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ON PRIME RINGS WITH INVOLUTION AND GENERALIZED DERIVATIONS

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Abstract

In this note we investigate some commutativity conditions on prime rings with involutions by using some generalized derivations. We have provided a counter example as well.

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1. Introduction

Throughout we assume that R is an associative ring with the center $Z(R)$. A ring R is said to be 2-torsion free, if for $x \in R$, $2x = 0$ then $x = 0$.

For each $x, y \in R$, the symbol $[x, y]$ will represent the commutator $xy - yx$ and the symbol $x \circ y$ stands for the skew-commutator $xy + yx$.

A ring R is prime if $aRb = \{0\}$ implies that $a = 0$ or $b = 0$ for any $a, b \in R$, it is called semiprime if $aRa = \{0\}$ implies that $a = 0$ for any $a \in R$. An ideal P of R is called a prime ideal if $P(\neq R)$ and if for any ideals I, J of R, $IJ \subseteq P$ implies that $I \subseteq P$ or $J \subseteq P$.

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By involution $*$ we mean an additive mapping $* : R \to R$ defined by $* : x \mapsto *(x) = x^*$ such that $(xy)^* = y^*x^*$ and $(x^*)^* = x$ hold for all $x, y \in R$. If R adheres to an involution $*$, then we briefly say that R is a $*$-ring. It is known that a $*$-prime ring is semiprime (for proof see [11] Fact E). Also, it can be easily seen that if R i a $*$-prime ring of characteristic not 2, then R is 2-torsion free (for proof see [2] Lemma7). Let R be a $*$-ring. A left (resp. right, two sided) ideal I of R is called a left (resp. right, two sided) $*$-ideal if $I^* = I$. An ideal P of R is called a $*$-prime ideal if $P(\neq R)$ is a $*$-ideal and for $*$-ideals I, J of R, $IJ \subseteq P$ implies that $I \subseteq P$ or $J \subseteq P$.

Now we will explore some examples.

Example 1.1. Let $\mathbb{Z}$ be the ring of integers. Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b \in \mathbb{Z} \right\}$. We define a map $* : R \longrightarrow R$ as follows: $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^* = \begin{pmatrix} c & -b \\ 0 & a \end{pmatrix}$. It is easy to check that R is a $*$-ring and that $I = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in \mathbb{Z} \right\}$ is a $*$-ideal of R.

Example 1.2. Let F be any field and $R = F[x]$ be the polynomial ring over F. Define $* : R \longrightarrow R$ by $(f(x))^* = f(-x)$ for all $f(x) \in R$. Then one can easily check that R is a $*$-ring and that xR is a $*$-prime ideal of R.

Note that an ideal I of R may not be a $*$-ideal. For instance,

Example 1.3. Let $\mathbb{Z}$ be the ring of integers and $R = \mathbb{Z} \times \mathbb{Z}$. Consider a map $* : R \longrightarrow R$ defined by $(a, b)^* = (b, a)$ for all $a, b \in R$. For an ideal $I = \mathbb{Z} \times \{0\}$ of R, I is not a $*$-ideal of R since $I^* = \{0\} \times \mathbb{Z} \neq I$.

A $*$-ring R is said to be a $*$-prime ring if for any $a, b \in R$, $aRb = aRb^* = \{0\}$ implies $a = 0$ or $b = 0$. Obviously, every prime ring equipped with involution $*$ is $*$-prime. Following example show that the converse need not be true in general.

Example 1.4. Let $\boldsymbol{\mathbb{Z}}$ be the ring of integers and let $R = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \mid a, b \in \boldsymbol{\mathbb{Z}} \right\}$. Define involution $* : R \to R$ by $*(A) = A^*$, where A^* is the transpose of $A \in R$. Let $A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}$. It is clear that $ARB = 0$, but neither $A = 0$ nor $B = 0$, where $A, B \in R$, it is follows that R is not prime. Further it is easily seen that $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} R$ $\begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}^* = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, implies that either $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = 0$ or $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = 0$. Hence R is a $*$-prime but not prime.

This example shows that $*$-prime rings constitute a more general class than that of prime rings.

On the other hand, some classes of rings do not adhere to any involution, for example,

Example 1.5. The ring

$$K_{2^n} := \langle x_1, \ldots, x_n |\ x_i x_j = x_i,\ 2x_i = 0,\ \forall i = 1, 2, \ldots, n\rangle$$

cannot adhere to any involution (see Example 2.5 [13]), though it has derivations and generalized derivations (see [1]). In particular, for $n = 2$, we can rewrite $K_{2^2} = \langle a, b\rangle = \{0, a, b, c\}$ with the relations as defined above (see Example2.1 [1]). Then $R = K_{2^2} \times K_{2^2}$ is a ring under the above operations. It is easy to check that R equipped with the exchange involution $*_{ex}$, defined by $*_{ex}(a, b) = (b, a)$, is $*_{ex}$-prime but not prime. Define I as $I = \{(0, 0), (a, 0), (b, 0), (c, 0)\}$, it is clear that I is an ideal of R but not $*_{ex}$-ideal of R. Moreover, the ideal $J = \{(0, 0), (c, 0), (0, c), (c, c)\}$, is a $*_{ex}$-ideal of R with the exchange involution $*_{ex}$. Also, note that $(c, c), (0, 0)$ are the symmetric and skew-symmetric elements of R.

We need the symmetric ($x^* = x$) and skew-symmetric ($x^* = -x$) elements in our work. Let us denote by $S(R) := \{x \in R \mid x^* = \pm x\}$.

Recall that an additive mapping $d : R \longrightarrow R$ is a derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. An additive mapping $F\colon\ R \to R$ is a generalized derivation associated with a derivation d if $F(xy) = F(x)y + xd(y)$ holds $\forall\ x, y \in R$. Over the last four decades, several authors have proved commutativity theorems for prime and semiprime rings that admitting some special mappings, particularly in the form of automorphisms, derivations and generalized derivations on an appropriate subset of R. Recently, some well-known results concerning prime rings have been extended for prime rings with involution (see [3, 4, 5, 8, **?**] and [14], for partial bibliography).

In this paper, we continue to investigate the commutativity of the $*$-prime ring R admitting generalized derivations (F, d) and (G, g) satisfying any one of the following properties: $\forall\ x, y \in I$, (i) $F(x)G(y) = 0$, (ii) $F(x)[x, y] = G[x, y]$, and (iii) $F(x)[x, y] = G(x \circ y)$.

2. Preliminary results

We shall use without explicit mention the following basic identities, that hold for any $x, y, z \in R$.

- $[xy, z] = x[y, z] + [x, z]y$

- $[x, yz] = y[x, z] + [x, y]z$
- $x \circ (yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z$
- $(xy) \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]$.

We begin our discussion with the following lemmas which are essential for developing the proof of our main results.

Lemma 2.1 ([7], Lemma 3.1)**.** Let R be a 2-torsion free $*$-prime ring and I a nonzero $*$-ideal of R. If $a, b \in R$ such that $aIb = aIb^* = 0$, then $a = 0$ or $b = 0$.

Lemma 2.2 ([12], Theorem 3.1)**.** Let R be a 2-torsion free $*$-prime ring with involution $*$, I a nonzero $*$-ideal of R. If R admits a generalized derivation F associated with a nonzero derivation d commuting with $*$ such that $F(x) = 0$ for all $x \in I$, then R is commutative.

Lemma 2.3 ([12], Lemma 2.4)**.** Let R be a 2-torsion free $*$-prime ring and I a nonzero $*$-ideal of R. If R admits a nonzero derivation d commuting with $*$ such that $[x, y]Id(x) = 0$ for all $x, y \in I$, then R is commutative.

3. Main results

In [2] in Theorem 1 the authors obtained that a $*$-prime ring R admitting a nonzero derivation d which commutes with $*$, such that $d(xoy) = d(x)oy$ or $d(x)oy = xoy$ for all x, y in a nonzero $*$-ideal of R is commutative. Also in Theorem 2 they proved that $d = 0$ when R is a $*$-prime ring admitting a derivation d commutes with $*$ such that $d(xoy) = xoy$ or $d(xoy) = d(x)oy$ for all x, y in a nonzero $*$-ideal of R. In Theorem 1 in the following we have restated Theorems 1 and 2 of [2] and provided a short and direct proof of Theorem 2 of [2].

Theorem 3.1 (Theorems 1 and 2 in [2])**.** *Let R be a $*$-prime ring of characteristic different than 2, I a nonzero $*$-ideal of R and d a derivation on R such that d commutes with $*$. If $d(xoy) = d(x)oy$ or $d(xoy) = xoy$ for all $x, y \in I$, then R is commutative and $d = 0$.*

Proof. Note that, if once it is established that under the situations mentioned in the theorem that R is commutative, then together with the Identity (3.1) in [2] it can directly be proved that the derivation involved is *zero*. Let us prove it here.

Identity (3.1) in [2] states that

$$d(x)yx = -xd(y)y; \text{ for all } x, y \in I.$$

Because R is commutative and $char(R) \neq 2$, so

$$d(x)yx = 0; \text{ for all } x, y \in I.$$

Then $\forall r \in R$, $rx \in I$ and so

$$d(rx)yrx = 0 \text{ for all } x, y \in I,$$

which implies that

$$d(r)xyrx + rd(x)yrx = d(r)rxyx = 0; \text{ for all } x, y \in I.$$

If $d \neq 0$, then $xyx = 0$ which eventually implies that $I = 0$. Hence we conclude that $d = 0$. ■

Theorem 3.2. *Let R be a 2-torsion free $*$-prime ring and I be a nonzero $*$-ideal of R. If R admits generalized derivations (F, d) and (G, g) in which g commutes with $*$ such that $F(x)G(y) = 0$ for all $x, y \in I$, then either $d = 0$ or $g = 0$.*

Proof. By hypothesis, we have

$$F(x)G(y) = 0 \text{ for all } x, y \in I. \tag{3.1}$$

Replacing y by yx in (3.1) and using (3.1), we get

$$F(x)Ig(x) = 0 \text{ for all } x \in I. \tag{3.2}$$

Since I is a nonzero $*$-ideal of R and $*g = g*$, then equation (3.2), yields that $F(x)I(g(x))^* = 0$ for all $x \in I \cap S(R)$. So, in view of Lemma 2.1, we have either $F(x) = 0$ or $g(x) = 0$. Using the fact that $x \pm x^* \in I \cap S(R)$ for all $x \in I$, we get $F(x \pm x^*) = 0$ or $g(x \pm x^*) = 0$. Hence, in view of the Equation (3.2) and Lemma (2.1), we get either $F(x) = 0$ or $g(x) = 0$, for all $x \in I$. We obtain that I is the set theoretic union of two proper subgroups, namely,

$$A := \{x \in I \mid F(x) = 0\}$$

and

$$B := \{x \in I \mid g(x) = 0\}.$$

But $(I, +)$ cannot be the set theoretic union of two proper subgroups, hence either $A = I$ or $B = I$. If $A = I$, then $F(I) = \{0\}$, so $F = 0$ and thus $d = 0$. On the other hand, if $B = I$, then $g(I) = 0$ which yields that $g = 0$. ■

Theorem 3.3. *Let R be a 2-torsion free $*$-prime ring and I be a nonzero $*$-ideal of R. If R admits generalized derivations F and G associated with nonzero derivations d and g, respectively, with $*g = g*$, such that $F(x)[x, y] = G[x, y]$ for all $x, y \in I$, then R is commutative.*

Proof. It is given that F and G are generalized derivations such that $F(x)[x,y] = G[x,y]$ for all $x,y \in I$. If $F = 0$, then for any $x,y \in I$, $G[x,y] = 0$. Replacing y by yx in the last relation and using it, we get $[x,y]g(x) = 0$. Now, replacing y by zy in the last relation and using this, we have

$$[x,z]Ig(x) = 0 \text{ for all } x,z \in I.$$

Thus, by Lemma (2.3), we get the required result. On the other hand, if $G = 0$, then for any $x,y \in I$ we have $F(x)[x,y] = 0$. Replacing y by yz in the last expression and using this, we get

$$F(x)I[x,z] = 0 \text{ for all } x,z \in I. \tag{3.3}$$

Since I is a nonzero $*$-ideal of R, then (3.3) yields that

$$F(x)I[x,z]^* = 0 \text{ for all } z \in I,\ x \in I \cap S(R).$$

So, in view of Lemma (2.1), we have $F(x) = 0$ or $[x,z] = 0$ for all $x,z \in I$. Using the fact that $x \pm x^* \in I \cap S(R)$ for all $x \in I$, then $F(x \pm x^*) = 0$ or $[x \pm x^*, z] = 0$ for all $z \in I$. Hence, we have either $F(x) = 0$ or $[x,z] = 0$ for all $x,z \in I$. This means that I is the union of two of its additive subgroups, defined by $A := \{x \in I \mid F(x) = 0\}$ and $B := \{x,z \in I \mid [x,z] = 0\}$. But a group cannot be the union of its two proper subgroups and hence either $I = A$ or $I = B$. If $I = A$, then $F(x) = 0$ for all $x \in I$ and hence by Lemma (2.2), we get the required result. On the other hand if $I = B$, then $[x,z] = 0$ for all $x,z \in I$ and hence by [[6], proof of Theorem (1.1)], R is commutative.

Henceforth, we shall assume that $F \neq 0$ and $G \neq 0$. We have

$$F(x)[x,y] = G[x,y] \text{ for all } x,y \in I. \tag{3.4}$$

Replacing y by yx in (3.4) and using (3.4), we get

$$[x,y]g(x) = 0 \text{ for all } x,y \in I. \tag{3.5}$$

Again, replacing y by zy in (3.5) and using (3.5), we get $[x,z]Ig(x) = 0$ for all $x,z \in I$. Hence Lemma (2.3), we get the require result. ■

Theorem 3.4. *Let R be a 2-torsion free $*$-prime ring and I be a nonzero $*$-ideal of R. If R admits a generalized derivation F associated with a nonzero derivation d and a generalized derivation G associated with a derivation g commuting with $*$, such that $F(x)[x,y] = G(x \circ y)$ for all $x,y \in I$, then R is commutative and $G = g = 0$.*

Proof. It is given that $F(x)[x,y] = G(x \circ y)$ for all $x, y \in I$. If $F = 0$, then we have

$$G(x \circ y) = 0 \tag{3.6}$$

Replacing y by yx in the last equation and using this, we get $(x \circ y)g(x) = 0$ for all $x, y \in I$. Again, replacing y by zy in the last equation and using this, we have

$$[x,z]Ig(x) = 0 \tag{3.7}$$

for all $x, z \in I$. Since I is a nonzero $*$-ideal of R and $*g = g*$, then (3.7) yields that

$$[x,z]I(g(x))^* = 0 \text{ for all } z \in I,\ x \in I \cap S(R).$$

So, in view of Lemma (2.1), we have $[x,z] = 0$ or $g(x) = 0$ for all $x, z \in I$. Using the fact that $x \pm x^* \in I \cap S(R)$ for all $x \in I$, then $[x \pm x^*, z] = 0$ or $g(x \pm x^*) = 0$ for all $z \in I$. Hence, we have either $[x,z] = 0$ or $g(x) = 0$ for all $x, z \in I$. Consider $[x,z] = 0$ for all $x, z \in I$. Using the same arguments as used in [[6], proof of Theorem (1.1)], we get that R is commutative. If R is commutative, then the equation (3.6) becomes $G(2xy) = 0$ for all $x, y \in I$, using the fact that R is 2-torsion free, we get $G(xy) = 0$. Now replacing y by yz in the last equation and using it, we get $xyg(z) = 0$ for all $x, y, z \in I$. Since g commutes with $*$, then the last equation leads to $xIg(z) = xI(g(z))^* = \{0\}$, since $I \neq \{0\}$ then we have $g(I) = 0$, thus $g = 0$. As $0 = G(xy) = G(x)y + xg(y)$ it follows that $G(I) = 0$. Accordingly, $G = 0 = g$. Now, if $G = 0$, then we have $F(x)[x,y] = 0$ for all $x, y \in I$. Using the similar arguments as used in the proof of the last theorem, we get the required result.

Hence onward, we assume that $F \neq 0$, we have

$$F(x)[x,y] = G(x \circ y) \text{ for all } x, y \in I. \tag{3.8}$$

Replacing y by yx in (3.8) and using (3.8), we get $(x \circ y)g(x) = 0$ for all $x, y \in I$. For any $z \in I$, replacing y by zy in the last expression and using this, we get $[x,z]Ig(x) = 0$ for all $x, z \in I$. Notice that the arguments given after equation (3.7) are still valid in the present situation and hence repeating the same process, we get the required result. ■

Example 3.1. Let $\mathbb{Z}$ be the ring of integers and let $R = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \mid a, b \in \mathbb{Z} \right\}$ and $I = \left\{ \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} \mid b \in \mathbb{Z} \right\}$. Define $* : R \to R$ by $*\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -b & a \end{pmatrix}$ for all $\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \in R$. and $F : R \to R$ by $F\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}$, $d : R \to R$

by $d\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}$, and $G : R \to R$ by $G\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, $g : R \to R$ by $g\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -b & 0 \end{pmatrix}$. Then R is a ring under usual operations, I is a $*$-ideal, and it is easy to see that F and G are generalized derivations of R associated with the derivations d and g, respectively, and g commutes with $*$ such that either (i) $F(x)G(y) = 0$, for all $x, y \in I$, but $d \neq 0$ and $g \neq 0$ or (ii) $F(x)[x, y] = G[x, y]$, for all $x, y \in I$, but R is not commutative or (iii) $F(x)[x, y] = G(x \circ y)$ for all $x, y \in I$, but R is not commutative and $G \neq g \neq 0$. Hence, in Theorems 3.2, 3.3 and 3.4, the hypothesis of $*$-primeness cannot be omitted.

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