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ON RIGHT INVERSE ORDERED SEMIGROUPS

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Abstract

A regular ordered semigroup S is called right inverse if every principal left ideal of S is generated by an $\mathcal{R}$-unique positive element of it. We prove that a regular ordered semigroup is right inverse if and only if any two inverses of an element $a \in S$ are $\mathcal{R}$-related. Furthermore the class of right Clifford ordered semigroups have been characterized by the class of right inverse ordered semigroups.

Keywords: ordered regular, ordered inverse, positive element, completely regular, right inverse.

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1. Introduction

Following Bailes [2], a right inverse semigroup is a regular semigroup with the property that each element has a unique left unit. Due to Bailes the class of right inverse semigroups are strictly between the class of orthodox semigroups and the class of inverse semigroups. Venkatesan [8] studied these semigroups under the name of right unipotent semigroups. He showed that a semigroup is right inverse if and only if every principal left ideal has a unique idempotent generator.

Bhuniya and Hansda [1] have deal with ordered semigroups in which any two inverses of an element are $\mathcal{H}$-related. These ordered semigroups are analogue to inverse semigroups. So it is a logical step to study ordered semigroups with the property that any two inverses of an element of it are $\mathcal{R}$-related. So the purpose of this paper is to characterise such ordered semigroups which we call right inverse ordered semigroups. We give a detailed exposition on the characterization of these ordered semigroups. Here we generalize such ordered semigroups into right inverse ordered semigroups. This paper is inspired by the works done by Venkatesan [8] and Bailes [2].

The presentation of this article is as follows: This section is followed by preliminaries. Definitions and basic properties of ordered semigroups are described in section 2. Section 3 is devoted to the right inverse ordered semigroups and its different characterizations.

2. Preliminaries

An ordered semigroup is a partiality ordered set $(S, \leq)$, and at the same time a semigroup $(S, \cdot)$ such that for all $a, b, x \in S$, $a \leq b$ implies $xa \leq xb$ and $ax \leq bx$. It is denoted by $(S, \cdot, \leq)$. For an ordered semigroup S and $H \subseteq S$, denote the download closure by

$$(H] := \{t \in S : t \leq h, \text{ for some } h \in H\}.$$

Let I be a non-empty subset of S. Then I is called a left(right) ideal [4] of S, if $SI \subseteq I(IS \subseteq I)$ and $(I] \subseteq I$. If I is both left and right ideal, then it is called an ideal of S. We call S a (left, right) simple ordered semigroup if it does not contain any proper (left, right) ideal. For $a \in S$, the smallest (left, right) ideal of S that contains a is denoted by $(L(a), R(a))I(a)$.

Following Kehayopulu [4] an ordered semigroup S is said to be regular (respectively, completely regular, right regular) if for every $a \in S$, $a \in (aSa]$ $(a \in (a^2Sa^2],\ a \in (a^2S])$. Due to Kehayopulu [4] Green's relations on a regular ordered semigroup given as follows:

$a\mathcal{L}b$ if $L(a) = L(b)$, $a\mathcal{R}b$ if $R(a) = R(b)$, $a\mathcal{J}b$ if $I(a) = I(b)$, $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$. These four relations $\mathcal{L}, \mathcal{R}, \mathcal{J}$, and $\mathcal{H}$ are equivalence relations.

An equivalence relation ρ is called left (right) congruence if for $a, b, c \in S$ $a\rho b$ implies $ca\rho cb$ $(ac\rho bc)$. By a congruence we mean both left and right congruence. A congruence ρ is called semilattice congruence if for all $a, b \in S$, $a\rho a^2$ and $ab\rho\, ba$. By a complete semilattice congruence we mean a semilattice congruence σ such that for $a, b \in S$, $a \leq b$ implies that $a\sigma ab$. An ordered semigroup S is called complete semilattice of subsemigroups of type τ if there exists a complete semilattice congruence ρ such that $(x)_\rho$ is a type τ subsemigroup of S.

A regular ordered semigroup S is said to be group-like (respectively, left group-like) [1] ordered semigroup if for every $a, b \in S$, $a \in (Sb]$ and $b \in (aS]$ (respectively, $a \in (Sb]$). Right group-like ordered semigroup can be defined dually. A regular ordered semigroup S is called a right (left) Clifford [1] if for all $a \in S$, $(Sa] \subseteq (aS]$, $((aS] \subseteq (Sa])$. Every right (left) group-like ordered semigroup is a right (left) Clifford ordered semigroup. An element $b \in S$ is said to be an inverse of $a \in S$ if $a \leq aba$ and $b \leq bab$. We denote the set of all inverses of an element a

by $V_{\leq}(a)$. In an ordered semigroup S, an element [7] $e \in S$ is said to be positive element if $e \leq e^2$. The set of all positive elements of S denoted by $E_{\leq}(S)$.

Following results are useful for the sake of this paper.

Theorem 1 [1]**.** *Let S be a regular ordered semigroup. Then the following statements are equivalent.*

(1) *S is right Clifford ordered semigroup;*

(2) *for all $e \in E_{\leq}(S), (Se] \subseteq (eS]$;*

(3) *for all $a \in S$, and $e \in E_{\leq}(S)$, there is $x \in S$ such that $ea \leq ax$;*

(4) *for all $a, b \in S$, there is $x \in S$ such that $ba \leq ax$;*

(5) *$\mathcal{L} \subseteq \mathcal{R}$ on S.*

Lemma 2 [1]**.** *Let S be a right Clifford ordered semigroup. Then the following conditions hold in S.*

(1) *$a \in (a^2Sa]$, for every $a \in S$;*

(2) *$ef \in (feSef]$, for every $e, f \in E_{\leq}(S)$.*

Theorem 3 [1]**.** *Let S be an ordered semigroup. Then S is right* (*left*) *Clifford ordered semigroup if and only if $\mathcal{R}(\mathcal{L})$ is the least complete semilattice congruence on S.*

Theorem 4 [1]**.** *Let S be a regular ordered semigroup. Then S is right* (*left*) *Clifford ordered semigroup if and only if it is a complete semilattice of right* (*left*) *group-like ordered semigroups.*

3. Right inverse ordered semigroup

Let S be an ordered semigroup and ρ be an equivalence relation on S. By the notion ρ-unique in S we mean the uniqueness in respect to the relation ρ.

Lemma 5. *Every principal left ideal in a regular ordered semigroup is generated by a positive element.*

Proof. Let S be a regular ordered semigroup. Consider a principal left ideal $J = (Sa]$ of S for $a \in S$. Since S is regular there is $x \in S$ such that $a \leq axa$. Then $xa \in E_{\leq}(S)$. Say $e = xa$. Clearly $a\mathcal{L}e$. Take $y \in J$. Then $y \leq za$ for some $z \in S$, so that $y \leq zaxa = zae$. Therefore $y \in (Se]$. Hence $J = (Se]$. ■

Lemma 6. *Let S be a regular ordered semigroup. Then S is a left group-like ordered semigroup if and only if any two positive elements of S are $\mathcal{L}$-related.*

Proof. Suppose that S is a left group-like ordered semigroup. Choose $e, f \in E_{\leq}(S)$. Then $e\mathcal{L}f$.

Conversely, assume that the given condition holds in S. Let $a, b \in S$. Since S is regular, there exists $s, t \in S$ such that $a \leq asa$ and $b \leq btb$. Clearly $as, sa, bt, tb \in E_{\leq}(S)$. Then by given condition, $sa \leq vtb$ for some $v \in S$. Then $a \leq asa \leq avtb$. Hence S is left group-like ordered semigroup. ■

Right group-like ordered semigroup can be characterized dually.

Let S be a regular ordered semigroup and $a \in S$. Then there is $x \in S$ such that $a \leq axa \leq a(xax)a$. Also $xax \leq xaxaxax$. So there is $a' \in V_{\leq}(a)$ such that $a \leq aa'a$. Now we have the following theorem.

Theorem 7. *Let S be a regular ordered semigroup. Then any two inverses of an element of S are $\mathcal{L}$-related if and only if for every $e, f \in E_{\leq}(S), ef \in (eSfSe]$.*

Proof. First suppose that the given condition holds in S. Let $a \in S$ and $a',\ a'' \in V_{\leq}(a)$. Now $a' \leq a'aa' \leq a'aa''aa'$. Clearly aa' and $aa'' \in E_{\leq}(S)$ so that by the given condition we have $aa''aa' \leq aa''raa'taa''$ for some $r, t \in S$. Therefore $a \leq a'aa''raa'taa''$, that is, $a' \leq ma''$, where $m = a'aa''raa'ta$. Similarly $a'' \leq na'$ for some $n \in S$. Hence $a'\mathcal{L}a''$.

Conversely, let $a \in S$ and $a',\ a'' \in V_{\leq}(a)$ such that $a'\mathcal{L}a''$. Let $e, f \in E_{\leq}(S)$. Since S is regular, $V_{\leq}(ef) \neq \phi$. Let $x \in V_{\leq}(ef)$. Then $x \leq xefx$ and $ef \leq efxef$. Now $fxe \leq f(xefx)e \leq fxe(ef)fxe$ and $ef \leq ef(fxe)ef$. Then $ef \in V_{\leq}(fxe)$. Also $fxe \in E_{\leq}(S)$, and so $fxe \in V_{\leq}(fxe)$. This yields that $ef \in V_{\leq}(fxe)$, so $ef\mathcal{L}fxe$ by the given condition. Hence $ef \leq ufxe$ for some $u \in S$. Hence from above $ef \leq ef^2xeufxe$. Thus $ef \in (eSfSe]$. ■

Definition. A regular ordered semigroup S is called right inverse if every principal left ideal is generated by an $\mathcal{R}$-unique positive element of S.

Example 3.1. The ordered semigroup $S = \{a, e, f\}$ defined by multiplication and order below.

$\cdot$	a	e	f
a	a	e	f
e	a	e	f
f	a	e	f

$$' \leq' := \{(a, a), (a, e), (a, f), (e, e), (f, f)\}.$$

$(S, \cdot, \leq)$ is a right inverse ordered semigroup.

The following lemma is obvious so we omit its proof.

Lemma 8. *A regular ordered semigroup S is a right inverse if and only if for any two positive elements $e, f \in E_{\leq}(S)$, $e\mathcal{L}f$ implies $e\mathcal{H}f$.*

Left group-like ordered semigroups are generalizations of group-like ordered semigroups. Every right inverse and left group-like ordered semigroups are group-like ordered semigroups.

Lemma 9. *Every right inverse left group-like ordered semigroup is a group-like ordered semigroup.*

Proof. This follows from Lemma 6 and Lemma 8. ∎

The characterization of right inverse ordered semigroups by the inverses of their elements has been given in the following theorem.

Theorem 10. *The following conditions are equivalent on a regular ordered semigroup S.*

(1) *S is right inverse;*

(2) *for every $a \in S$ and $a', a'' \in V_{\leq}(a)$, $a'\mathcal{R}a''$;*

(3) *for every $e, f \in E_{\leq}(S)$, $ef \in (fSeSf]$;*

(4) *for every $e, f \in E_{\leq}(S)$, $(eS] \cap (fS] = (efS]$;*

(5) *for $e \in E_{\leq}(S)$ and $x \in (Se]$ implies $x' \in (eS]$, where $x \in S$ and $x' \in V_{\leq}(x)$.*

Proof. (1) $\Rightarrow$ (2): Let S be a right inverse semigroup and $a \in S$. Consider $a', a'' \in V_{\leq}(a)$. Then $a'a, a''a \in E_{\leq}(S)$. Say $L = (Sa]$ and let $x \in (Sa]$. Then there is $s \in S$ such that $x \leq sa$. Thus $x \leq saa'a$, which implies that $x \in (Sa'a]$ and so $(Sa] \subseteq (Sa'a]$. Also $(Sa'a] \subseteq (Sa]$. And thus $(Sa] = (Sa'a]$. Similarly $(Sa] = (Sa''a]$. Since S is right inverse, we have $a'a\mathcal{R}a''a$. Now $a'' \leq a''aa'' \leq a'az_1a''$ for some $z_1 \in S$. Hence $a'' \leq a't_1$, where $t_1 = az_1a''$. Also $a' \leq a'aa'' \leq a''az_2a'$ for some $z_2 \in S$. Hence $a' \leq a''t_2$, where $t_2 = az_2a'$. Hence $a'\mathcal{R}a''$.

(2) $\Rightarrow$ (3): Suppose that for every $a \in S$ and $a', a'' \in V_{\leq}(a)$, we have $a'\mathcal{R}a''$. Since S is regular, $V_{\leq}(ef) \neq \phi$. Let $x \in V_{\leq}(ef)$. Then $x \leq xefx$ and $ef \leq efxef$ and so $fxe \leq fxefxe \leq fxe(ef)fxe$ and $ef \leq ef^2xe^2f$. Thus

$$ef \leq ef(fxe)ef \tag{1}$$

and thus $ef \in V_{\leq}(fxe)$. Also $fxe \leq (fxe)^2$, so that is $fxe \in V_{\leq}(fxe)$. Hence $ef, fxe \in V_{\leq}(fxe)$. Then by the condition (2), we have $ef\mathcal{R}fxe$, so $ef \leq fxeu$ for some $u \in S$. Also $ef \leq fxeufxe^2f$ from inequality (1). Hence $ef \in (fSeSf]$.

(3) $\Rightarrow$ (4): Let $x \in (eS] \cap (fS]$. Then there are $s_1, s_2 \in S$ such that $x \leq es_1$ and $x \leq fs_2$. Since S is regular there is $z \in V_{\leq}(x)$ such that $x \leq xzx$, so $x \leq es_1zfs_2$. Also $es_1z \leq es_1zxz \leq es_1zes_1z$, thus $es_1z \in E_{\leq}(S)$. Now by condition (3) $es_1zf \in (fSes_1zSf]$, and thus $es_1zf \leq fs_3es_1zs_4f$ for some $s_3, s_4 \in S$. Therefore $es_1zf \leq fm$, where $m = s_3es_1zs_4f$. Finally $x \leq es_1zfs_2 \leq e(es_1zf)s_2 \leq efms_2$. Thus $x \in (efS]$ and hence $(eS] \cap (fS] \subseteq (efS]$.

Next let $y \in (efS]$ then $y \in (eS]$. Also $ef \in (fSeSf]$, by given condition. Now $y \in (efS]$ implies that $y \leq efq$ for some $q \in S$. Then there are $s_6, s_7 \in S$ such that $y \leq fs_6es_7fq$. Thus $y \in (fS]$, and so $y \in (eS] \cap (fS]$. Therefore $(efS] \subseteq (eS] \cap (fS]$ and hence $(eS] \cap (fS] = (efS]$.

(4) $\Rightarrow$ (5): Given $x \in (Se]$. Then there is $s \in S$ such that $x \leq se$. Also $x \leq xx'x$ and $x' \leq x'xx'$ for some $x' \in V_{\leq}(x)$. Now $x'se \leq x'sex'se$, so $x'se \in E_{\leq}(S)$. Then for $x'se,\ e \in E_{\leq}(S)$ we have that $(x'seeS] = (eS] \cap (x'seS]$, by condition (4). Also from $x' \leq x'xx'$ we have $x' \leq x'sex'$. Therefore $x' \in (x'SeeS]$ and hence $x' \in (eS]$.

(5) $\Rightarrow$ (1): Since S is regular, by Lemma 5 we have that every principal left ideal is generated by a positive element. Let $e, f \in E_{\leq}(S)$ such that $(Se] = (Sf]$. Also $e \leq ee$, so $e \in (Se] = (Sf]$. Now $e \in V_{\leq}(e)$. So from the given condition $e \in (fS]$. Similarly $f \in (eS]$ and thus $e\mathcal{R}f$. Hence S is a right inverse ordered semigroup. ■

Corollary 3.2. *Let S be a right inverse ordered semigroup. Then any two positive elements $e, f \in E_{\leq}(S)$ are $\mathcal{H}$-commutative if and only if $(Se] \cap (Sf] = (Sef]$.*

Proof. This follows from Theorem 10 and ([1], Theorem 3.10). ■

Let S be a regular ordered semigroup. For $A \subseteq S$, denote $A' = \{x \in S : x \in V_{\leq}(y) \text{ where } y \in A\}$, the set of all ordered inverses of the elements of A.

In the following lemma we characterize these subsets A' in an ordered semigroup.

Lemma 11. *Let S be a regular ordered semigroup. Then following are true in S.*

(1) *For any subset R of S, $(R')' \subseteq R$.*

(2) *For any subset A, B of S, $A \subseteq B$ implies $A' \subseteq B'$.*

Proof. (1) Let $x \in (R')'$. So there $x' \in R'$ such that $x \in V_{\leq}(x')$. Also $x' \in V_{\leq}(x)$ which implies that $x \in R$. So $(R')' \subseteq R$.

(2) Let $x \in A'$. So there exists $y \in A$ such that $x \in V_{\leq}(y)$. Now $y \in A$ implies $y \in B$. So $x \in B'$. So $A' \subseteq B'$. ■

In the following theorem we characterize a right inverse ordered semigroup S by the set of all inverses of elements of $\mathcal{R}$-class of a positive element of S.

Theorem 12. *Let S be a regular ordered semigroup. Then S is a right inverse ordered semigroup if and only if $L_e \subseteq (R_e)'$ for any $e \in E_{\leq}(S)$.*

Proof. Let $e \in E_{\leq}(S)$. Say $R = R_e$ and $L = L_e$. Suppose that S is a right inverse ordered semigroup. Let $x \in L_e$ and $x' \in V_{\leq}(x)$. Then $x' \in (L_e)'$ and $x'x \in E_{\leq}(S)$. Now $x'x \leq (x'xx')x$ and $x \leq x(x'x)$, which gives that $x\mathcal{L}x'x$.

Hence $x'x$ is a positive element in L_e. Therefore $L = (Se] = (Sx'x]$. Since S is a right inverse ordered semigroup, so $e\mathcal{R}x'x$, by definition. Also $x'\mathcal{R}x'x$, then $x' \in R_e$ which implies that $x \in (R_e)'$ and hence $L \subseteq R'$.

Conversely assume that the given condition holds in S. Since S is regular, by Lemma 5 we have every principal left ideal is generated by a positive element. Let $e, f \in E_{\leq}(S)$ such that $(Se] = (Sf]$. Clearly $e, f \in L_e$. Also $L_e \subseteq (R_e)'$ implies $(L_e)' \subseteq ((R_e)')' \subseteq R_e$ by Lemma 11. Now $e, f \in L_e$ implies $e, f \in (L_e)'$. So $e, f \in R_e$. Hence S is right inverse ordered semigroup. ∎

Theorem 13. *An ordered semigroup S is right Clifford if and only if S is right inverse and for every $a \in S$, $a \in (a^2Sa]$.*

Proof. First suppose that S is a right inverse ordered semigroup and for every $a \in S$, $a \in (a^2Sa]$. Then $a \leq a^2xa$ for some $x \in S$. So we have $a^2x \leq a^2xa^2x$, which implies that $a^2x \in E_{\leq}(S)$. Let $e \in E_{\leq}(S)$. Since S is right inverse, there are $x_1, x_2 \in S$ such that $ea^2x \leq a^2xx_1ex_2a^2x$ by Theorem 2. Now $ea \leq ea^2xa \leq a^2xx_1ex_2a^2xa = a(axx_1ex_2a^2xa) = ax_3$, where $x_3 = axx_1ex_2a^2xa$. Thus $ea \in (aS]$. So S is a right Clifford ordered semigroup.

Conversely, assume that S is a right Clifford ordered semigroup. Then from Lemma 2, $a \in (a^2Sa]$ and $ef \in (feSef]$. So there is $x_4 \in S$ such that $ef \leq fex_4ef \leq f(e)e(x_4e)f = f(e)e(x_5)f$, where $x_5 = x_4e$ and thus $ef \in (fSeSf]$. Hence S is right inverse ordered semigroup, by Theorem 10. ∎

Bhuniya and Hansda [1] showed that every Clifford ordered semigroup is a union of group-like ordered semigroups. Here we have shown that a left Clifford ordered semigroup is a union of group-like ordered semigroups in right inverse ordered semigroup.

Theorem 14. *Let S be a right inverse ordered semigroup. If S is left Clifford then S is a union of group-like ordered semigroups.*

Proof. Suppose that S is a left Clifford ordered semigroup. Then $\mathcal{L}$ is a semilattice congruence on S by Theorem 4. Let $a \in S$ and $a' \in V_{\leq}(a)$. Then $a \leq aa'a$ and $a' \leq a'aa'$. Let us denote $aa' = e$ and $a'a = f$. Clearly $e, f \in E_{\leq}(S)$. Also $a \leq aa'a$ and $a'a \leq a'aa'a$, which implies that $a\mathcal{L}f$. Since $\mathcal{L}$ is a congruence on S, $ea\mathcal{L}ef$. Also $a \leq aa'a = ea \leq eea$ and $ea \leq eea$, so $a\mathcal{L}ea$. Thus $a\mathcal{L}ef$ implies that $f\mathcal{L}ef$. Similarly it can be shown that $a'\mathcal{L}e$ and $e\mathcal{L}fe$. Since $\mathcal{L}$ is a semilattice congrence, so $ef\mathcal{L}fe$. Hence $ef\mathcal{L}fe\mathcal{L}a\mathcal{L}a'\mathcal{L}f\mathcal{L}e$.

Also $a\mathcal{R}e$ and $a'\mathcal{R}f$. Hence $a\mathcal{H}e$ and $a'\mathcal{H}f$. Now let $x \in (Se]$. Hence there are $s_1, s_2 \in S$ such that $x \leq s_1e \leq s_1s_2f$, since $e\mathcal{L}f$.

So $x \in (Sf]$. Hence $(Se] \subseteq (Sf]$. Similarly it can be shown that $(Sf] \subseteq (Se]$. Hence $(Se] = (Sf]$. Since S is right inverse ordered semigroup, we have $e\mathcal{R}f$. Hence $e\mathcal{H}f$. So $a\mathcal{H}f\mathcal{H}e\mathcal{H}a'$.

Since $a\mathcal{H}f$ and $a\mathcal{H}a'$, so there is $x_1, x_2 \in S$ such that $a \leq aa'a \leq aa'aa'a = afa'a \leq aax_1x_2aa = a^2x_1x_2a^2$.

So, $a \in (a^2Sa^2]$. Thus S is a completely regular ordered semigroup and hence S is a union of group-like ordered semigroups by Theorem 4.8 of [1]. ∎

In the following we show that in a right inverse ordered semigroup $\mathcal{R}$ is a congruence if and only if $\mathcal{L} = \mathcal{H}$.

Theorem 15. *Let S be a right inverse ordered semigroup. Then following conditions are equivalent.*

(1) *$\mathcal{R}$ is a congruence on S;*

(2) *$\mathcal{L} = \mathcal{H}$;*

(3) *S is a complete semilattice of right group-like ordered semigroups.*

Proof. (1) $\Rightarrow$ (2). Let $\mathcal{R}$ be a congruence on S. Since S is right inverse, S is regular. So for every $a \in S$ there is $a' \in V_{\leq}(a)$. Now $a \leq aa'a$ and $a' \leq a'aa'$. Denote $aa' = e$ and $a'a = f$. Therefore $a\mathcal{R}e$ and $a'\mathcal{R}f$. Since $\mathcal{R}$ is a congruence on S, $aa'\mathcal{R}ef$. That is $e\mathcal{R}ef$.

Again $a' \leq (a'e)e$ and $a'e \leq a'(ee)$. Therefore $a'\mathcal{R}a'e$ which implies that $a'\mathcal{R}fe$. Thus $f\mathcal{R}fe$. So there is $z \in S$ such that $f \leq fez$. Since S is right inverse, there are $x, y \in S$ such that $ef \leq fxeyf \leq fezxeyf$. Hence $ef \leq fet_1$, where $t_1 = zxeyf$. Similarly $fe \leq eft_2$, for some $t_2 \in S$. Therefore $ef\mathcal{R}fe$. Hence $a\mathcal{R}e\mathcal{R}ef\mathcal{R}fe\mathcal{R}f\mathcal{R}a'$. Now $a \leq aa'a \leq aawa$, for some $w \in S$. Hence $a \in (a^2Sa]$. So S is a right clifford ordered semigroup. Hence $\mathcal{L} \subseteq \mathcal{R}$. Thus $\mathcal{L} = \mathcal{H}$.

(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1). These implications follows from Theorem 4 and Theorem 3. ∎

Our paper ends up with the following corollary which follows from Theorem 15 and Theorem 14. This gives a condition for a right inverse ordered semigroup to become a completely regular ordered semigroup.

Corollary 3.3. *Let S be a right inverse and left regular ordered semigroup. Then following conditions are equivalent.*

(1) *$\mathcal{R}$ is a congruence on S;*

(2) *$\mathcal{L} = \mathcal{H}$;*

(3) *S is a complete semilattice of right group-like ordered semigroups;*

(4) *S is completely regular.*

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