ need not be equal to E.

Example 20. Consider the lattice $L = \{0, a, b, c, 1\}$ where the Hasse diagram of L is shown in Figure 2. Let $A = \begin{pmatrix} 1 & b \\ a & 1 \end{pmatrix}$, then $A^T = \begin{pmatrix} 1 & a \\ b & 1 \end{pmatrix}$. Consider $S = A \vee A^T = \begin{pmatrix} 1 & b \\ a & 1 \end{pmatrix} \vee \begin{pmatrix} 1 & a \\ b & 1 \end{pmatrix} = \begin{pmatrix} 1 & c \\ c & 1 \end{pmatrix}$. $T = A^T \to A = \begin{pmatrix} 1 \to 1 & a \to b \\ b \to a & 1 \to 1 \end{pmatrix} = \begin{pmatrix} 1 & b \\ a & 1 \end{pmatrix}$. So, we have $S \wedge T = \begin{pmatrix} 1 & b \\ a & 1 \end{pmatrix} = A$ and $S \vee T = \begin{pmatrix} 1 & c \\ c & 1 \end{pmatrix} \neq E$.

Suppose in the representation of A as $A = S \wedge T$, S is symmetric and T is weakly tranjugate, we can observe the uniqueness of the representation $A = S \wedge T$ in the following theorem.

Theorem 21. *Suppose S is symmetric and T is weakly tranjugate such that $A = S \wedge T$, then S has a unique representation and T is the least upper bound of all such weakly tranjugate matrices which satisfy above representation.*

Proof. Let $S = A \vee A^T$ be symmetric and $T = A^T \to A$ be weakly tranjugate. Then clearly, $A = S \wedge T$. Now consider any other representation of A as $A = X \wedge Y$, where X is symmetric and Y is weakly tranjugate. Then $S = A \vee A^T = (X \wedge Y) \vee (X^T \wedge Y^T) = X \wedge (Y \vee Y^T) = X$.

Therefore $S = A \vee A^T$ has unique representation. And consider $T = A^T \to A = (X^T \wedge Y^T) \to (X \wedge Y) = (X \wedge Y^T) \to (X \wedge Y) = [(X \wedge Y^T) \to X] \wedge [(X \wedge Y^T) \to Y] = (X \wedge Y^T) \to Y = X \to (Y^T \to Y) = X \to Y \geq Y$.

Clearly T is upper bound of set of weakly tranjugate matrices $\{Y_i,\ i \in I\}$ which satisfy the above representation.

Then $T \geq \bigvee_{i \in I} Y_i$ but always $\bigvee_{i \in I} Y_i \geq T$. Therefore $T = \bigvee_{i \in I} Y_i$. T is the least upper bound of $Y_i, i \in I$. In this sense $T = A^T \to A$ has unique representation. ∎

Example 22. Consider the lattice $L = \{0, a, b, c, d, e, f, g, h, i, j, 1\}$ where the Hasse diagram of L is shown below:

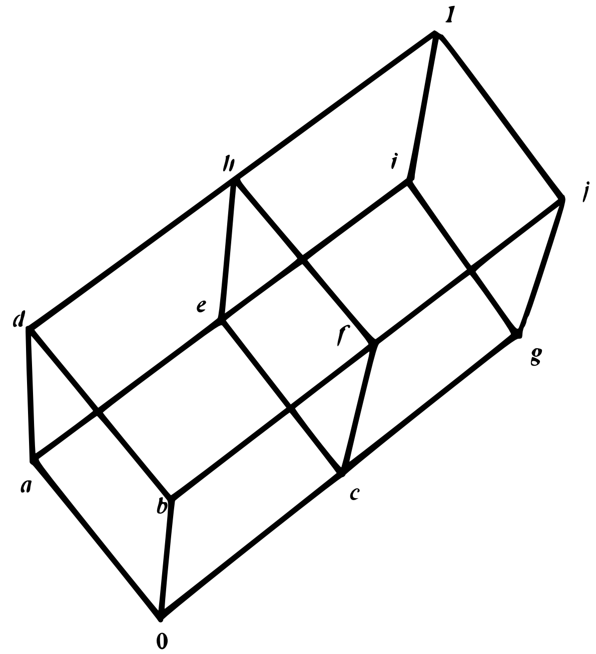

Figure 4.

Let $A = \begin{pmatrix} c & d & e \\ a & h & j \\ i & d & f \end{pmatrix}$. Let $S = A \vee A^T = \begin{pmatrix} c & d & i \\ d & h & 1 \\ i & 1 & f \end{pmatrix}$ and $T = A^T \rightarrow A = \begin{pmatrix} 1 & 1 & h \\ i & 1 & j \\ 1 & d & 1 \end{pmatrix}$. So, clearly $S \wedge T = A$ and S is symmetric and T is weakly tranjugate. But we have other representations $Y_1 = \begin{pmatrix} 1 & h & h \\ i & 1 & j \\ 1 & d & 1 \end{pmatrix}$, $Y_2 = \begin{pmatrix} 1 & h & h \\ e & 1 & j \\ 1 & d & 1 \end{pmatrix}$, $\cdots$ such that $S \wedge Y_1 = S \wedge Y_2 = \cdots = A$.

We can obtain that $T = \bigvee_{i \in I} Y_i$.

Conclusion

In this paper, we extended the concept of tranjugate lattice matrices. Some properties of elements of a complete and completely distributive lattice were studied. We have proven that a square lattice matrix can be written as meet (or greatest lower bound or infimum) of symmetric and tranjugate lattice matrices is given and we have discussed their uniqueness.

Acknowledgments

The author thank Dr. DPRV Subba Rao for his support and encouragement and the management of Icfai Tech, IFHE, Hyderabad for providing the necessary facilities.

REFERENCES

[1] G. Birkhoff, Lattice Theory, American Mathematical Society 3^{rd} edition (Providence, RI, USA, 1967).

[2] T.S. Blyth, *Pseudo-complementation, Stone and Heyting algebras*, in: T.S. Blyth, Lattices and Ordered Algebraic Structures (Springer Science and Business Media, 2006), 103–118.

[3] W.-K. Chen, *Boolean matrices and switching nets*, Math. Magazine **39** (1) (1966) 1–8.
https://doi.org/10.2307/2688986

[4] Y. Give'on, *Lattice matrices*, Information and Control **7** (4) (1964) 477–84.
https://doi.org/10.1016/S0019-9958(64)90173-1

[5] G. Gratzer, General Lattice Theory (Academic Press, New York, San Francisco, 1978).

[6] R. Gudepu and D.P.R.V.S. Rao, *A public key cryptosystem based on lattice matrices*, J. Math. Comput. Sci. **10** (6) (2020) 2408–2421.
https://doi.org/10.28919/jmcs/4882

[7] G. Joy, A Study on Lattice Matrices (Mahatma Gandhi University, India, 2018).
http://hdl.handle.net/10603/273450.

[8] S. Lang, Introduction to Linear Algebra, 2^{nd} edition (Springer, United States of America, 1985).

[9] R.D. Luce, *A note on Boolean matrix theory*, Proc. Amer. Math. Soc. **3** (3) (1952) 382–388.
https://doi.org/10.1090/S0002-9939-1952-0050559-1

[10] H. Rasiowa, An Algebraic Approch to Non-Clasical Logics (North-Holland Publishing Company, Amsterdam, London, 1974).

[11] Y.-J. Tan, *Eigenvalues and eigenvectors for matrices over distributive lattices*, Linear Algebra Appl. **283** (1998) 257–272.
https://doi.org/10.1016/S0024-3795(98)10105-2

[12] Y.-J. Tan, *On compositions of lattice matrices*, Fuzzy Sets and Systems **129** (2002) 19–28.
https://doi.org/10.1016/S0165-0114(01)00246-9

Received 7 November 2020
Revised 24 December 2020
Accepted 17 June 2021