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GENERALIZED ANDRÁSFAI GRAPHS

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Abstract

In this paper, we introduce a new family of circulants $GA(t,k)$, called Generalized Andrásfai graphs, where $t,k \geq 2$ are integers. We study various parameters like diameter, girth, domination number etc. of $GA(t,k)$. Moreover, we find the full automorphism group of $GA(t,k)$ and compute its determining number.

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1. Introduction

Andrásfai graphs, denoted by $And(k)$, is a family of triangle-free circulant graphs, named after Bela Andrásfai [1]. It is a Cayley graph on $\mathbb{Z}_n$ where $n = 3k-1$ and the connection set $S = \{[1],[4],\ldots,[3k-2]\}$, where $[x]$ denote the equivalence class of x in $\mathbb{Z}_n$. In this paper, we introduce a generalization of this family of circulants and study its automorphisms and structural properties. For definitions and terminologies, readers are referred to [4].

Definition 1.1. Let $t,k \geq 2$ be two positive integers and set $n = t(k-1)+2$. Define $GA(t,k)$ to be the Cayley graph on $\mathbb{Z}_n$ with connection set $S = \{[1],[t+1],[2t+1],\ldots,[t(k-1)+1]\}$.

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Clearly, $GA(3,k) = And(k)$ and hence $GA(t,k)$ is a generalization of the family of Andrásfai graphs. Being a circulant graph, $GA(t,k)$ is a k-regular, vertex-transitive, Hamiltonian graph.

To avoid ambiguity, by $[x]$, we denote the equivalence class of x in $\mathbb{Z}_n$ and by x, we denote the corresponding integer in the range $0 \leq x \leq t(k-1)+1$. While denoting any vertex, we use both symbols $[x]$ and x, where they have the above meanings.

2. Structural properties of $GA(t,k)$

In this section, we study some structural properties of $GA(t,k)$, like girth, diameter, chromatic number, domination number etc. Before that, we prove two lemmas which will be crucial throughout the paper.

Lemma 2.1. Let $[x]$ and $[y]$ be two vertices of $GA(t,k)$ with $0 \leq x, y \leq t(k-1)+1$. If $[x]$ is adjacent to $[y]$, then $x - y \equiv \pm 1 \pmod{t}$.

Proof. *Case* 1. $x > y$. In this case, we have $[x-y] = [x] - [y] \in S$. Therefore, $[x-y] = [st+1]$ where $0 \leq s \leq (k-1)$, i.e., $x - y \equiv st + 1 \pmod{n}$, i.e., $(x-y)-(st+1)$ is a multiple of n. As $0 < x-y, st+1 < n$, we have $x-y = st+1$ (as integers), i.e., $x - y \equiv 1 \pmod{t}$.

Case 2. $y > x$. Proceeding similarly as above, we get $y - x \equiv 1 \pmod{t}$, i.e., $x - y \equiv -1 \pmod{t}$. ■

It is to be noted that the converse of the above lemma does not hold. For example, let $t = k = 4$, then $n = 14$. Also let $x = 9$ and $y = 6$. Then $x - y = 3 \equiv -1 \pmod{4}$, but $[6] \not\sim [9]$, as $S = \{[1], [5], [9], [13]\}$. This observation motivates us to get a better understanding of the adjacency condition in light of Lemma 2.1.

Let $[x]$ and $[y]$ be two adjacent vertices of $GA(t,k)$ with $0 \leq x, y \leq t(k-1)+1$. Then there exists integers l_1, l_2, r_1, r_2 such that $x = tl_1 + r_1$ and $y = tl_2 + r_2$ where $0 \leq l_1, l_2 \leq k-1$ and $0 \leq r_1, r_2 \leq t-1$. Note that if $l_i = k-1$, then $r_i \in \{0,1\}$. Suppose $x > y$. Then by Lemma 2.1, $x - y = t(l_1 - l_2) + (r_1 - r_2) \equiv \pm 1 \pmod{t}$. Again, as $[x] \sim [y]$, we have $x - y = st + 1$, i.e., $x - y \equiv 1 \pmod{t}$. Similarly, if $x < y$, we have $x - y \equiv -1 \pmod{t}$. This gives us the following lemma, which is a sort of converse to Lemma 2.1.

Lemma 2.2. Let $[x]$ and $[y]$ be two vertices of $GA(t,k)$ with $0 \leq x, y \leq n-1$ and $x - y \equiv \pm 1 \pmod{t}$. Then $[x] \sim [y]$ if and only if either $x - y \equiv 1 \pmod{t}$ and $x > y$ or $x - y \equiv -1 \pmod{t}$ and $x < y$.

Proposition 2.3. $GA(t,k)$ is triangle-free and hence of girth 4.

Proof. We first show that $GA(t,k)$ has no triangle containing the vertex $[0]$. Let $[g]$ and $[h]$ be two neighbours of $[0]$, which forms a triangle. Then $[g],[h] \in S$. Let $g = l_1t+1$ and $h = l_2t+1$. Without loss of generality, let $g > h$. Then so $l_1 \geq l_2$. Thus $[g]-[h] = [g-h] = [(l_1-l_2)t] \notin S$. Thus $[g] \not\sim [h]$. So $GA(t,k)$ has no triangle with $[0]$ as a vertex. As $GA(t,k)$ is vertex transitive, it is triangle free.

As $GA(t,k)$ is a Cayley graph on $\mathbb{Z}_n$, so girth of $GA(t,k) \leq 4$ and it is triangle-free, hence $GA(t,k)$ is of girth 4. ■

Theorem 2.1. $GA(t,k)$ is bipartite if and only if t is even.

Proof. Let t be even, then n is even. Let $A = \{[0],[2],[4],\ldots,[t],[t+2],\ldots,[2t],[2t+2],\ldots,[(k-1)t]\}$ and $B = \{[1],[3],[5],\ldots,[t-1],[t+1],\ldots,[2t+1],\ldots,[(k-1)t+1]\}$. As difference between any two elements of A (or B) is even, it can not be congruent to $\pm 1 \pmod t$. Hence A and B form partite sets of $GA(t,k)$. Hence $GA(t,k)$ is bipartite.

Let t be odd. Consider the cycle $[0] \sim [1] \sim [2] \sim \cdots \sim [t] \sim [t+1] \sim [0]$ of length $t+2$. As t is odd, it is an odd cycle and hence $GA(t,k)$ is not bipartite. ■

Corollary 2.4. If $t = 2$, then $GA(t,k)$ is a complete bipartite graph with equal partite-sets.

Proof. If $t = 2$, by previous theorem, $GA(t,k)$ is bipartite, where $A = \{[0],[2],[4],\ldots,[2(k-1)]\}$ and $B = \{[1],[3],[5],\ldots,[2(k-1)+1]\}$ forms the bipartition and $|A| = |B| = k$. Now, as $GA(t,k)$ is k-regular and no two vertices in A (or B) are adjacent to each other, $GA(t,k)$ is a complete bipartite graph. ■

Proposition 2.5. If t is odd, the odd girth of $GA(t,k)$ is $t+2$.

Proof. Let t be odd. In the proof of Theorem 2.1, we have seen the existence of an odd cycle of length $t+2$. Let, if possible, $C : [0] \sim [x_1] \sim [x_2] \sim \cdots \sim [x_{t-2m-1}] \sim [0]$ be a cycle of length $t-2m$ where $0 \leq m \leq \frac{t-5}{2}$.

$$\begin{array}{rcl} x_1 & \equiv & 1 \pmod t \\ x_2 - x_1 & \equiv & \pm 1 \pmod t \\ x_3 - x_2 & \equiv & \pm 1 \pmod t \\ \vdots & \vdots & \vdots \\ x_{t-2m-1} - x_{t-2m-2} & \equiv & \pm 1 \pmod t. \end{array}$$

Adding this relations we have $x_{t-2m-1} \equiv \beta \pmod t$. As $[0] \sim [x_{t-2m-1}]$, we have $x_{t-2m-1} \equiv \beta \equiv 1 (\bmod t)$. Since there are $t-2m-1$ relations, so $-t+2m+3 \leq \beta \leq t-2m-1$. This implies $\beta = 1$. So half of $x_{i+1} - x_i \equiv 1 \pmod t$ and half of $x_{i+1} - x_i \equiv -1 \pmod t$. Thus $t-2m-2$ must be even, which contradicts that t is odd. Therefore, $GA(t,k)$ has no odd cycle of length $< t+2$. So odd girth of $GA(t,k)$ is $t+2$. ■

Theorem 2.2. The chromatic number of $GA(t,k)$ is $= \begin{cases} 2, & \text{if } t \text{ is even,} \\ 3, & \text{otherwise.} \end{cases}$

Proof. If t is even, then $GA(t,k)$ is bipartite, thus chromatic number $= 2$. If t is odd, then $GA(t,k)$ is not bipartite and hence $\chi \geq 3$. Now, consider the sets

$A = \{[0],[2],[4],\ldots,[t-1],[t+2],\ldots,[2t-1],[2t+2],\ldots,[(k-1)t-1]\}$,
$B = \{[1],[3],[5],\ldots,[t],[t+3],\ldots,[2t],[2t+3],\ldots,[(k-1)t]\}$ and
$C = \{[t+1],[2t+1],[3t+1],\ldots,[(k-1)t+1]\}$.

We claim that A, B, C are independent sets, as that will prove $GA(t,k)$ is 3-colourable. Let $[g],[h] \in A$ be arbitrary, and if possible $[g] \sim [h]$. So $[g] = l_1t + i_1$ and $[h] = l_2t + i_2$ where $0 \leq l_1, l_2 \leq k-1$ and $i_1, i_2 \in \{0,2,4,\ldots,t-1\}$. Without loss of generality, let $i_1 \geq i_2$. Thus $[g]-[h] = [g-h] = (l_1-l_2)t + (i_1-i_2) \equiv i_1 - i_2 \pmod{t} \not\equiv \pm 1 (\text{mod } t)$, except when $i_1 = t-1, i_2 = 0$, as i_1, i_2 both are even and $0 \leq i_1 - i_2 \leq t-1$. Hence, by Lemma 2.1, $[g] \nsim [h]$.

If $i_1 = t-1, i_2 = 0$, then $[g] = [l_1t + (t-1)]$ and $[h] = [l_2t]$. However this implies $[h] = [0]$. Now, all the neighbours of $[0]$ are in C and hence $[g] \nsim [h]$, and A is an independent set. In a similar way, B is also an independent set. Since $GA(t,k)$ is triangle-free and vertices of C are neighbours of $[0]$, there is no edge between them and hence C is also an independent set. Therefore, $\chi = 3$. ■

Now, we prove two lemmas which serves as the distance formulae for any two vertices in $GA(t,k)$. These lemmas will be used to compute the diameter in Theorem 2.3 as well as it will be useful in ascertaining the automorphism group of $GA(t,k)$ in Section 3.

Lemma 2.6. Let t be even and $[x]$ be a vertex in $GA(t,k)$ such that $x = lt + i$, where $0 \leq l \leq k-1$ and $0 \leq i \leq t-1$. (If $l = k-1$, then $i = 0$ or 1.) Then

$$d([0],[x]) = \begin{cases} 2, & \text{if } i = 0, \\ i, & \text{if } 1 \leq i \leq \frac{t}{2}+1, \\ t+2-i, & \text{if } \frac{t}{2}+1 \leq i \leq t-1. \end{cases}$$

Proof. *Case* 1. If $i = 0$, then $x = lt$ and hence $[0] \nsim [x]$. Also $[0] \sim [lt+1] \sim [lt]$ is a path of length 2 and hence $d([0],[x]) = 2$.

Case 2. Let $1 \leq i \leq \frac{t}{2}+1$. Then there exists a path $[0] \sim [lt+1] \sim [lt+2] \sim \cdots \sim [lt+i] = [x]$ of length i joining $[0]$ and $[x]$. If possible, let $d([0],[x]) < i$, i.e., there exists a path of length $p < i$ between $[0]$ and $[x]$, such that $[0] \sim [y_1] \sim [y_2] \sim \cdots \sim [y_p] = [lt+i]$. Then, by Lemma 2.1, we have

$$\begin{array}{rcl} y_1 & \equiv & 1 \pmod{t} \\ y_2 - y_1 & \equiv & \pm 1 \pmod{t} \\ y_3 - y_2 & \equiv & \pm 1 \pmod{t} \\ \vdots & \vdots & \vdots \\ y_p - y_{p-1} & \equiv & \pm 1 \pmod{t}. \end{array}$$

Adding these congruences, we get $y_p \equiv \beta \pmod{t}$, where $-p+2 \leq \beta \leq p$. Also, we have $y_p \equiv i \pmod{t}$. However, as $p < i$, this is a contradiction. Therefore, $d([0],[x]) = i$.

Case 3. Let $\frac{t}{2}+1 \leq i \leq t-1$, i.e., $3 \leq t+2-i \leq \frac{t}{2}+1$. We can find a path $[x] = [lt+i] \sim [lt+(i+1)] \sim [lt+(i+2)] \sim \cdots \sim [lt+(t+1)] \sim [0]$ of length $t+2-i$ between $[0]$ and $[x]$. If possible, let $d([0],[x]) < t+2-i$, i.e., there exists a path of length $q < t+2-i$ between $[0]$ and $[x]$, such that $[0] \sim [y_1] \sim [y_2] \sim \cdots \sim [y_q] = [lt+i]$. Then, using Lemma 2.1, as in Case 1 and adding the congruences, we get $y_q \equiv \beta \pmod{t}$ where $-q+2 \leq \beta \leq q$. Also, we have $y_q \equiv i \pmod{t}$, i.e., $\beta \equiv i \pmod{t}$. Note that as $q < t+2-i$ and $\frac{t}{2}+1 \leq i \leq t-1$, we have $-t+i < -q+2 \leq \beta \leq q < t+2-i \leq i$, i.e., $-t+i < \beta < i$. Now, the range of β shows that $\beta \not\equiv i \pmod{t}$, a contradiction.

This completes the proof and we have $d([0],[x]) = t+2-i$. ■

Lemma 2.7. Let t be odd and $[x]$ be a vertex in $GA(t,k)$ such that $x = lt+i$, where $0 \leq l \leq k-1$ and $0 \leq i \leq t-1$. (If $l = k-1$, then $i = 0$ or 1.) Then

$$d([0],[x]) = \begin{cases} 2, & \text{if } i = 0, \\ i, & \text{if } 1 \leq i \leq \frac{t+1}{2}, \\ t+2-i, & \text{if } \frac{t+3}{2} \leq i \leq t-1. \end{cases}$$

Proof. It can be shown as that of Lemma 2.6. ■

Theorem 2.3. The diameter of $GA(t,k)$ is $\lceil \frac{t+1}{2} \rceil$.

Proof. From Lemma 2.6, we have when t is even, $\max_{v \in V(G)} d([0],[v]) = \frac{t}{2}+1$ and from Lemma 2.7, we have when t is odd, $\max_{v \in V(G)} d([0],[v]) = \frac{t+1}{2}$. As $GA(t,k)$ is vertex transitive, combining these two, we have for any t, diameter of $GA(t,k)$ is $\lceil \frac{t+1}{2} \rceil$. ■

Theorem 2.4. The domination number of $GA(t,k)$ is less than or equal to t. If $k > 2t+1$, then it is equal to t.

Proof. It is easy to see that $A = \{[0],[1],[2],\ldots,[t-1]\}$ is a dominating set of $GA(t,k)$. Thus $\gamma \leq t$. Also, if $k > 2t+1$ and as $\gamma \geq \frac{n}{1+\Delta}$, then we have

$$\gamma \geq \frac{n}{k+1} = \frac{t(k-1)+2}{k+1} = t - \frac{2(t+1)}{k+1} > t-1, \text{ i.e., } \gamma = t.$$

■

3. Automorphism group of $GA(t,k)$

We denote the automorphsim group of $GA(t,k)$ by $A(t,k)$. If $t = 2$, i.e., $n = 2k$, then $GA(t,k)$ is a complete bipartite graph with equal partite sets each of size k

and hence $A(2,k) \cong S_k \times S_k \times \mathbb{Z}_2$. Also, if $k = 2$, then $GA(t,k) \cong C_n$ is a cycle of length n and hence $A(t,2) \cong D_n$. Thus for the rest of the section, we assume $t, k \geq 3$. Since $GA(t,k)$ is a Cayley graph, $A(t,k)$ contains a regular subgroup isomorphic to $\mathbb{Z}_n$. Note that $\varphi_a : GA(t,k) \to GA(t,k)$ defined by $\varphi_a([x]) = [x+a]$ is an automorphism of $GA(t,k)$, and $\{\varphi_a : 0 \leq a \leq n-1\}$ forms a regular cyclic subgroup of $A(t,k)$ generated by φ_1 which is isomorphic to $\mathbb{Z}_n$. Also note that $\tau : GA(t,k) \to GA(t,k)$ given by $\tau([x]) = [n-x]$ forms an automorphism of $GA(t,k)$ and $\tau\varphi_1\tau = \varphi_1^{-1}$. Thus $D_n \cong H = \langle \varphi_1, \tau \rangle$ forms a subgroup of $A(t,k)$. However, we claim that for $t \neq 2$, $A(t,k) = H$. In this section, we establish this claim.

Consider the stabilizer of $[0]$, $\mathsf{Stab}_{[0]}$ in $A(t,k)$. Clearly, id and τ are two elements of H which belong to $\mathsf{Stab}_{[0]}$, i.e., $|\mathsf{Stab}_{[0]}| \geq 2$. We will prove that $|\mathsf{Stab}_{[0]}| = 2$, i.e., $\mathsf{Stab}_{[0]} = \{\mathsf{id}, \tau\}$. As $GA(t,k)$ is vertex-transitive, this will prove that $|A(t,k)| = |GA(t,k)| \cdot |\mathsf{Stab}_{[0]}| = 2n$, i.e., $A(t,k) = H$. Thus, now we focus on proving $\mathsf{Stab}_{[0]} = \{\mathsf{id}, \tau\}$.

Let $\varphi \in \mathsf{Stab}_{[0]}$. Consider the following cycle $C_0 : [0] \sim [1] \sim [2] \sim \cdots \sim [t] \sim [t+1] \sim [0]$ of length $t+2$. Then $\varphi(C_0)$ is a cycle of length $t+2$ and it is of the form $\varphi(C_0) : [0] \sim [x_1] \sim [x_2] \sim \cdots \sim [x_t] \sim [x_{t+1}] \sim [0]$, where $0 < x_i \leq t(k-1)+1$. Then by Lemma 2.1, we get the following $t+1$ congruences:

$$\begin{array}{rcl} x_1 & \equiv & 1 \pmod{t} \\ x_2 - x_1 & \equiv & \pm 1 \pmod{t} \\ x_3 - x_2 & \equiv & \pm 1 \pmod{t} \\ \vdots & \vdots & \vdots \\ x_{t+1} - x_t & \equiv & \pm 1 \pmod{t}. \end{array}$$

Adding these congruences, we get

$$x_{t+1} \equiv \alpha_0 \pmod{t}, \ \textit{where} \ -t+1 \leq \alpha_0 \leq t+1. \tag{1}$$

But, as $[x_{t+1}] \sim [0]$, we have $x_{t+1} \equiv 1 \pmod{t}$, i.e., $\alpha_0 \equiv 1 \pmod{t}$. Now, given the possible range of α_0, it can assume only one of three values, namely $-t+1, 1, t+1$. If $\alpha_0 = t+1$, then all the above congruences (with R.H.S ± 1) take the value $+1$. Similarly, if $\alpha_0 = -t+1$, then all the above congruences (with R.H.S ± 1) take the value -1. However, if $\alpha_0 = 1$, half of the t congruences (with R.H.S ± 1) are $+1 \pmod{t}$ and half of them are $-1 \pmod{t}$ and this implies that t is even. Thus, if t is odd, the only possible value of α_0 is $t+1$ or $-t+1$. For $t =$ even and $t \neq 2$, we prove separately that $\alpha_0 = 1$ can not hold. For that, we first prove some lemmas.

Lemma 3.1. Let $t \neq 2$ be even. If $\varphi([i]) = [l_i t + i]$ for $i = 1$ and 2, then $\varphi([j]) = [l_j t + j]$ for $3 \leq j \leq t+1$.

Proof. We will proof this lemma by method of induction.

Base Step. As $[2] \sim [3] \Rightarrow \varphi([2]) \sim \varphi([3])$. Then by Lemma 2.1, $\varphi([3]) = [l_3t+3]$ or $[l_3t+1]$. If $\varphi([3]) = [l_3t+1]$, then $\varphi([0]) \sim \varphi([3])$, which holds only if $t = 2$, hence $\varphi([3]) = [l_3t+3]$.

Inductive Step. Let $\varphi([j]) = [l_jt+j]$ for $3 \leq j \leq p \leq t$. As $[0] \sim [1] \sim [2] \sim \cdots \sim [p]$, we have $[0] \sim \varphi([1]) \sim \varphi([2]) \sim \cdots \sim \varphi([p])$, i.e., $[0] \sim [l_1t+1] \sim [l_2t+2] \sim \cdots \sim [l_pt+p]$. Therefore, by Lemma 2.2, we have $l_1 \leq l_2 \leq \cdots \leq l_p$. Again, as $[p+1] \sim [p]$, by Lemma 2.1, $\varphi([p+1]) = [l_{p+1}t+(p+1)]$ or $[l_{p+1}t+(p-1)]$. We will show that $\varphi([p+1]) = [l_{p+1}t+(p+1)]$.

If possible, let

$$\varphi([p+1]) = [l_{p+1}t+(p-1)] \tag{2}$$

and hence, by Lemma 2.2, $l_{p+1} \leq l_p$. Again, this implies that $l_{p+1} < l_{p-2}$, because if $l_{p+1} \geq l_{p-2}$, as $\varphi([p-2]) = [l_{p-2}t+(p-2)]$ then by Lemma 2.2, $\varphi([p-2]) \sim \varphi([p+1])$, i.e., $[p-2] \sim [p+1]$, i.e., $[0] \sim [3]$, which holds only if $t = 2$. Thus, we have

$$l_{p-1} \geq l_{p-2} > l_{p+1}. \tag{3}$$

Now, by Lemma 2.6, we have

$$d([0], \varphi([p+1])) = d([0], [l_{p+1}t+(p-1)]) = \begin{cases} 2, & \text{if } p-1 = 0, \text{ i.e., } p = 1, \\ p-1, & \text{if } 1 \leq p-1 \leq \frac{t}{2}+1, \\ t+3-p, & \text{if } \frac{t}{2}+1 \leq p-1 \leq t-1. \end{cases}$$

We will prove that none of these distance formulae hold.

Case 1. Since, $p \geq 3$, the case $p = 1$ does not arise. Therefore $d([0], \varphi([p+1])) \neq 2$.

Case 2. Let $\frac{t}{2}+1 \leq p-1 \leq t-1$, i.e., $\frac{t}{2}+3 \leq p+1 \leq t+1$ and hence $d([0], \varphi([p+1])) = t+3-p$. Again, by Lemma 2.6, note that

$$d([0], [p+1]) = \begin{cases} t+1-p, & \text{if } \frac{t}{2}+3 \leq p+1 \leq t-1, \\ 2, & \text{if } p+1 = t, \text{ i.e., } p = t-1, \\ 1, & \text{if } p+1 = t+1, \text{ i.e., } p = t. \end{cases}$$

Now, as $d([0], \varphi([p+1])) = d([0], [p+1])$, we have $t+3-p = 1$ or 2 or $t+1-p$. If $t+3-p = t+1-p$, then $2 = 0$, a contradiction. If $t+3-p = 2$ and in this case $p = t-1$, we have $2 = 0$, a contradiction. If $t+3-p = 1$ and in this case $p = t$, we have $2 = 0$, a contradiction. Thus $d([0], \varphi([p+1])) \neq t+3-p$.

Case 3. Finally, let $1 \leq p-1 \leq \frac{t}{2}+1$, i.e., $3 \leq p+1 \leq \frac{t}{2}+3$ and $d([0], \varphi([p+1])) = p-1$. Again, note that

$$d([0],[p+1]) = \begin{cases} p+1, & \text{if } 3 \leq p+1 \leq \frac{t}{2}+1, \\ t+1-p, & \text{if } p+1 = \frac{t}{2}+2 \text{ or } \frac{t}{2}+3. \end{cases} \tag{4}$$

Now, as $d([0], \varphi([p+1])) = d([0],[p+1])$, we have $p-1 = p+1$ or $t+1-p$. Clearly $p-1 = p+1$ can not hold. If $p-1 = t+1-p$, then $p = \frac{t}{2}+1$. This implies

$$d([0], \varphi([p+1])) = \frac{t}{2}, \text{ as } \varphi([p+1]) = \left[l_{p+1}t + \frac{t}{2}\right]. \tag{5}$$

As $[p+1] \sim [p+2]$, then from Equation 5 and Lemma 2.1, $\varphi([p+2]) = \left[l_{p+2}t + \left(\frac{t}{2}-1\right)\right]$ or $\left[l_{p+2}t + \left(\frac{t}{2}+1\right)\right]$.

If $\varphi([p+2]) = \left[l_{p+2}t + \left(\frac{t}{2}+1\right)\right]$, then $d([0],[p+2]) = d\left([0], \left[\frac{t}{2}+3\right]\right) = \frac{t}{2}-1$ (by Lemma 2.6) and $d([0], \varphi([p+2])) = d\left([0], [l_{p+2}t + \frac{t}{2}+1]\right) = \frac{t}{2}+1 \neq d([0],[p+2])$, a contradiction, hence $\varphi([p+2]) \neq \left[l_{p+2}t + \left(\frac{t}{2}+1\right)\right]$.

If $\varphi([p+2]) = \left[l_{p+2}t + \left(\frac{t}{2}-1\right)\right] = [l_{p+2}t + (p-2)]$, as $p = \frac{t}{2}+1$. Note that this, along with Equation 3, implies that $l_{p-1} \geq l_{p-2} > l_{p+1} \geq l_{p+2}$.

Now, as $\varphi([p-1]) = [l_{p-1}t + (p-1)]$ and $\varphi([p+2]) = [l_{p+2}t + (p-2)]$, by Lemma 2.2, we have $\varphi([p-1]) \sim \varphi([p+2])$, i.e., $[p-1] \sim [p+2]$, i.e., $[0] \sim [3]$, a contradiction, as $t \neq 2$. This establishes that $d([0], \varphi([p+1])) \neq p-1$, contradicting Equation 4. Thus Equation 2 does not hold and $\varphi([p+1]) = [l_{p+1}t + (p+1)]$. Hence, the lemma. ■

Lemma 3.2. Let $t \neq 2$ be even. If $\varphi([1]) = [l_1 t + 1]$ and $\varphi([2]) = [l_2 t]$, then $\varphi([i]) = [l_i t + \{t-(i-2)\}]$ for $3 \leq i \leq t+1$.

Proof. We will proof this lemma by method of induction.

Base Step. As $[2] \sim [3] \Rightarrow \varphi([2]) \sim \varphi([3])$. Then by Lemma 2.1, $\varphi([3]) = [l_3 t+1]$ or $[l_3' t-1] = [(l_3'-1)t+(t-1)] = [l_3 t+(t-1)]$(where $l_3'-1 = l_3$). If $\varphi([3]) = [l_3 t+1]$, then $\varphi([0]) \sim \varphi([3])$, which holds only if $t = 2$. Hence $\varphi([3]) = [l_3 t + (t-1)]$.

Inductive Step. Let $\varphi([j]) = [l_j t + \{t-(j-2)\}]$ for $3 \leq j \leq p \leq t$. As $[0] \sim [1] \sim [2] \sim \cdots \sim [p]$, we have $[0] \sim \varphi([1]) \sim \varphi([2]) \sim \cdots \sim \varphi([p])$, i.e., $[0] \sim [l_1 t+1] \sim [l_2 t] \sim \cdots \sim [l_p t + \{t-(p-2)\}]$. Therefore, by Lemma 2.2, we have $l_1 \geq l_2 \geq \cdots \geq l_p$. Again, as $[p+1] \sim [p]$, by Lemma 2.1, $\varphi([p+1]) = [l_{p+1}t + (t-p+1)]$ or $[l_{p+1}t + (t-p+3)]$. We will show that $\varphi([p+1]) = [l_{p+1}t + (t-p+1)] = [l_{p+1}t + \{t-(p+1-2)\}]$.

If possible, let

$$\varphi([p+1]) = [l_{p+1}t + (t-p+3)] \tag{6}$$

and hence, by Lemma 2.2, $l_{p+1} \geq l_p$. Again, this implies that $l_{p+1} > l_{p-2}$, because if $l_{p+1} \leq l_{p-2}$,as $\varphi([p-2]) = [l_{p-2}t + (t-p+4)]$ then by Lemma 2.2, $\varphi([p-2]) \sim \varphi([p+1])$, i.e., $[p-2] \sim [p+1]$, i.e., $[0] \sim [3]$, which holds only if $t = 2$. Thus, we have

$$l_{p-1} \leq l_{p-2} < l_{p+1}. \tag{7}$$

Now, by Lemma 2.6, we have

$$d([0], \varphi([p+1])) = d([0], [l_{p+1}t + (t-p+3)])$$
$$= \begin{cases} 2, & \text{if } t-p+3 = 0, \text{ i.e., } p = t+3, \\ t-p+3, & \text{if } 1 \leq t-p+3 \leq \frac{t}{2}+1, \\ (t+2)-(t-p+3) = p-1, & \text{if } \frac{t}{2}+1 \leq t-p+3 \leq t-1. \end{cases}$$

We will prove that none of these distance formulae hold.

Case 1. Since, $p \leq t$, the case $p = t+3$ does not arise. Therefore $d([0], \varphi([p+1])) \neq 2$.

Case 2. Let $1 \leq t-p+3 \leq \frac{t}{2}+1$, i.e., $\frac{t}{2}+3 \leq p+1 \leq t+3$ and hence $d([0], \varphi([p+1])) = t-p+3$. Again, note that

$$d([0], [p+1]) = \begin{cases} t+1-p, & \text{if } \frac{t}{2}+3 \leq p+1 \leq t-1, \\ 2, & \text{if } p+1 = t, t+2, \text{ i.e., } p = t-1, t+1, \\ 1, & \text{if } p+1 = t+1, \text{ i.e., } p = t, \\ 3, & \text{if } p+1 = t+3, \text{ i.e., } p = t+2. \end{cases}$$

As we consider $p \leq t$, the cases $p = t+1, t+2$ do not arise. Now, as $d([0], \varphi([p+1])) = d([0], [p+1])$, we have $t+3-p = 1$ or 2 or $t+1-p$.

If $t+3-p = t+1-p$, then $2 = 0$, a contradiction. If $t+3-p = 2$ and in this case $p = t-1$, we have $2 = 0$, a contradiction. If $t+3-p = 1$ and in this case $p = t$, we have $2 = 0$, a contradiction. Thus $d([0], \varphi([p+1])) \neq t+3-p$.

Case 3. Finally, let $\frac{t}{2}+1 \leq t-p+3 \leq t-1$, i.e., $5 \leq p+1 \leq \frac{t}{2}+3$ and $d([0], \varphi([p+1])) = p-1$. Again, note that

$$d([0], [p+1]) = \begin{cases} p+1, & \text{if } 5 \leq p+1 \leq \frac{t}{2}+1, \\ t+1-p, & \text{if } p+1 = \frac{t}{2}+2 \text{ or } \frac{t}{2}+3. \end{cases} \tag{8}$$

Now, as $d([0], \varphi([p+1])) = d([0], [p+1])$, we have $p-1 = p+1$ or $t+1-p$. Clearly $p-1 = p+1$ can not hold. If $p-1 = t+1-p$, then $p = \frac{t}{2}+1$. This implies

$$\varphi([p+1]) = [l_{p+1}t + (t+3-p)] = \left[l_{p+1}t + \left(\frac{t}{2}+2\right)\right]. \tag{9}$$

As $[p+1] \sim [p+2]$, then from Equation 9, $\varphi([p+2]) = \left[l_{p+2}t + \left(\frac{t}{2}+1\right)\right]$ or $\left[l_{p+2}t + \left(\frac{t}{2}+3\right)\right]$.

If $\varphi([p+2]) = \left[l_{p+2}t + \left(\frac{t}{2}+1\right)\right]$, then $d\left([0],[p+2]\right) = d\left([0],\left[\frac{t}{2}+3\right]\right) = \frac{t}{2}-1$ (by Lemma 2.6).

Also $d\left([0],\varphi\left([p+2]\right)\right) = d\left([0],[l_{p+2}t+\frac{t}{2}+1]\right) = \frac{t}{2}+1 \neq d\left([0],[p+2]\right)$, a contradiction, hence $\varphi([p+2]) \neq \left[l_{p+2}t + \left(\frac{t}{2}+1\right)\right]$.

If $\varphi([p+2]) = \left[l_{p+2}t + \left(\frac{t}{2}+3\right)\right] = [l_{p+2}t + (p+2)]$, as $p = \frac{t}{2}+1$. Note that this, along with Equation 7, implies that $l_{p-1} \leq l_{p-2} < l_{p+1} \leq l_{p+2}$.

Now, as $\varphi([p-1]) = \varphi([\frac{t}{2}]) = [l_{p-1}t + \frac{t}{2}+2] = [l_{p-1}t + (p+1)]$ and $\varphi([p+2]) = [l_{p+2}t + (p+2)]$, by Lemma 2.2, we have $\varphi([p-1]) \sim \varphi([p+2])$, i.e., $[p-1] \sim [p+2]$, i.e., $[0] \sim [3]$, a contradiction, as $t \neq 2$. This establishes that $d([0],\varphi([p+1])) \neq p-1$, contradicting Equation 8. Thus Equation 6 does not hold and $\varphi([p+1]) = [l_{p+1}t + (t-p+1)] = [l_{p+1}t + \{t-(p+1-2)\}]$. Hence, the lemma. ■

Note. From the Equation 1, we concluded that $\alpha_0 = 1$ can occur only if t is even. Now by Lemma 3.1 and 3.2, we get for any $t(\neq 2)$, $\alpha_0 = t+1$ or $-t+1$. Analogously corresponding to the cycle $C_i : [0] \sim [it+1] \sim [it+2] \sim \cdots \sim [(i+1)t] \sim [(i+1)t+1] \sim [0]$ of length $t+2$, we have $x_{(i+1)t+1} \equiv \alpha_i \pmod{t}$ for $0 \leq i \leq k-2$, where $-t+1 \leq \alpha_i \leq t+1$ and hence α_i can take the value either of $-t+1, 1, t+1$. (See Figure 1. The blue vertex in the center is $[0]$ and the red vertices are elements of S.) In the next lemma, we determine the values of α_i in terms of α_0.

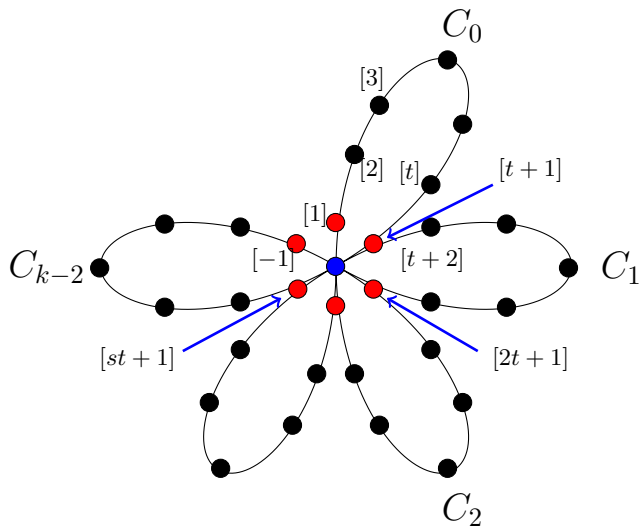

Figure 1. The cycles $C_0, C_1, \ldots, C_{k-2}$.

Lemma 3.3. If $\alpha_0 = t+1$ or $-t+1$, then $\alpha_i = t+1$ or $-t+1$ respectively for $1 \leq i \leq k-2$.

Proof. Let $\alpha_0 = t+1$. We will proof this by method of induction.

Base Step. For $i = 1$ we have $x_{2t+1} \equiv \alpha_1 \pmod{t}$, where $-t+1 \leq \alpha_1 \leq t+1$. Then $\alpha_1 = t+1$ or $-t+1$. We will prove that $\alpha_1 = t+1$.

If possible let $\alpha_1 = -t+1$, then by Lemma 2.2, we have $l_{t+1} \geq l_{t+2} \geq \cdots \geq l_{2t+1}$. Now,

$$\begin{aligned} x_{t+2} - x_{t+1} &\equiv -1 \pmod{t} \\ x_{t+1} - x_t &\equiv 1 \pmod{t} \\ x_t - x_{t-1} &\equiv 1 \pmod{t}. \end{aligned}$$

Adding these congruences, we get $x_{t+2} - x_{t-1} \equiv 1 \pmod{t}$. If $l_{t+2} \geq l_{t-1}$, then by Lemma 2.2, we have $x_{t+2} \sim x_{t-1}$, which holds only if $t = 2$, hence $l_{t+2} < l_{t-1}$. Thus, we have

$$l_t \geq l_{t-1} > l_{t+2} \geq l_{t+3}. \tag{10}$$

Now,

$$\begin{aligned} x_{t+3} - x_{t+2} &\equiv -1 \pmod{t} \\ x_{t+2} - x_{t+1} &\equiv -1 \pmod{t} \\ x_{t+1} - x_t &\equiv 1 \pmod{t}. \end{aligned}$$

Adding these congruences, we get $x_{t+3} - x_t \equiv -1 \pmod{t}$, i.e., $x_t - x_{t+3} \equiv 1 \pmod{t}$, then by Equation 10 and Lemma 2.2, we get $x_t \sim x_{t+3}$, which holds only if $t = 2$, which is a contradiction, hence $\alpha_1 = t+1$.

Inductive Step. Let $\alpha_i = t+1$ for $1 \leq i \leq p \leq k-3$, we will show that $\alpha_{p+1} = t+1$.

If possible let $\alpha_{p+1} = -t+1$, then by Lemma 2.2, we have $l_{pt+1} \geq l_{pt+2} \geq \cdots \geq l_{(p+1)t+1}$. Now,

$$\begin{aligned} x_{pt+2} - x_{pt+1} &\equiv -1 \pmod{t} \\ x_{pt+1} - x_{pt} &\equiv 1 \pmod{t} \\ x_{pt} - x_{pt-1} &\equiv 1 \pmod{t}. \end{aligned}$$

Adding these congruences, we get $x_{pt+2} - x_{pt-1} \equiv 1 \pmod{t}$. If $l_{pt+2} \geq l_{pt-1}$, then by Lemma 2.2, we have $x_{pt+2} \sim x_{pt-1}$, which holds only if $t = 2$, hence $l_{pt+2} < l_{pt-1}$. Thus, we have

$$l_t \geq l_{pt-1} > l_{pt+2} \geq l_{pt+3} \tag{11}$$

Now,

$$\begin{aligned} x_{pt+3} - x_{pt+2} &\equiv -1 \pmod{t} \\ x_{pt+2} - x_{pt+1} &\equiv -1 \pmod{t} \\ x_{pt+1} - x_{pt} &\equiv 1 \pmod{t}. \end{aligned}$$

Adding these congruences, we get $x_{pt+3} - x_{pt} \equiv -1 \pmod{t}$, i.e., $x_{pt} - x_{pt+3} \equiv 1 \pmod{t}$, then by Equation 11 and Lemma 2.2, we get $x_{pt} \sim x_{pt+3}$, which holds only if $t = 2$, which is a contradiction, hence $\alpha_{p+1} = t + 1$.

Similarly, it can be proved that if $\alpha_0 = -t + 1$, then $\alpha_i = -t + 1$ for all $1 \leq i \leq k - 2$. ∎

Now, we are in a position to prove the main theorem that $|\mathsf{Stab}_{[0]}| = 2$, i.e., $\mathsf{Stab}_{[0]} = \{\mathsf{id}, \tau\}$. Recall that, in the begining of this section, we started with an arbitrary element $\varphi \in \mathsf{Stab}_{[0]}$. In the next theorem, we prove that $\varphi = \mathsf{id}$ or τ.

Theorem 3.1. If $\alpha_0 = t + 1$ or $-t + 1$ then $\varphi = \mathsf{id}$ or τ respectively, and hence for $t \neq 2$, $A(t, k) = \langle \varphi_1, \tau \rangle \cong D_n$.

Proof. *Case* 1. Let $\alpha_0 = t + 1$, we will show that $\varphi = \mathsf{id}$. Let consider the cycle $[0] \sim \varphi([1]) = [l_1 t + 1] \sim \varphi([2]) = [l_2 t + 2] \sim \cdots \sim \varphi([i]) = [l_i t + i] \sim \cdots \sim \varphi([n-1]) = [l_{n-1} t + (n-1)] \sim 0$. Then by Lemma 2.2, we have

$$0 \leq l_1 \leq l_2 \leq \cdots \leq l_{n-1} \leq k - 1. \tag{12}$$

We will show that $l_i = 0 \ \forall i = 1, 2, \ldots, n-1$.

If possible let $\exists i \in \{1, 2, \ldots, n-1\}$ such that $l_i = p (> 0)$, then by Equation 12, we have $l_j \geq p \ \ \forall j > i \ \Rightarrow \ \varphi([n-1]) \geq pt + (n-1) = (k-1+p)t + 1$, as $n = (k-1)t + 2$ then $l_{n-1} \geq k - 1 + p \gneq k - 1$, which is a contradiction, hence we have $l_i = 0 \ \ \forall i = 1, 2, \ldots, n-1$. Therefore we have $\varphi([i]) = [i] \ \ \forall i = 0, 1, \ldots, n-1$, hence $\varphi = \mathsf{id}$.

Case 2. Let $\alpha_0 = -t + 1$, we will show that $\varphi = \tau$. Let us consider the cycle $[0] \sim \varphi([1]) = [l_1 t + 1] \sim \varphi([2]) = [l_2 t] \sim \varphi([3]) = [l'_3 t - 1] = [(l'_3 - 1)t + (t-1)] = [l_3 t + (t-1)] \sim \cdots \sim \varphi([i]) = [l'_i t - (i-2)] = [(l'_i - 1)t + t - (i-2)] = [l_i t + \{t - (i-2)\}] \sim \cdots \sim \varphi([n-1]) = [(l'_{n-1} - 1)t + t - (n-3)] = [l_{n-1} t + \{t - (n-3)\}] \sim 0$, where $l_i = l'_i - 1 \ \ \forall \ i = 3, \ldots, n-1$. Then by Lemma 2.2, we have

$$k - 1 \geq l'_1 \geq l'_2 \geq \cdots \geq l'_{n-1} \geq 0. \tag{13}$$

We will show that $l'_i = k - 1 \ \ \forall i = 1, 2, \ldots, n-1$.

If possible let $\exists i \in \{1, 2, \ldots, n-1\}$ such that $l'_i = p (< k - 1)$, then by Equation 13, we have $l'_j \leq p \ \ \forall j > i \ \Rightarrow \ \varphi([n-1]) \leq (p-1)t + t - (n-3) = \{p - (k-1)\}t + 1$, as $n = (k-1)t + 2$ then $l'_{n-1} \leq p - (k-1) \lneq 0$, which is a contradiction, hence we have $l'_i = k - 1 \ \ \forall i = 1, 2, \ldots, n-1$. Therefore we have $\varphi([i]) = [(k-2)t + \{t - (i-2)\}] = [n - i] \ \ \forall i = 0, 1, \ldots, n-1$, hence $\varphi = \tau$.

The last part of the theorem follows from the discussion in the begining of this section. ∎

Corollary 3.4. *For $t \neq 2$, $GA(t, k)$ is a normal Cayley graph.*

Proof. As $GA(t,k) = Cay(\mathbb{Z}_n, S)$ and $A(t,k) \cong D_n$, the regular subgroup of $A(t,k)$ is of index 2 and hence, normal in $A(t,k)$. ■

Corollary 3.5. *$GA(t,k)$ is edge-transitive if and only if $t = 2$ or $k = 2$. If $t, k > 2$, it has $\lceil \frac{k}{2} \rceil$ many edge-orbits.*

Proof. If $t = 2$, then $GA(t,k)$ is a complete bipartite graph with balanced bipartite set and hence is edge-transitive. If $k = 2$, the $GA(t,k)$ is a cycle and hence, it is edge-transitive. Let $t, k > 2$. Consider the set of k edges which are adjacent to $[0]$, i.e., $\{e_0 = ([0],[1]), e_1 = ([0],[t+1]), e_2 = ([0],[2t+1]), \ldots, e_{k-1} = ([0],[t(k-1)+1])\}$. As $A(t,k) \cong D_n$, only e_i and e_{k-1-i} lie in the same edge-orbit. Thus there are at least $\lceil \frac{k}{2} \rceil$ many edge-orbits. As $GA(t,k)$ is vertex-transitive, there are exactly $\lceil \frac{k}{2} \rceil$ many edge-orbits. Thus, for $t, k \geq 3$, there are at least 2 edge-orbits and hence $GA(t,k)$ is not edge-transitive. ■

At this junction, we recall another graph parameter related to the automorphism group of a graph, namely determining number of a graph [2]. The *determining number of a graph $G = (V,E)$* is the minimum size of a set $S \subseteq V$ such that the pointwise stabilizer of S in the full automorphism group of G is trivial.

Corollary 3.6. *For $t \neq 2$, the determining number of $GA(t,k)$ is 2.*

Proof. Let $S = \{[0],[1]\}$. We first show that S is a determining set for $GA(t,k)$, i.e., $\mathsf{Stab}_{[0]} \cap \mathsf{Stab}_{[1]} = \{\mathsf{id}\}$. Let $\varphi_1^i \tau^j$ be an arbitrary element of $A(t,k)$ which stabilizes both $[0]$ and $[1]$, where $0 \leq i \leq n-1$ and $0 \leq j \leq 1$. If $j = 1$, we have $\varphi_1^i \tau([0]) = [0]$ and $\varphi_1^i \tau([1]) = [1]$, i.e., $[i] = [0]$ and $[i-1] = [1]$. But this implies $[0] = [2]$, a contradiction. Thus $j = 0$ and so we have $\varphi_1^i([0]) = [0]$ and $\varphi_1^i([1]) = [1]$, i.e., $[i] = [0]$ and hence $i = 0$. Thus $\mathsf{Stab}_{[0]} \cap \mathsf{Stab}_{[1]} = \{\mathsf{id}\}$ and determining number is less than or equal to 2. Moreover, the determining number can not be 1, otherwise as $GA(t,k)$ is vertex-transitive, we would have $|\mathsf{Stab}_{[0]}| = 1$, a contradiction. This proves the corollary. ■

4. Conclusion and open issues

In this paper, we introduced a new family of circulants, called Generalized Andrásfai graphs, and studied its various parameters like diameter, girth, domination number etc. The full automorphism group and determining number is also computed.

As circulant graphs are an important class of interconnection networks in parallel and distributed computing, two important questions pertaining to generalized Andrásfai graphs for further research are to determining its metric dimension and spectrum. It is worth mentioning that recently in [6], authors computed the metric dimension of Andrásfai graphs.

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References

[1] B. Andrásfai, *Graphentheoretische Extremalprobleme*, Acta Math. Acad. Sci. Hungar. **15** (1964) 413–438.
https://doi.org/10.1007/BF01897150

[2] D.L. Boutin, *Identifying graph automorphisms using determining sets*, Electron. J. Combin. **13** 2006.
https://doi.org/10.37236/1104

[3] O. Ebsen and M. Schacht, *Homomorphism threshold for odd cycles*, Combinatorica **40** (2020) 39–62.
https://doi.org/10.1007/s00493-019-3920-8

[4] C. Godsil and G.F. Royle, Algebraic Graph Theory, Graduate Texts in Mathematics 207 (Springer-Verlag, 2001).

[5] S. Letzter and R. Snyder, *The homomorphism threshold of* $\{C_3, C_5\}$*-free graphs*, J. Graph Theory **90** (2019) 83–106.
https://doi.org/10.1002/jgt.22369

[6] S.B. Pejman, S. Payrovi and A. Behtoei, *Metric dimension of Andrásfai Graphs*, Opuscula Math. **39** (3) (2019) 415–423.
https://doi.org/10.7494/OpMath.2019.39.3.415

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