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(M, N)-DOUBLE-FRAMED SOFT *bi*-IDEALS OF ABEL GRASSMANN'S GROUPOIDS

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Abstract

The left invertive law makes Abel Grassmann's groupoids (briefly AG-groupoids) a very interesting structure to study. In this paper, we define (M, N)-double-framed soft bi-ideals (briefly (M, N)-DFS bi-ideals) and (M, N)-double-framed soft generalized bi-ideals (briefly (M, N)-DFS generalized bi-ideals) of AG-groupoids and study some of its properties. We obtain some interesting results of these notions in intra-regular AG-groupoids.

Keywords: DFS-set, (M, N)-DFS AG-groupoid, (M, N)-DFS generalized bi-ideal, (M, N)-DFS bi-ideal, DFS int-uni product.

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1. Introduction

The traditional mathematical framework couldn't allow for the uncertainties that appear in economics, engineering, environmental, medical and social sciences to be resolved within its compass. Molodtsov [31], therefore,conceptualized a framework known as soft set that could account for the uncertainties in the sciences mentioned above. The theory of soft set has undergone rapid growth popularity as there could be various applications of this theory. The number of articles published in recent years by reputable publishing platforms stands as witness to the fact that the soft set theory has gained a niche for itself in dealing with complex topics in the fields as varied as mentioned above. For example, soft set theory is applied in the field of optimization by Kovkov in [28], decision making problems have been studied in [29, 36]. Soft set theory has been applied to different algebraic structures. We refer the reader to the papers [1,2,4,8,9,13,15–18,23,30,38].

The idea of generalization of a commutative semigroup, (known as left almost semigroup) was introduced by Kazim and Naseeruddin in 1972 (see [19]). Some other names have also been used in literature for left almost semigroups. Cho *et al.* [5] studied this structure under the name of right modular groupoid. Holgate [10] studied it as left invertive groupoid. Similarly, Stevanovic and Protic [35] called this structure an Abel-Grassmann groupoid (or simply AG-groupoid), which is the primary name under which this structure in known nowadays. There are many important applications of AG-groupoids in the theory of flocks [34]. For more study of AG-groupoids, the reader is invited to read [7, 25–27, 35, 39].

Motivated by the idea of Zhan *et al.* [40, 41] and the idea of double-framed soft sets given by Jun *et al.* [14], we extend our work [21] and introduce the concept of (M, N)-double-framed soft bi-ideals (briefly (M, N)-DFS bi-ideals) and (M, N)-double-framed soft generalized bi-ideals (briefly (M, N)-DFS generalized bi-ideals) of AG-groupoids. We discuss some of its properties in intra-regular AG-groupoids. Some important results on double-framed soft sets on AG-groupoids have been obtained in [3, 12, 24].

2. Preliminaries

A groupoid $(S, \cdot)$ is called an AG-groupoid if it satisfies the left invertive law, that is, $(ab)c = (cb)a$ for all $a, b, c \in S$.

Every AG-groupoid S satisfies the medial law, that is, $(ab)(cd) = (ac)(bd)$ for all $a, b, c, d \in S$.

It is basically a non-associative algebraic structure in between a groupoid and a commutative semigroup. It is important to mention here that if an AG-groupoid contains identity or even right identity, then it becomes a commutative monoid. An AG-groupoid may or may not contain left identity. If there exists left identity in an AG-groupoid, then it is unique. [32].

Every AG-groupoid S with left identity satisfies the paramedial law, that is, $(ab)(cd) = (db)(ca)$ for all $a, b, c, d \in S$.

In an AG-groupoid S with left identity, using the paramedial law, it is easy to prove that

$$(ab)(cd) = (dc)(ba), \text{ for all } a, b, c, d \in S.$$

Moreover,in an AG-groupoid S with left identity, we have

$$a(bc) = b(ac), \text{ for all } a, b, c \in S.$$

Throughout this paper, S will represent an AG-groupoid unless otherwise stated.

For non-empty subsets A and B of S we denote by $AB := \{ab|a \in A \text{ and } b \in B\}$. If $A = \{a\}$, then we write aB instead of $\{a\}B$. A nonempty subset A of an AG-groupoid S is called *sub AG-groupoid* of S if $A^2 \subseteq A$. A nonempty subset A of an AG-groupoid S is called *left* (resp. *right*) *ideal* of S if $SA \subseteq A$ (resp. $AS \subseteq A$). If A is both a left and a right ideal of S, then it is called a *two-sided ideal* or simply an *ideal* of S. A non-subset A of an AG-groupoid S is called *generalized bi-ideal* of S if $(AS)A \subseteq A$. A non-empty subset A of an AG-groupoid S is called a *bi-ideal* if:

(i) A is a sub AG-groupoid of S,

(ii) A is a generalized bi-ideal of S.

A subset A of an AG-groupoid S is called semiprime if for all $a \in S$, whenever $a^2 \in A$ then $a \in A$.

3. Soft set (basic operations)

In [2], Atagun and Sezgin introduced some new operations on soft set theory and defined soft sets in the following way.

Let U be an initial universe set, E a set of parameters, $P(U)$ the power set of U and $A \subseteq E$. Then a soft set f_A over U is a function defined by:

$$f_A : E \longrightarrow P(U) \text{ such that } f_A(x) = \emptyset \text{ if } x \notin A.$$

Here f_A is called an approximate function. A soft set over U can be represented by the set of ordered pairs

$$f_A := \{(x, f_A(x)) : x \in E,\ f_A(x) \in P(U)\}.$$

It is clear that a soft set is a parameterized family of subsets of U. The set of all soft sets over U is denoted by $S(U)$.

Definition. Let $f_A, f_B \in S(U)$. Then f_A is a soft subset of f_B, denoted by $f_A \widetilde{\subseteq} f_B$ if $f_A(x) \subseteq f_B(x)$ for all $x \in E$. Two soft sets f_A, f_B are said to be equal soft sets if $f_A \widetilde{\subseteq} f_B$ and $f_B \widetilde{\subseteq} f_A$ and is denoted by $f_A \widetilde{=} f_B$.

Definition. Let $f_A, f_B \in S(U)$. Then the union of f_A and f_B, denoted by $f_A \widetilde{\cup} f_B$, is defined by $f_A \widetilde{\cup} f_B = f_{A\cup B}$, where $f_{A\cup B}(x) = f_A(x) \cup f_B(x)$, for all $x \in E$.

Definition. Let $f_A, f_B \in S(U)$. Then the intersection of f_A and f_B, denoted by $f_A \widetilde{\cap} f_B$, is defined by $f_A \widetilde{\cap} f_B = f_{A\cap B}$, where $f_{A\cap B}(x) = f_A(x) \cap f_B(x)$, for all $x \in E$.

Definition [37]**.** Let $f_A, f_B \in S(U)$. Then the soft product of f_A and f_B, denoted by $f_A \widetilde{\circ} f_B$, is defined by

$$\left(f_A \widetilde{\circ} f_B\right)(x) := \begin{cases} \bigcup\limits_{x=yz} \left\{f_A(y) \cap f_B(z)\right\}, & \text{if } \exists\; y, z \in S \text{ such that } x = yz \\ \emptyset & \text{otherwise.} \end{cases}$$

Throughout this paper, let $E = S$, where S is an AG-groupoid and $A, B, C, \ldots$ are sub AG-groupoids, unless otherwise stated.

Definition [14]**.** A *double-framed soft pair* $\langle(\alpha_A, \beta_A); A\rangle$ is called a *double-framed soft set* of A over U (briefly, *DFS*-set), where α_A and β_A are mappings from A to $P(U)$. The set of all DFS-sets of S over U will be denoted by $DFS(U)$.

Define an order relation $\sqsubseteq_{[M,N]}$ on $DFS(U)$ as follows. For any $\langle(f_S, g_S); S\rangle$ and $\langle(p_S, q_S); S\rangle$, $\emptyset \subseteq M \subset N \subseteq U$, we define $\langle(f_S, g_S); S\rangle \sqsubseteq_{[M,N]} \langle(p_S, q_S); S\rangle$ $\Leftrightarrow (f_S(x) \cap N) \cup M \subseteq (p_S(x) \cap N) \cup M$ and $(g_S(x) \cup M) \cap N \supseteq (q_S(x) \cup M) \cap N$ for all $x \in S$.

If in case $\langle(f_S, g_S); S\rangle \sqsubseteq_{[M,N]} \langle(p_S, q_S); S\rangle$ and $\langle(p_S, q_S); B\rangle \sqsubseteq_{[M,N]} \langle(f_S, g_S); S\rangle$ then we say $\langle(f_S, g_S); A\rangle =_{[M,N]} \langle(p_S, q_S); B\rangle$.

Let $\langle(\alpha_S, \beta_S); S\rangle$ and $\langle(f_S, g_S); S\rangle$ be two double-framed soft sets of S over U. Then the *int-uni soft product* [20] is denoted by $\langle(\alpha_S, \beta_S); S\rangle \diamond \langle(f_S, g_S); S\rangle$ and is defined as a double framed soft set $\langle(\alpha_S \widetilde{\circ} f_S, \beta_S \widetilde{\circ} g_S); S\rangle$ defined to be a double-framed soft set over U, in which $\alpha_S \widetilde{\circ} f_S$, and $\beta_S \widetilde{\circ} g_S$ are soft mappings from S to $P(U)$, given as follows:

$$\alpha_S \widetilde{\circ} f_S : S \longrightarrow P(U), x \longmapsto \begin{cases} \bigcup\limits_{x=yz} \left\{\alpha_S(y) \cap f_S(z)\right\} & \text{if } \exists y, z \in S \text{ such that } x = yz, \\ \emptyset & \text{otherwise,} \end{cases}$$

$$\beta_S \widetilde{\circ} g_S : S \longrightarrow P(U), x \longmapsto \begin{cases} \bigcap\limits_{x=yz} \left\{\beta_S(y) \cup g_S(z)\right\} & \text{if } \exists\; y, z \in S \text{ such that } x = yz, \\ U & \text{otherwise.} \end{cases}$$

One can easily see that the operation "$\diamond$"is well-defined.

For any two DFS sets $\langle(\alpha_S,\beta_S);S\rangle$ and $\langle(f_S,g_S);S\rangle$ of S over U, the *DFS intersection* [14] of $\langle(\alpha_S,\beta_S);S\rangle$ and $\langle(f_S,g_S);S\rangle$, is defined to be a DFS set $\left\langle\left(\alpha_S\tilde{\cap}f_S,\beta_S\tilde{\cup}g_S\right);S\right\rangle$ where$\alpha_S\tilde{\cap}f_S$, and $\beta_S\tilde{\cup}g_S$ are mappings given by $\alpha_S\tilde{\cap}f_S : S\to P(U), x\to\alpha_S(x)\cap f_S(x)$ and $\beta_S\tilde{\cup}g_S : S\to P(U), x\to\beta_S(x)\cup g_S(x)$ for all $x\in S$.

It is denoted by $\langle(\alpha_S,\beta_S);S\rangle\sqcap\langle(f_S,g_S);S\rangle=\left\langle\left(\alpha_S\tilde{\cap}f_S,\beta_S\tilde{\cup}g_S\right);S\right\rangle$.

For any two DFS sets $\langle(\alpha_S,\beta_S);S\rangle$ and $\langle(f_S,g_S);S\rangle$ of S over U, the *DFS union* of $\langle(\alpha_S,\beta_S);S\rangle$ and $\langle(f_S,g_S);S\rangle$, is defined to be a DFS set $\langle(\alpha_S\tilde{\cup}f_S, \beta_S\tilde{\cap}g_S);S\rangle$ where$\alpha_S\tilde{\cup}f_S$, and $\beta_S\tilde{\cap}g_S$ are mappings given by $\alpha_S\tilde{\cup}f_S : S\to P(U)$, $x\to\alpha_S(x)\cup f_S(x)$ and $\beta_S\tilde{\cap}g_S : S\to P(U), x\to\beta_S(x)\cap g_S(x)$ for all $x\in S$. It is denoted by $\langle(\alpha_S,\beta_S);S\rangle\sqcup\langle(f_S,g_S);S\rangle=\left\langle\left(\alpha_S\tilde{\cup}f_S,\beta_S\tilde{\cap}g_S\right);S\right\rangle$.

For a non-empty subset A of S, the DFS set $\mathbf{X}_A$=$(\chi_A,\chi_A^c;A)$ is called the double framed characteristic soft set where

$$\chi_A : S\to P(U),\ x\to\begin{cases} U \text{ if } x\in A\\ \emptyset \text{ if } x\notin A\end{cases} \text{ and } \chi_A^c : S\to P(U),\ x\to\begin{cases}\emptyset \text{ if } x\in A\\ U \text{ if } x\notin A.\end{cases}$$

The following lemmas are due to Izhar *et al.* [11].

Lemma 1. *The set $(DFS(U),\Diamond)$ forms an AG-groupoid.*

Lemma 2. *If S is an AG-Groupoid, then the medial law holds in $DFS(U)$. That is for $\langle(\alpha_S,\beta_S);S\rangle,\langle(f_S,g_S);S\rangle$, $\langle(h_S,k_S);S\rangle$ and $\langle(p_S,q_S);S\rangle\in DFS(U)$, we have*

$$\begin{aligned}&(\langle(\alpha_S,\beta_S);S\rangle\Diamond\langle(f_S,g_S);S\rangle)\Diamond(\langle(h_S,k_S);S\rangle\Diamond\langle(p_S,q_S);S\rangle)\\&=_{[M,N]}(\langle(\alpha_S,\beta_S);S\rangle\Diamond\langle(h_S,k_S);S\rangle)\Diamond(\langle(f_S,g_S);S\rangle\Diamond\langle(p_S,q_S);S\rangle).\end{aligned}$$

Lemma 3. *If S is an AG-groupoid with left identity, then the paramedial law holds in $DFS(U)$. That is for all $\langle(\alpha_S,\beta_S);S\rangle,\langle(f_S,g_S);S\rangle$, $\langle(h_S,k_S);S\rangle$ and $\langle(p_S,q_S);S\rangle\in DFS(U)$,*

$$\begin{aligned}&(\langle(\alpha_S,\beta_S);S\rangle\Diamond\langle(f_S,g_S);S\rangle)\Diamond(\langle(h_S,k_S);S\rangle\Diamond\langle(p_S,q_S);S\rangle)\\&=_{[M,N]}(\langle(p_S,q_S);S\rangle\Diamond\langle(f_S,g_S);S\rangle)\Diamond(\langle(h_S,k_S);S\rangle\Diamond\langle(\alpha_S,\beta_S);S\rangle).\end{aligned}$$

Lemma 4. *If S is an AG-groupoid with left identity, then for all $\langle(f_S,g_S);S\rangle$, $\langle(h_S,k_S);S\rangle$ and $\langle(p_S,q_S);S\rangle\in DFS(U)$, we have*

$$\begin{aligned}&\langle(f_S,g_S);S\rangle\Diamond(\langle(h_S,k_S);S\rangle\Diamond\langle(p_S,q_S);S\rangle)\\&=_{[M,N]}\langle(h_S,k_S);S\rangle\Diamond(\langle(f_S,g_S);S\rangle\Diamond\langle(p_S,q_S);S\rangle).\end{aligned}$$

Lemma 5. *If S is an AG-Groupoid and $\langle(\alpha_S,\beta_S);S\rangle,\langle(f_S,g_S);S\rangle,\langle(h_S,k_S);S\rangle$ and $\langle(p_S,q_S);S\rangle\in DFS(U)$, then the following holds.*

(i) $\langle(\alpha_S,\beta_S);S\rangle \lozenge(\langle(f_S,g_S);S\rangle \sqcap \langle(h_S,k_S);S\rangle) =_{[M,N]} \langle(\alpha_S,\beta_S);S\rangle \lozenge\langle(f_S,g_S);S\rangle \sqcap \langle(\alpha_S,\beta_S);S\rangle \lozenge \langle(h_S,k_S);S\rangle)$.

(ii) $\langle(\alpha_S,\beta_S);S\rangle \lozenge(\langle(f_S,g_S);S\rangle \sqcup \langle(h_S,k_S);S\rangle) =_{[M,N]} \langle(\alpha_S,\beta_S);S\rangle \lozenge\langle(f_S,g_S);S\rangle \sqcup \langle(\alpha_S,\beta_S);S\rangle \lozenge \langle(h_S,k_S);S\rangle)$.

(iii) *If* $\langle(\alpha_S,\beta_S);S\rangle \sqsubseteq_{[M,N]} \langle(f_S,g_S);S\rangle$, *then* $\langle(\alpha_S,\beta_S);S\rangle \lozenge \langle(h_S,k_S);S\rangle \sqsubseteq_{[M,N]} \langle(f_S,g_S);S\rangle \lozenge \langle(h_S,k_S);S\rangle$.

(iv) *If* $\langle(\alpha_S,\beta_S);S\rangle \sqsubseteq_{[M,N]} \langle(f_S,g_S);S\rangle$ *and* $\langle(h_S,k_S);S\rangle \sqsubseteq_{[M,N]} \langle(p_S,q_S);S\rangle$, *then* $\langle(\alpha_S,\beta_S);S\rangle \lozenge \langle(h_S,k_S);S\rangle \sqsubseteq_{[M,N]} \langle(f_S,g_S);S\rangle \lozenge \langle(p_S,q_S);S\rangle$.

Lemma 6. *Let A and B be two non empty subsets of an AG-groupoid S then the following properties hold:*

(i) $A \subseteq B$ *if and only if* $\mathbf{X}_A \sqsubseteq_{[M,N]} \mathbf{X}_B$.

(ii) $\mathbf{X}_A \sqcap \mathbf{X}_B =_{[M,N]} \mathbf{X}_{A\cap B}$.

(iii) $\mathbf{X}_A \diamond \mathbf{X}_B =_{[M,N]} \mathbf{X}_{AB}$.

Lemma 7. *If S is an AG-groupoid, then* $\mathbf{X}_S \lozenge \mathbf{X}_S \sqsubseteq_{[M,N]} \mathbf{X}_S$.

Lemma 8. *If S is an AG-groupoid with left identity e, then* $\mathbf{X}_S \lozenge \mathbf{X}_S =_{[M,N]} \mathbf{X}_S$.

The following definitions have been taken from [21].

Definition. Let S be an AG-groupoid and $\langle(f_A,g_A);A\rangle$ be a DFS-set of A over U. Then $\langle(f_A,g_A);A\rangle$ is called *double-framed soft AG-groupoid* (briefly, DFS AG-groupoid) of A over U if

$$f_A(xy) \supseteq f_A(x) \cap f_A(y) \text{ and } g_A(xy) \subseteq g_A(x) \cup g_A(y) \text{ for all } x,y \in A.$$

Definition. A double-framed soft set $\langle(f_A,g_A);A\rangle$ of A over U is called a double-framed soft generalized bi-ideal (briefly, DFS generalized bi-ideal) of A over U if it satisfies: $f_A((xa)y) \supseteq f_A(x) \cap f_A(y)$ and $g_A((xa)y) \subseteq g_A(x) \cup g_A(y)$ for all $a,x,y \in A$.

Definition. A double-framed soft set $\langle(f_A,g_A);A\rangle$ of A over U is called a double-framed soft bi-ideal (briefly, DFS bi-ideal) of A over U if it satisfies:

(i) $f_A(xy) \supseteq f_A(x) \cap f_A(y)$ and $g_A(xy) \subseteq g_A(x) \cup g_A(y)$ for all $x,y \in A$.

(ii) $f_A((xa)y) \supseteq f_A(x) \cap f_A(y)$ and $g_A((xa)y) \subseteq g_A(x) \cup g_A(y)$ for all $a,x,y \in A$.

4. $(M,\ N)$-DFS Bi-ideals of AG-groupoids

In this section, we give concept of $(M,\ N)$-DFS bi-ideal of AG-groupoids and discuss their properties. From now on $\emptyset \subseteq M \subset N \subseteq U$.

Definition [11]**.** Let S be an AG-groupoid and $\langle (f_A, g_A); A\rangle$ be a DFS-set of A over U. Then $\langle (f_A, g_A); A\rangle$ is called $(M,\ N)$-*double-framed soft AG-groupoid* (briefly, $(M,\ N)$-DFS AG-groupoid) of A over U if

$$f_A(xy) \cup M \supseteq f_A(x) \cap f_A(y) \cap N \text{ and } g_A(xy) \cap N \subseteq g_A(x) \cup g_A(y) \cup M$$

for all $x, y \in A$.

Definition [11]**.** Let S be an AG-groupoid and $\langle (f_A, g_A); A\rangle$ be a DFS-set of A over U. Then $\langle (f_A, g_A); A\rangle$ is called $(M,\ N)$-*double-framed soft left (resp. right) ideal* (briefly, $(M,\ N)$-DFS left (resp. right)) ideal of A over U if

$$f_A(ab) \cup M \supseteq f_A(b) \cap N \text{ (resp. } f_S(ab) \cup M \supseteq f_S(a) \cap N$$

and

$$g_A(ab) \cap N \subseteq g_A(b) \cup M \text{ (resp. } g_A(ab) \cap N \subseteq g_A(a) \cup M)$$

for all $a, b \in A$.

A DFS-set $\langle (f_A, g_A); A\rangle$ of A over U is called $(M,\ N)$-double-framed soft two sided ideal (briefly, $(M,\ N)$-DFS two-sided ideal) of S over U if it is both $(M,\ N)$-DFS left and $(M,\ N)$-DFS right ideal of A over U.

Definition. A DFS-set $\langle (f_A, g_A); A\rangle$ is called a $(M,\ N)$-*double-framed soft semiprime* (briefly, $(M,\ N)$-DFS semiprime) if $\alpha(a) \cup M \supseteq \alpha(a^2) \cap N$ and $\beta(a) \cap N \subseteq \beta(a^2) \cup M$.

Definition. Let S be an AG-groupoid and $\langle (f_A, g_A); A\rangle$ be a DFS-set of A over U. Then $\langle (f_A, g_A); A\rangle$ is called $(M,\ N)$-*double-framed soft generalized bi-ideal* of A over U if

$$f_A((ax)b) \cup M \supseteq f_A(a) \cap f_A(b) \cap N \text{ and } g_A((ax)b) \cap N \subseteq g_A(a) \cup g_S(b) \cup M$$

for all $a, b, x \in A$.

Definition. Let S be an AG-groupoid and $\langle (f_A, g_A); A\rangle$ be a DFS-set of A over U. Then $\langle (f_A, g_A); A\rangle$ is called $(M,\ N)$-*double-framed soft bi-ideal* of A over U if;

(i) $\langle (f_A, g_A); A\rangle$ is $(M,\ N)$-DFS AG-groupoid,

(ii) $\langle (f_A, g_A); A\rangle$ is $(M,\ N)$-DFS generalized bi-ideal.

Example 1. Let us consider $S = \{a, b, c\}$ with the following multiplication table.

$\cdot$	a	b	c
a	b	c	b
b	b	b	b
c	b	b	b

Then $(S, \cdot)$ is an AG-groupoid. Clearly S is non-commutative and non-associative since $ab \neq ba$ and $(ab)b \neq a(bb)$.

Let $M = 16\mathbb{Z}$ and $N = 8\mathbb{Z}$ and consider a DFS set $\langle (f_S, g_S)\,; S\rangle$ of S over $U = \mathbb{Z}$ defined by $f_S(a) = 8\mathbb{Z}$, $f_S(b) = 2\mathbb{Z}$, $f_S(c) = 4\mathbb{Z}$ and $g_S(a) = 4\mathbb{Z}$, $g_S(b) = 16\mathbb{Z}$, $g_S(c) = 8\mathbb{Z}$.

By routine checking, $\langle (f_S, g_S)\,; S\rangle$ is $(M,\ N)$-DFS bi-ideal of S over U.

It is important to note that every (M, N)-DFS bi-ideal is a (M, N)-DFS generalized bi-ideal, but the converse is not true in general.

Let us consider $M = 16\mathbb{Z}$ and $N = 8\mathbb{Z}$ and DFS set $\langle (f_S, g_S)\,; S\rangle$ of S over $U = \mathbb{Z}$. Defined by $f_S(a) = 8\mathbb{Z}$, $f_S(b) = 4\mathbb{Z}$, $f_S(c) = 16\mathbb{Z}$ and $g_S(a) = 16\mathbb{Z}$, $g_S(b) = 32\mathbb{Z}$, $g_S(c) = 8\mathbb{Z}$.

Clearly $\langle (f_S, g_S)\,; S\rangle$ is DFS generalized bi-ideal but it is not DFS bi-ideal because $f(c) \cup M = f(ab) \cup M \not\supseteq f(a) \cap f(b) \cap N$ and/or $g(c) \cap N = g(ab) \cap N \not\subseteq g(a) \cup g(b) \cup M$.

It is interesting to note that if an AG-groupoid is intra-regular and contains left identity, then every DFS generalized bi-ideal becomes DFS bi-ideal. We will prove this in Theorem 19

Proposition 1 [11]**.** *A DFS set $\langle (f_S, g_S); S\rangle$ of an AG-groupoid S over U is (M, N)-DFS left ideal of S if and only if $\mathbf{X}_S \diamond \langle (f_S, g_S); S\rangle \sqsubseteq_{[M,N]} \langle (f_S, g_S); S\rangle$.*

Proposition 2 [11]**.** *A DFS set $\langle (f_S, g_S); S\rangle$ of an AG-groupoid S over U is (M, N)-DFS right ideal of S if and only if $\langle (f_S, g_S); S\rangle \Diamond \mathbf{X}_S \sqsubseteq_{[M,N]} \langle (f_S, g_S); S\rangle$.*

Proposition 3 [11]**.** *A DFS-set $\langle (f_S, g_S)\,; S\rangle$ of an AG groupoid S over U is an $(M,\ N)$-DFS AG groupoid if and only if*

$$\langle (f_S, g_S)\,; S\rangle \Diamond \langle (f_S, g_S)\,; S\rangle \sqsubseteq_{[M,N]} \langle (f_S, g_S)\,; S\rangle\,.$$

Theorem 9. *Let $\langle (f_S, g_S)\,; S\rangle$ be a DFS-set of S over U. Then $\langle (f_S, g_S)\,; S\rangle$ is a $(M,\ N)$-DFS generalized bi-ideal of S over U if and only if*

$$(\langle (f_S, g_S)\,; S\rangle \diamond \mathbf{X}_S) \diamond \langle (f_S, g_S)\,; S\rangle \sqsubseteq_{[M,N]} \langle (f_S, g_S)\,; S\rangle\,. \tag{1}$$

Proof. Assume that $\langle (f_S, g_S)\,; S\rangle$ is a DFS generalized bi-ideal of S over U. Let $x \in S$. If there are no y, z in S such that $x = yz$, then $((((f_S \widetilde{\circ} \chi_S)\, \widetilde{\circ} f_S)\,(x)) \cap N) \cup M = M \subseteq (f_S\,(x) \cap N) \cup M$ and $((((g_S \widetilde{\circ} \chi_S^c)\, \widetilde{\circ} g_S)\,(x)) \cup M) \cap N = N \supseteq (g_S(x) \cup M) \cap N$. Let x can be expressed as a product of $y, z \in S$ such that $x = yz$, then

$$
\begin{aligned}
&((((f_S\widetilde{\circ}\chi_S)\,\widetilde{\circ}f_S)\,(x))\cap N)\cup M\\
&=\left(\left(\bigcup_{x=yz}\{(f_S\widetilde{\circ}\chi_S)\,(y)\cap f_S\,(z)\}\right)\cap N\right)\cup M\\
&=\left(\left(\bigcup_{x=yz}\left\{\bigcup_{y=pq}\{f_S\,(p)\cap \chi_S\,(q)\}\cap f_S\,(z)\right\}\right)\cap N\right)\cup M\\
&=\left(\left(\bigcup_{x=(pq)z}\{f_S\,(p)\cap f_S\,(z)\}\right)\cap N\right)\cup M\\
&=\left(\left(\bigcup_{x=(pq)z}\{f_S\,(p)\cap f_S\,(z)\cap N\}\right)\cap N\right)\cup M\\
&\subseteq\left(\left(\bigcup_{x=(pq)z}\{f_S\,((pq)\,z)\cup M\}\right)\cap N\right)\cup M\\
&=\left(\bigcup_{x=(pq)z}\{f_S\,((pq)\,z)\cup M\}\right)\cap (N\cup M)\\
&=(f_S(x)\cup M)\cap(N\cup M)=(f_S(x)\cup M)\cap N=(f_S(x)\cap N)\cup M
\end{aligned}
$$

and

$$
\begin{aligned}
&((((g_S\widetilde{\circ}\chi_S^c)\,\widetilde{\circ}g_S)\,(x))\cup M)\cap N\\
&=\left(\left(\bigcap_{x=yz}\{(g_S\widetilde{\circ}\chi_S^c)\,(y)\cup g_S\,(z)\}\right)\cup M\right)\cap N\\
&=\left(\left(\bigcap_{x=yz}\left\{\bigcap_{y=pq}\{g_S\,(p)\cup \chi_S^c\,(q)\}\cup g_S\,(z)\right\}\right)\cup M\right)\cap N\\
&=\left(\left(\bigcap_{x=(pq)z}\{g_S\,(p)\cup g_S\,(z)\}\right)\cup M\right)\cap N\\
&=\left(\left(\bigcap_{x=(pq)z}\{g_S\,(p)\cup g_S\,(z)\cup M\}\right)\cup M\right)\cap N\\
&\supseteq\left(\left(\bigcap_{x=(pq)z}\{g_S\,(pq)\,z)\cap N\}\right)\cup M\right)\cap N\\
&=\left(\bigcap_{x=(pq)z}\{g_S\,(pq)\,z)\cap N\}\right)\cup (M\cap N)\\
&=(g_S(x)\cap N)\cup M=(g_S(x)\cup M)\cap N.
\end{aligned}
$$

Conversely, suppose that Equation 1 holds. Let $x, y, z \in S$. Then by hypothesis,

$$\begin{aligned} f_S((xy)z) \cup M &\supseteq (f_S((xy)z) \cup M) \cap N = (f_S((xy)z) \cap N) \cup M \\ &\supseteq ((((f_S \widetilde{\circ} \chi_S) \widetilde{\circ} f_S)((xy)z)) \cap N) \cup M \\ &\supseteq (((f_S \widetilde{\circ} \chi_S)(xy) \cap f_S(z)) \cap N) \cup M \\ &\supseteq ((f_S(x) \cap \chi_S(y) \cap f_S(z)) \cap N) \cup M \\ &= (f_S(x) \cap f_S(z) \cap N) \cup M \supseteq f_S(x) \cap f_S(z) \cap N \end{aligned}$$

and

$$\begin{aligned} g_S((xy)z) \cap N &\subseteq (g_S((xy)z) \cap N) \cup M = (g_S((xy)z) \cup M) \cap N \\ &\subseteq ((((g_S \widetilde{\circ} \chi_S^c) \widetilde{\circ} g_S)((xy)z)) \cup M) \cap N \\ &= (((g_S \widetilde{\circ} \chi_S^c)(xy) \cup g_S(z)) \cup M) \cap N \\ &= ((g_S(x) \cup \chi_S^c(y) \cup g_S(z)) \cup M) \cap N \\ &= (g_S(x) \cup g_S(z) \cup M) \cap N \subseteq g_S(x) \cup g_S(z) \cup M. \end{aligned}$$

Hence $\langle (f_S, g_S); S \rangle$ is a DFS generalized bi-ideal of S over U. ■

By Proposition 3 and Theorem 9, we have the following theorem.

Theorem 10. *Let $\langle (f_S, g_S); S \rangle$ be a DFS-set an AG-groupoid S over U. Then $\langle (f_S, g_S); S \rangle$ is a (M, N)-DFS bi-ideal of S over U if and only if*

$$(\langle (f_S, g_S); S \rangle \diamond \mathbf{X}_S) \diamond \langle (f_S, g_S); S \rangle \sqsubseteq_{[M,N]} \langle (f_S, g_S); S \rangle$$

and

$$\langle (f_S, g_S); S \rangle \Diamond \langle (f_S, g_S); S \rangle \sqsubseteq_{[M,N]} \langle (f_S, g_S); S \rangle .$$

Theorem 11. *Let A be a nonempty subset of an AG-groupoid S. Then A is a bi-ideal of S if and only if the DFS-set $\mathbf{X}_A = \langle (\chi_A, \chi_A^c); A \rangle$ is a (M, N)-DFS bi-ideal of S over U.*

Proof. Suppose that A is a bi-ideal of S. Then $AA \subseteq A$ and $(AS)A \subseteq A$. Thus by Lemma 6, we have

$$\mathbf{X}_{AA} \sqsubseteq_{[M,N]} \mathbf{X}_A \Longrightarrow \mathbf{X}_A \Diamond \mathbf{X}_A \sqsubseteq_{[M,N]} \mathbf{X}_A$$

and

$$\mathbf{X}_{(AS)A} \sqsubseteq_{[M,N]} \mathbf{X}_A \Longrightarrow (\mathbf{X}_{AS}) \Diamond \mathbf{X}_A \sqsubseteq_{[M,N]} \mathbf{X}_A \Longrightarrow (\mathbf{X}_A \Diamond \mathbf{X}_S) \Diamond \mathbf{X}_A \sqsubseteq_{[M,N]} \mathbf{X}_A.$$

Hence by Theorem 10, $\mathbf{X}_A = \langle (\chi_A, \chi_A^c); A \rangle$ is (M, N)-DFS bi-ideal of S over U.

Conversely, let the DFS-set $\mathbf{X}_A = \langle(\chi_A, \chi_A^c)\,;A\rangle$ is a $(M,\ N)$-DFS bi-ideal of S over U. then by Lemma 6, we have

$$\mathbf{X}_A \Diamond \mathbf{X}_A \sqsubseteq_{[M,N]} \mathbf{X}_A \Longrightarrow \mathbf{X}_{AA} \sqsubseteq_{[M,N]} \mathbf{X}_A \Longrightarrow AA \subseteq A$$

and

$$\begin{aligned}&(\mathbf{X}_A \Diamond \mathbf{X}_S) \Diamond \mathbf{X}_A \sqsubseteq_{[M,N]} \mathbf{X}_A \Longrightarrow (\mathbf{X}_{AS})\Diamond \mathbf{X}_A \sqsubseteq_{[M,N]} \mathbf{X}_A\\ &\Longrightarrow \mathbf{X}_{(AS)A} \sqsubseteq_{[M,N]} \mathbf{X}_A \Longrightarrow (AS)A \subseteq A.\end{aligned}$$

Hence A is bi-ideal of S. ■

Corollary 1. *Let A be a nonempty subset of an AG-groupoid S. Then A is a generalized bi-ideal of S if and only if the DFS-set $\mathbf{X}_A$ is a $(M,\ N)$-DFS generalized bi-ideal of S over U.*

Lemma 12. *Every $(M,\ N)$-DFS right ideal of an AG-groupoid is $(M,\ N)$-DFS bi-ideal.*

Proof. Assume that S is an AG-groupoid and $\langle(f_S, g_S)\,;S\rangle$ is a $(M,\ N)$-DFS right ideal of S. Now,

$$\begin{aligned}(\langle(f_S, g_S)\,;S\rangle \Diamond \mathbf{X}_S) \Diamond \langle(f_S, g_S)\,;S\rangle &\sqsubseteq_{[M,N]} \langle(f_S, g_S)\,;S\rangle \Diamond \langle(f_S, g_S)\,;S\rangle\\ &\sqsubseteq \langle(f_S, g_S)\,;S\rangle \Diamond \mathbf{X}_S \sqsubseteq_{[M,N]} \langle(f_S, g_S)\,;S\rangle\end{aligned}$$

and

$$\langle(f_S, g_S)\,;S\rangle \Diamond \langle(f_S, g_S)\,;S\rangle \sqsubseteq_{[M,N]} \langle(f_S, g_S)\,;S\rangle \Diamond \mathbf{X}_S \sqsubseteq_{[M,N]} \langle(f_S, g_S)\,;S\rangle.$$

Thus by Theorem 10, $\langle(f_S, g_S)\,;S\rangle$ is $(M,\ N)$-DFS bi-ideal. ■

Corollary 2. *Every $(M,\ N)$-DFS ideal of an AG-groupoid is $(M,\ N)$-DFS bi-ideal.*

Theorem 13. *Let $\langle(f_S, g_S)\,;S\rangle$ and $\langle(p_S, q_S)\,;S\rangle$ be $(M,\ N)$-DFS bi-ideals of an AG-groupoids S with left identity. Then $\langle(f_S, g_S)\,;S\rangle \Diamond \langle(p_S, q_S)\,;S\rangle$ and $\langle(p_S, q_S)\,;S\rangle \Diamond \langle(f_S, g_S)\,;S\rangle$ are $(M,\ N)$-DFS bi-ideal of S over U.*

Proof. We prove that $\langle(f_S, g_S)\,;S\rangle \Diamond \langle(p_S, q_S)\,;S\rangle$ is $(M,\ N)$-DFS bi-ideal of S over U. We first show that $\langle(f_S, g_S)\,;S\rangle \Diamond \langle(p_S, q_S)\,;S\rangle$ is $(M,\ N)$-DFS AG-groupoid.

$$\begin{aligned}&(\langle(f_S, g_S)\,;S\rangle \Diamond \langle(p_S, q_S)\,;S\rangle) \Diamond (\langle(f_S, g_S)\,;S\rangle \Diamond \langle(p_S, q_S)\,;S\rangle)\\ &=_{[M,N]} (\langle(f_S, g_S)\,;S\rangle \Diamond \langle(f_S, g_S)\,;S\rangle) \Diamond (\langle(p_S, q_S)\,;S\rangle \Diamond \langle(p_S, q_S)\,;S\rangle\\ &\sqsubseteq_{[M,N]} \langle(f_S, g_S)\,;S\rangle \Diamond \langle(p_S, q_S)\,;S\rangle\end{aligned}$$

Moreover,

$$(((\langle (f_S, g_S)\,;S\rangle \lozenge \langle (p_S, q_S)\,;S\rangle) \lozenge \mathbf{X}_S) \lozenge \langle (f_S, g_S)\,;S\rangle \lozenge \langle (p_S, q_S)\,;S\rangle$$

$$=_{[M,N]} (((\langle (f_S, g_S)\,;S\rangle \lozenge \langle (p_S, q_S)\,;S\rangle) \lozenge (\mathbf{X}_S \lozenge \mathbf{X}_S)) \lozenge \langle (f_S, g_S)\,;S\rangle \lozenge \langle (p_S, q_S)\,;S\rangle$$

$$=_{[M,N]} (((\langle (f_S, g_S)\,;S\rangle \lozenge \mathbf{X}_S) \lozenge (\langle (p_S, q_S)\,;S\rangle \lozenge \mathbf{X}_S)) \lozenge \langle (f_S, g_S)\,;S\rangle \lozenge \langle (p_S, q_S)\,;S\rangle$$

$$\sqsubseteq_{[M,N]} ((\langle (f_S, g_S)\,;S\rangle \lozenge \langle (p_S, q_S)\,;S\rangle) \lozenge (\langle (f_S, g_S)\,;S\rangle \lozenge \langle (p_S, q_S)\,;S\rangle)\,.$$

It follows that $\langle (f_S, g_S)\,;S\rangle \lozenge \langle (p_S, q_S)\,;S\rangle$ is $(M,\ N)$-DFS bi-ideal of S over U. In a similar way we can show that $\langle (p_S, q_S)\,;S\rangle \lozenge \langle (f_S, g_S)\,;S\rangle$ is a DFS bi-ideal of S over U. ■

Lemma 14 [11]**.** *Let A be a nonempty subset of an AG-groupoid S. Then*

(i) *A is a left (resp. right) of S if and only if the DFS-set $\mathbf{X}_A$ is a $(M,\ N)$-DFS left (resp. right ideal) of S over U.*

(ii) *A is a semiprime if and only if the DFS-set $\mathbf{X}_A$ is a (M, N)-DFS semiprime.*

5. Intra-regular AG-groupoids

In this section, we discuss properties of $(M,\ N)$-DFS ideals and $(M,\ N)$-DFS bi-ideals in intra-regular AG-groupoids.

An AG-groupoid S is called intra-regular if for all $a \in S$, there exist $x, y \in S$ such that $a = (xa^2)y$.

Example 2 [7]**.** Let us consider the set $S = \{1, 2, 3, 4, 5, 6\}$ with the following multiplication table.

$\cdot$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	1	1	1	1
3	1	1	5	6	3	4
4	1	1	4	5	6	3
5	1	1	3	4	5	6
6	1	1	6	3	4	5

It is not difficult to verify that $(S, \cdot)$ is an AG-groupoid and it is intra-regular, since $1 = (1 \cdot 1^2) \cdot 1$, $2 = (2 \cdot 2^2) \cdot 2$, $3 = (3 \cdot 3^2) \cdot 5$, $4 = (6 \cdot 4^2) \cdot 3$, $5 = (5 \cdot 5^2) \cdot 5$, $6 = (4 \cdot 6^2) \cdot 3$.

Lemma 15 [33]**.** *If an AG-groupoid S contains left identity e, then $S = S^2$ and $S = eS = Se$.*

Lemma 16 [33]**.** *If S is an AG-groupoid with left identity e, then $(xy)^2 = x^2y^2 = y^2x^2$ for all $x, y \in S$.*

Lemma 17. *If S is an intra-regular AG-groupoid with left identity e, then every $a \in S$ can be expressed as*

(i) $a = (a^2t)a^2$ *for some* $t \in S$.

(ii) $a = a(ua^2)$ *for some* $u \in S$.

(iii) $a = a((a^2q)a^2)$ *for some* $q \in S$.

(iv) $a = (ta)a$ *for some* $t \in S$.

(v) $a = ((a(xa))p)a$ *for some* $p \in S$.

Proof. (i) Let $a \in S$. Then since S is intra-regular, there exists $x, y \in S$ such that $a = (xa^2)y$. Using Equation 2, left invertive law, paramedial law and medial law, we can further write, $a = (xa^2)y = (x(aa))y = (a(xa))y = (y(x((xa^2)y)))a = (x(y((xa^2)y)))a = (x((xa^2)y^2))a = ((xa^2)(xy^2))a = (x^2(a^2y^2))a = (a^2(x^2y^2))a = (a(x^2y^2))a^2 = (((xa^2)y)(x^2y^2))a^2 = ((y^2y)(x^2(xa^2)))a^2 = ((y^2x^2)(y(xa^2)))a^2 = ((y^2x^2)((y_1y_2)(xa^2)))a^2 = ((y^2x^2)((a^2y_2)(xy_1)))a^2 = ((y^2x^2)((a^2x)(y_2y_1)))a^2 = ((y^2x^2)(((y_2y_1)x)(aa)))a^2 = ((y^2x^2)(a^2(x(y_2y_1))))a^2 = (a^2((y^2x^2)(x(y_2y_1)))a^2 = (a^2t)a^2$ where $t = ((y^2x^2)(x(y_2y_1))$. Hence $a = (a^2t)a^2$ for some $t \in S$ which is required result.

(ii) Let $a \in S$. Then since S is intra-regular so there exist $x, y \in S$ such that $a = (xa^2)y$. Using Equation 2, left invertive law, paramedial law, Equation 2 and medial law, we can further write $a = (x(aa))y = (a(xa))y = (y(xa))a = (y(xa))((xa^2)y) = (y(xa^2))((xa)y) = ((y_1y_2)(xa^2))((xa)(y_1y_2)) = ((a^2x)(y_2y_1))((y_2y_1)(ax)) = ((a^2x)(y_2y_1))(a((y_2y_1)x)) = a(((a^2x)(y_2y_1))((y_2y_1)x)) = a((a^2(x(y_2y_1))((y_2y_1)x)) = a((x(y_2y_1))((x(y_2y_1))a^2)) = a(ua^2)$ where $u = (x(y_2y_1))((x(y_2y_1))$. Hence $a = a(ua^2)$ for some $u \in S$.

(iii) Let $a \in S$, then by using (ii), there exists $u \in S$ such that $a = a(ua^2)$. Now using Equation 2, left invertive law, paramedial law, Equation 2, medial law and Lemma 2.4, we have $ua^2 = u((xa^2)y)^2 = u((xa^2)^2y^2) = (xa^2)^2(uy^2) = (x^2(a^2)^2)(uy^2) = ((a^2)^2x^2)(uy^2) = ((uy^2)x^2)(a^2)^2 = ((uy^2)x^2)(a^2a^2) = (a^2a^2)(x^2(uy^2)) = ((x^2(uy^2))a^2)a^2 = (a^2((uy^2)x^2))a^2 = (a^2q)a^2$ where $q = (uy^2)x^2$. Hence $a = a((a^2q)a^2)$ for some $q \in S$.

(iv) Let $a \in S$, there exist $x, y \in S$ such that $a = (xa^2)y$. Using Equation 2, left invertive law, paramedial law, Equation 2 and medial law, we can further write $a = (x(aa))y = (a(xa))y = (y(xa))a = ((ey)(xa))a = ((ax)(ye))a = (((ye)x)a)a = (ta)a$ where $t = (ye)x$. Hence $a = (ta)a$ for some $t \in S$.

(v) Let $a \in S$, then by using (iv), there exists $t \in S$ such that $a = (ta)a$. Thus using Equation 2, left invertive law, paramedial law, Equation 2 and medial law, we can further write $a = (ta)a = (t((xa^2)y))a = ((xa^2)(ty))a =$

$((x(aa))(ty))a = ((a(xa))(ty))a = ((a(xa))p)a$ where $p = ty$. Hence $a = a = ((a(xa))p)a$ for some $p \in S$. ■

Proposition 4. *If S is an intra-regular AG-groupoid with left identity e, then for any DFS set $\langle (f_S, g_S)\,; S\rangle$ of S over U, $\langle (f_S, g_S)\,; S\rangle \sqsubseteq_{[M,N]} \mathbf{X}_S \diamond \langle (f_S, g_S)\,; S\rangle$.*

Proof. Suppose S is intra-regular AG-groupoid with left identity e and $\langle (f_S, g_S)\,; S\rangle$ be a DFS set of S over U. By Lemma 17(iv) for any $a \in S$, there exists $t \in S$ such that $a = (ta)a$ or $a = ua$ for some $u \in S$. Now

$$\begin{aligned}
&((\chi_S \widetilde{\circ} f_S)(a) \cap N) \cup M \\
&= \left(\left(\bigcup_{a=mn} \{\chi_S(m) \cap f_S(n)\}\right) \cap N\right) \cup M \\
&= \left(\bigcup_{a=mn} \{\chi_S(m) \cap f_S(n) \cap N\}\right) \cup M \supseteq (\chi_S(u) \cap f_S(a) \cap N) \cup M \\
&= (f_S(a) \cap N) \cup M
\end{aligned}$$

and

$$\begin{aligned}
&((\chi_S^c \widetilde{\circ} g_S)(a) \cup M) \cap N \\
&= \left(\left(\bigcap_{a=mn} \{\chi_S^c(m) \cup g_S(n)\}\right) \cup M\right) \cap N \\
&= \left(\bigcap_{a=mn} \{\chi_S^c(m) \cup g_S(n) \cup M\}\right) \cap N \subseteq (\chi_S^c(u) \cup g_S(a) \cup M) \cap N \\
&= (g_S(a) \cup M) \cap N.
\end{aligned}$$

Hence $\langle (f_S, g_S)\,; S\rangle \sqsubseteq_{[M,N]} \mathbf{X}_S \diamond \langle (f_S, g_S)\,; S\rangle$. ■

Proposition 5. *If S is an intra-regular AG-groupoid with left identity e, then for any DFS set $\langle (f_S, g_S)\,; S\rangle$ of S over U, $\langle (f_S, g_S)\,; S\rangle \sqsubseteq_{[M,N]} \langle (f_S, g_S)\,; S\rangle \diamond \mathbf{X}_S$.*

Proof. Suppose S is intra-regular AG-groupoid with left identity e and $\langle (f_S, g_S)\,; S\rangle$ be a DFS set of S over U. By Lemma 17(ii) for any $a \in S$, there exists $u \in S$ such that $a = a(ua^2)$ or $a = at$ where $t = ua^2 \in S$. Now

$$\begin{aligned}
&((f_S \widetilde{\circ} \chi_S)(a) \cap N) \cup M \\
&= \left(\left(\bigcup_{a=mn} \{f_S(m) \cap \chi_S(n)\}\right) \cap N\right) \cup M \\
&= \left(\bigcup_{a=mn} \{f_S(m) \cap \chi_S(n) \cap N\}\right) \cup M \supseteq (f_S(a) \cap \chi_S(t) \cap N) \cup M \\
&= (f_S(a) \cap N) \cup M
\end{aligned}$$

and

$$
\begin{aligned}
&((g_S \widetilde{\circ} \chi_S^c)(a) \cup M) \cap N \\
&= \left(\left(\bigcap_{a=mn} \{g_S(m) \cup \chi_S^c(n)\} \right) \cup M \right) \cap N \\
&= \left(\bigcap_{a=mn} \{g_S(m) \cup \chi_S^c(n) \cup M\} \right) \cap N \subseteq (g_S(a) \cup \chi_S^c(t) \cup M) \cap N \\
&= (g_S(a) \cup M) \cap N.
\end{aligned}
$$

Hence $\langle (f_S, g_S)\,; S\rangle \sqsubseteq_{[M,N]} \langle (f_S, g_S)\,; S\rangle \diamond \mathbf{X}_S$. ∎

As a consequence of Proposition 4, 1 and Proposition 5, 2 we have the following result.

Theorem 18. *In an intra-regular AG-groupoid S with left identity e, for any $(M,\ N)$-DFS left (resp. right ideal) $\langle (f_S, g_S); S\rangle$ of S over U, we have $\mathbf{X}_S \diamond \langle (f_S, g_S); S\rangle =_{[M,N]} \langle (f_S, g_S); S\rangle$ (resp. $\langle (f_S, g_S); S\rangle \diamond \mathbf{X}_S =_{[M,N]} \langle (f_S, g_S); S\rangle$).*

Theorem 19. *In an intra-regular AG-groupoid S with left identity e, the following statements are equivalent:*

(i) *$\langle (f_S, g_S)\,; S\rangle$ is $(M,\ N)$-DFS bi-ideal of S over U.*

(ii) *$\langle (f_S, g_S)\,; S\rangle$ is $(M,\ N)$-DFS generalized bi-ideal of S over U.*

Proof. (i)$\Longrightarrow$(ii) is obvious.

(ii)$\Longrightarrow$(i) Let $\langle (f, g)\,; S\rangle$ is $(M,\ N)$-DFS generalized bi-ideal of S over U. Let $p, q \in S$. Since S is intra-regular so for $p \in S$, then there exist u, v in S such that $p = (up^2)v$, so by using medial law, paramedial law, Equation 2 and Equation 2, we have

$$
\begin{aligned}
&f(pq) \cup M = f((up^2)v)q) \cup M = f(((up^2)(ev))q) \cup M = f(((ve)(p^2u))q) \cup M \\
&= f((p^2((ve)u))q) \cup M = f(((pp)((ve)u))q) \cup M = f(((u(ve))(pp))q) \cup M \\
&= f((p((u(ve))p))q) \cup M \supseteq f(p) \cap f(q) \cap N
\end{aligned}
$$

and

$$
\begin{aligned}
&g(pq) \cap N = g((up^2)v)q) \cap N = g(((up^2)(ev))q) \cap N = g(((ve)(p^2u))q) \cap N \\
&= g((p^2((ve)u))q) \cap N = g(((pp)((ve)u))q) \cap N = g(((u(ve))(pp))q) \cap N \\
&= g((p((u(ve))p))q) \cap N \subseteq g(p) \cup g(q) \cup M.
\end{aligned}
$$

Since p, q are arbitrary elements of S, then we conclude that $\langle (f_S, g_S)\,; S\rangle$ is (M, N)-DFS bi-ideal. ∎

Theorem 20. *In an intra-regular AG-groupoid S with left identity e, the following statements are equivalent:*

(i) $\langle (f_S, g_S)\,; S\rangle$ *is* $(M,\, N)$*-DFS left* (*resp. right*) *ideal of* S *over* U.

(ii) $\langle (f, g)\,; S\rangle$ *is* $(M,\, N)$*-DFS bi-ideal of* S *over* U.

Proof. (i)$\Longrightarrow$(ii) Let $\langle (f_S, g_S)\,; S\rangle$ is $(M,\, N)$-DFS left ideal of S over U. Let $a, b, c \in S$ then using left invertive law, we have $f((ab)c) \cup M = f((cb)a) \cup M \supseteq f(a) \cap N \supseteq f(a) \cap f(c) \cap N$, and $g((ab)c) \cap N = g((cb)a) \cap N \subseteq g(a) \cup M \subseteq g(a) \cup g(c) \cup M$. Hence $\langle (f_S, g_S)\,; S\rangle$ is $(M,\, N)$-DFS generalized bi-ideal. By Theorem 19, $\langle (f_S, g_S)\,; S\rangle$ is $(M,\, N)$-DFS bi-ideal of S over U.

(ii)$\Longrightarrow$(i) Suppose that $\langle (f_S, g_S)\,; S\rangle$ is $(M,\, N)$-DFS bi-ideal of S over U and let $a, b \in S$. Since S is intra-regular, then there exist x, y and u, v in S such that $a = (xa^2)y$ and $b = (ub^2)v$. Therefore by using left invertive law, medial law, paramedial law, Equation 2 and Equation 2, we have

$$
\begin{aligned}
&f(ab) \cup M = f(a((ub^2)v)) \cup M = f((ub^2)(av)) \cup M = f((va)(b^2u)) \cup M \\
&= f(b^2((va)u)) \cup M = f((bb)((va)u)) \cup M = f((((va)u)b)b) \cup M \\
&= f((((va)u)((ub^2)v))b) \cup M = f(((ub^2)(((va)u)v))b) \cup M \\
&= f(((v((va)u))(bu^2))b) \cup M = f((b^2((v((va)u))u))b) \cup M \\
&= f(((bb)((v((va)u))u))b) \cup M = f(((u(v((va)u)))(bb))b) \cup M \\
&= f((b((u(v((va)u)))b))b) \cup M \supseteq f(b) \cap f(b) \cap N = f(b) \cap N,
\end{aligned}
$$

$$
\begin{aligned}
&\text{and } g(ab) \cap N = g(a((ub^2)v)) \cap N = g((ub^2)(av)) \cap N = g((va)(b^2u)) \cap N \\
&= g(b^2((va)u)) \cap N = g((bb)((va)u)) \cap N = g((((va)u)b)b) \cap N \\
&= g((((va)u)((ub^2)v))b) \cap N = g(((ub^2)(((va)u)v))b) \cap N \\
&= g(((v((va)u))(bu^2))b) \cap N = g((b^2((v((va)u))u))b) \cap N \\
&= g(((bb)((v((va)u))u))b) \cap N = g(((u(v((va)u)))(bb))b) \\
&= g((b((u(v((va)u)))b))b) \cap N \subseteq g(b) \cup g(b) \cup M = g(b) \cup M.
\end{aligned}
$$

Hence $\langle (f_S, g_S)\,; S\rangle$ is $(M,\, N)$-DFS left ideal of S over U. ∎

Corollary 3. *In an intra-regular AG-groupoid S with left identity e, the following statements are equivalent.*

(i) $\langle (f_S, g_S)\,; S\rangle$ *is* $(M,\, N)$*-DFS ideal of* S *over* U.

(ii) $\langle (f_S, g_S)\,; S\rangle$ *is* $(M,\, N)$*-DFS bi-ideal of* S *over* U.

Theorem 21. *In an intra-regular AG-groupoid with left identity e, every (M,N)-DFS bi-ideal is idempotent.*

Proof. Let S be intra-regular AG-groupoid with left identity e. Then $S = S^2$. That is, for all $x \in S$, there exist $u, v \in S$ such that $x = uv$. Assume that $\langle (f_S, g_S)\,; S\rangle$ is $(M,\, N)$-DFS bi-ideal of S, so by Theorem 10.

$$\langle (f_S, g_S)\,; S\rangle \lozenge \langle (f_S, g_S)\,; S\rangle \sqsubseteq_{[M,N]} \langle (f_S, g_S)\,; S\rangle\,.$$

Using left invertive law, medial law, paramedial law, Equation 2 and Equation 2, for any $a \in S$, there exist $x, y \in S$ such that $a = (xa^2)ya = ((uv)a^2)y = ((uv)(aa))y = ((aa)(vu))y = (y(vu))(aa) = (ya)((vu)a) = (a(vu))(ay) = ((ay)(vu))a$. We have

$$\begin{aligned}
((f_S \widetilde{\circ} f_S)(a) \cap N) \cup M &= \left(\left(\bigcup_{a=pq} \{f_S(p) \cap f_S(q)\}\right) \cap N\right) \cup M \\
&= ((f_S((ay)(vu)) \cap f_S(a)) \cap N) \cup M \\
&= (f_S((ay)(vu) \cup M) \cap (f_S(a) \cup M) \cap N \\
&= ((f_S((ay)(vu) \cup M) \cup M) \cap (f_S(a) \cup M) \cap N \\
&\supseteq ((f_S(ay) \cap N) \cup M) \cap (f_S(a) \cup M) \cap N \text{ [Corollary 3]} \\
&\supseteq ((f_S(a) \cup M) \cup M) \cap (f_S(a) \cup M) \cap N \text{ [Corollary 3]} \\
&= (f_S(a) \cup M) \cap (f_S(a) \cup M) \cap N \\
&= (f_S(a) \cup M) \cap N = (f_S(a) \cap N) \cup M.
\end{aligned}$$

In a similar way we can prove that $((g_S \widetilde{\circ} g_S)\,(a) \cup M) \cap N \subseteq (\dot{g}_S(a) \cup M) \cap N$. Hence $\langle (f_S, g_S)\,; S\rangle \sqsubseteq_{[M,N]} \langle (f_S, g_S)\,; S\rangle \lozenge \langle (f_S, g_S)\,; S\rangle$. This proves that the DFS bi-ideal $\langle (f_S, g_S)\,; S\rangle$ is idempotent. ■

Theorem 22. *In an intra-regular AG-groupoid S with left identity e, for every $(M,\, N)$-DFS bi-ideal $\langle (f_S, g_S)\,; S\rangle$ of S, $\langle (f_S, g_S)\,; S\rangle\,(a) =_{[M,N]} \langle (f_S, g_S)\,; S\rangle\,(a^2)$ for all $a \in S$.*

Proof. Suppose S is intra-regular AG-groupoid with left identity and $\langle (f_S, g_S)\,; S\rangle$ is $(M,\, N)$-DFS bi-ideal of S. Let $a \in S$, then by Lemma 17, there exists t such that $a = (a^2t)a^2$. Now $(f_S(a) \cap N) \cup M = (f_S((a^2t)a^2) \cap N) \cup M = (f_S((a^2t)a^2) \cup M) \cap N = ((f_S((a^2t)a^2) \cup M) \cup M) \cap N \supseteq ((f_S(a^2) \cap f_S(a^2) \cap N) \cup M) \cap N = (f_S(a^2) \cap N) \cup M$, so $(f_S(a^2) \cap N) \cup M \subseteq (f_S(a) \cap N) \cup M$ and $(g_S(a) \cup M) \cap N = (g_S((a^2t)a^2) \cup M) \cap N = ((g_S((a^2t)a^2) \cap N) \cap N) \cup$

$M \subseteq \big((g_S(a^2) \cup g_S(a^2) \cup M\big) \cap N\big) \cup M = \big(g_S(a^2) \cup M\big) \cap N$, so $\big(g_S(a^2) \cup M\big) \cap N \supseteq (g_S(a) \cup M) \cap N$. Hence $\langle (f_S, g_S) ; S\rangle (a^2) \sqsubseteq_{[M,N]} \langle (f_S, g_S) ; S\rangle (a)$. Also since $\langle (f_S, g_S) ; S\rangle$ is (M, N)-DFS bi-ideal of S, by Corollary 3, $\langle (f_S, g_S) ; S\rangle$ is (M, N)-DFS ideal of S, thus $\big(f_S(a^2) \cap N\big) \cup M = \big(f_S(a^2) \cup M\big) \cap N = ((f_S(aa) \cup M) \cup M) \cap N \supseteq ((f_S(a) \cap N) \cup M) \cap N = (f_S(a) \cap N) \cup M$ and $(g_S(a^2) \cup M) \cap N = (g_S(a^2) \cap N) \cup M = ((g_S(aa) \cap N) \cap N) \cup M \subseteq ((g_S(a) \cup M) \cap N) \cup M = (g_S(a) \cup M) \cap N$. Hence $\langle (f_S, g_S) ; S\rangle (a^2) \sqsubseteq_{[M,N]} \langle (f_S, g_S) ; S\rangle (a)$ and so $\langle (f_S, g_S) ; S\rangle (a) =_{[M,N]} \langle (f_S, g_S) ; S\rangle (a^2)$ for all $a \in S$. ■

Corollary 4. *In an intra-regular AG-groupoid S with left identity e, for every (M, N)-DFS bi-ideal $\langle (f_S, g_S) ; S\rangle$ of S, $\langle (f_S, g_S) ; S\rangle (e) \sqsubseteq_{[M,N]} \langle (f_S, g_S) ; S\rangle (a^2)$ for all $a \in S$.*

Proof. By Theorem 22, $\big(f_S(a^2) \cap N\big) \cup M = (f_S(aa) \cap N) \cup M = (f_S((ee)(aa)) \cap N) \cup M = (f_S((aa)(ee)) \cap N) \cup M = (f_S((e(aa))e) \cap N) \cup M = (f_S((e(aa))e) \cup M) \cap N = ((f_S((e(aa))e) \cup M) \cup M) \cap N \supseteq ((f_S(e) \cap f_S(e) \cap N) \cup M) \cap N = (f_S(e) \cap N) \cup M$.

Also $\big(g_S(a^2) \cup M\big) \cap N = (g_S(aa) \cap N) \cup M = (g_S((ee)(aa)) \cap N) \cup M = (g_S((aa)(ee)) \cap N) \cup M = (g_S((e(aa))e) \cap N) \cup M = ((g_S((e(aa))e) \cap N) \cap N) \cup M \subseteq ((g_S(e) \cup g_S(e) \cup M) \cap N) \cup M = (g_S(e) \cup M) \cap N$. Hence $\langle (f_S, g_S) ; S\rangle (e) \sqsubseteq_{[M,N]} \langle (f_S, g_S) ; S\rangle (a^2)$ for all $a \in S$. ■

Theorem 23. *Let $\langle (f_S, g_S) ; S\rangle$ be a (M, N)-DFS bi-ideal of an intra-regular AG groupoid S with left identity e then $\langle (f_S, g_S) ; S\rangle (ab) =_{[M,N]} \langle (f_S, g_S) ; S\rangle (ba)$ for all $a, b \in S$.*

Proof. Suppose S is intra-regular AG-groupoid with left identity e, and $\langle (f_S, g_S); S\rangle$ a DFS bi-ideal of S over U then,

$$\begin{aligned}
(f_S(ab) \cap N) \cup M &= \big(f_S((ab)^2) \cap N\big) \cup M \text{ (By Theorem 22)} \\
&= (f_S((ab)(ab)) \cap N) \cup M \\
&= (f_S((ba)(ba)) \cap N) \cup M \text{ (using Equation 2)} \\
&= (f_S((ba)(ba)) \cup M) \cap N \\
&= ((f_S((ba)(ba)) \cup M) \cup M) \cap N \\
&\supseteq ((f_S(ba) \cap N) \cup M) \cap N \text{ (By Corollary 3)} \\
&= (f_S(ba) \cap N) \cup M
\end{aligned}$$

and

$$\begin{aligned}
(g_S(ab) \cup M) \cap N &= \big(g_S((ab)^2) \cup M\big) \cap N \text{ (By Theorem 22)} \\
&= (g_S((ab)(ab)) \cup M) \cap N
\end{aligned}$$

$$= (g_S((ba)(ba)) \cup M) \cap N \text{ (using Equation 2)}$$

$$= (g_S((ba)(ba)) \cap N) \cup M$$

$$= ((g_S((ba)(ba)) \cap N) \cap N) \cup M$$

$$\subseteq ((g_S(ba) \cup M) \cap N) \cup M \text{(By Corollary 3)}$$

$$= (g_S(ba) \cup M) \cap N.$$

Hence $\langle (f_S, g_S) ; S\rangle (ba) \sqsubseteq_{[M,N]} \langle (f_S, g_S) ; S\rangle (ab)$. Similarly as above we can prove $\langle (f_S, g_S) ; S\rangle (ab) \sqsubseteq_{[M,N]} \langle (f_S, g_S) ; S\rangle (ba)$. The required result is proved from these inclusions. ■

Theorem 24. *In an intra-regular AG-groupoid S with left identity e, for every DFS subset $\langle (f_S, g_S) ; S\rangle$ and (M, N)-DFS bi-ideal $\langle (p_S, q_S) ; S\rangle$ of S over U, $\langle (f_S, g_S) ; S\rangle \sqcap \langle (p_S, q_S) ; S\rangle \sqsubseteq_{[M,N]} \langle (f_S, g_S) ; S\rangle \diamond \langle (p_S, q_S) ; S\rangle$.*

Proof. Assume S is intra-regular. Let $\langle (f_S, g_S) ; S\rangle$ be a DFS set of S and $\langle (p_S, q_S) ; S\rangle$ a (M, N)-DFS bi-ideal of S over U. Now let $a \in S$. Then by Lemma 17, there exists $q \in S$, such that $a = a((a^2 q)a^2)$.

Now

$$((f_S \widetilde{\circ} p_S)(a) \cap N) \cup M$$

$$= \left(\left(\bigcup_{a=st} \{f_S(s) \cap p_S(t)\} \right) \cap N \right) \cup M \supseteq \big(\big(f_S(a) \cap p_S((a^2 q)a^2)\big) \cap N\big) \cup M$$

$$= ((f_S(a) \cap N) \cap (p_S((a^2 q)a^2) \cap N)) \cup M$$

$$= ((f_S(a) \cap N) \cup M) \cap ((p_S((a^2 q)a^2) \cap N) \cup M)$$

$$= ((f_S(a) \cap N) \cup M) \cap ((p_S((a^2 q)a^2) \cup M) \cap N)$$

$$= ((f_S(a) \cap N) \cup M) \cap (((p_S((a^2 q)a^2) \cup M) \cup M) \cap N)$$

$$\supseteq ((f_S(a) \cap N) \cup M) \cap (((p_S(a^2) \cap N) \cup M) \cap N)$$

$$= ((f_S(a) \cap N) \cup M) \cap (((p_S(a^2) \cup M) \cap N) \cap N)$$

$$= ((f_S(a) \cap N) \cup M) \cap (((p_S(a^2) \cup M) \cup M) \cap N) \cap N$$

$$\supseteq ((f_S(a) \cap N) \cup M) \cap (((p_S(a) \cap N) \cup M) \cap N) \cap N \text{ [Corollary 3]}$$

$$= ((f_S(a) \cap N) \cup M) \cap ((p_S(a) \cap N) \cup M) \cap N$$

$$= ((f_S(a) \cap N) \cup M) \cap ((p_S(a) \cap N) \cup M$$

$$= ((f_S(a) \cap p_S(a)) \cap N) \cup M.$$

In a similar way, we can prove that $((g_S \cup q_S) \cup M) \cap N \supseteq ((g_S \widetilde{\circ} q_S) \cup M) \cap N$. Hence $\langle (f_S, g_S); S\rangle \sqcap \langle (p_S, q_S); S\rangle \sqsubseteq_{[M,N]} \langle (f_S, g_S); S\rangle \diamond \langle (p_S, q_S); S\rangle$. ■

Theorem 25. *In an intra-regular AG-groupoid S with left identity e, for all (M, N)-DFS bi-ideal $\langle (f_S, g_S); S\rangle$ and any DFS set $\langle (p_S, q_S); S\rangle$ of S over U, $\langle (f_S, g_S); S\rangle \sqcap \langle (p_S, q_S); S\rangle \sqsubseteq_{[M,N]} \langle (f_S, g_S); S\rangle \diamond \langle (p_S, q_S); S\rangle$.*

Proof. Assume S is intra-regular AG-groupoid with left identity e. Let $\langle (f_S, g_S); S\rangle$ be a DFS bi-ideal and $\langle (p_S, q_S); S\rangle$ be DFS left ideal of S over U. For any $a \in S$. By Lemma 17 (v), there exists p such that $a = ((a(xa))p)a$. Since (M, N)-DFS bi-ideal is DFS ideal in intra-regular AG-groupoid with left identity, then we have

$$\begin{aligned}
&((f_S \widetilde{\circ} p_S)(a) \cap N) \cup M \\
&= \left(\left(\bigcup_{a=uv} \{f_S(u) \cap p_S(v)\}\right) \cap N\right) \cup M \supseteq ((f_S((a(xa))p) \cap p_S(a)) \cap N) \cup M \\
&= ((f_S((a(xa))p) \cap N) \cap (p_S(a) \cap N)) \cup M \\
&= ((f_S((a(xa))p) \cap N) \cup M) \cap ((p_S(a) \cap N) \cup M) \\
&= ((f_S((a(xa))p) \cup M) \cap N) \cap ((p_S(a) \cap N) \cup M) \\
&= (((f_S((a(xa))p) \cup M) \cup M) \cap N) \cap ((p_S(a) \cap N) \cup M) \\
&\supseteq (((f_S(a(xa)) \cap N) \cup M) \cap N) \cap ((p_S(a) \cap N) \cup M) \quad \text{[Corollary 3]} \\
&= (((f_S(a(xa)) \cup M) \cap N) \cap N) \cap ((p_S(a) \cap N) \cup M) \\
&= ((f_S(a(xa)) \cup M) \cap N) \cap ((p_S(a) \cap N) \cup M) \\
&= (((f_S(a(xa)) \cup M) \cup M) \cap N) \cap ((p_S(a) \cap N) \cup M) \\
&\supseteq (((f_S(a) \cap N) \cup M) \cap N) \cap ((p_S(a) \cap N) \cup M) \quad \text{[Corollary 3]} \\
&= ((f_S(a) \cap N) \cup M) \cap ((p_S(a) \cap N) \cup M) \\
&= ((f_S(a) \cap p_S(a)) \cap N) \cup M.
\end{aligned}$$

In a similar way, we can prove that $((g_S(a) \cup q_S(a)) \cup M) \cap N \supseteq ((g_S \widetilde{\circ} q_S)(a) \cup M) \cap N$. Hence $\langle (f_S, g_S); S\rangle \sqcap \langle (p_S, q_S); S\rangle \sqsubseteq_{[M,N]} \langle (f_S, g_S); S\rangle \diamond \langle (p_S, q_S); S\rangle$ for all (M, N)-DFS generalized bi-ideal $\langle (f_S, g_S); S\rangle$ and (M, N)-DFS left ideal $\langle (p_S, q_S); S\rangle$ of S over U. ■

Theorem 26. *In an intra-regular AG-groupoid S with left identity e, every (M, N)-DFS generalized bi-ideal is (M, N)-DFS semiprime.*

Proof. Assume S is intra-regular and $\langle(f_S, g_S)\,; S\rangle$ be $(M,\ N)$-DFS generalized bi-ideal of S over U. Let $a \in S$, since S is intra-regular, by Lemma 17 (i), there exists $t \in S$ such that $a = (a^2t)a^2$.

Now $f_S(a) \cup M = f_S((a^2t)a^2) \cup M \supseteq f_S(a^2) \cap f_S(a^2) \cap N = f_S(a^2) \cap N$ and $g_S(a) \cap N = g_S((a^2t)a^2) \cap N \subseteq g_S(a^2) \cup g_S(a^2) \cup M = g_S(a^2) \cup M$. Hence (M, N)-DFS generalized bi-ideal $\langle(f_S, g_S)\,; S\rangle$ is (M, N)-DFS semiprime. ■

Theorem 27. *In an intra-regular AG-groupoid S with left identity e, $\langle(f_S, g_S)\,; S\rangle \sqcap \langle(p_S, q_S)\,; S\rangle \sqsubseteq \langle(f_S, g_S)\,; S\rangle \diamond \langle(p_S, q_S)\,; S\rangle$ for all $(M,\ N)$-DFS bi-ideals $\langle(f_S, g_S); S\rangle$ and $\langle(p_S, q_S)\,; S\rangle$ of S over U.*

Proof. Assume that S is intra-regular AG-groupoid with left identity e. Let $\langle(f_S, g_S)\,; S\rangle$ and $\langle(p_S, q_S)\,; S\rangle$ are $(M,\ N)$-DFS bi-ideals of S over U. For any $a \in S$, there exist $x, y \in S$ such that $a = (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a$.

Now $y(xa) = y(x((xa^2)y)) = y((xa^2)(xy)) = (xa^2)(xy^2) = (a(xa))(xy^2) = ((xy^2)(xa))a = ((xy^2)(x((xa^2)y))a = ((xy^2)((xa^2)(xy)))a = ((xa^2)((xy^2))(a^2x))a = (((xy)(xy^2))(a^2x))a = (a^2(((xy)(xy^2))x))a = ((x((xy)(xy^2)))(aa))a = (a(x((xy)(xy^2)))a)a$. Thus $a = (y(xa))a = (a(x((xy)(xy^2)))a)a$. We have

$$
\begin{aligned}
((f_S \widetilde{\circ} p_S)(a) \cap N) \cup M &= \left(\left(\bigcup_{a=mn} \{f_S(m) \cap p_S(n)\}\right) \cap N\right) \cup M \\
&\supseteq \big(\big(f_S(a(x((xy)(xy^2)))a) \cap p_S(a)\big) \cap N\big) \cup M \\
&= \big(\big(f_S(a(x((xy)(xy^2)))a) \cap N\big) \cap (p_S(a) \cap N)\big) \cup M \\
&= \big(\big(f_S(a(x((xy)(xy^2)))a) \cap N\big) \cup M) \cap ((p_S(a) \cap N) \cup M) \\
&= \big(\big(f_S(a(x((xy)(xy^2)))a) \cup M\big) \cap N\big) \cap ((p_S(a) \cap N) \cup M) \\
&= \big(\big(\big(f_S(a(x((xy)(xy^2)))a) \cup M\big) \cup M) \cap N\big) \cap ((p_S(a) \cap N) \cup M) \\
&\supseteq (((f_S(a) \cap f_S(a) \cap N) \cup M) \cap N) \cap ((p_S(a) \cap N) \cup M) \\
&= ((f_S(a) \cap N) \cup M) \cap ((p_S(a) \cap N) \cup M) \\
&= ((f_S(a) \cap p_S(a)) \cap N) \cup M.
\end{aligned}
$$

In a similar way we can prove $((g_S(a) \cup q_S(a)) \cup M) \cap N \supseteq ((g_S \widetilde{\circ} q_S)(a) \cup M) \cap N$. Hence $\langle(f_S, g_S)\,; S\rangle \sqcap \langle(p_S, q_S)\,; S\rangle \sqsubseteq_{[M,N]} \langle(f_S, g_S)\,; S\rangle \diamond \langle(p_S, q_S)\,; S\rangle$ for all $(M,\ N)$-DFS bi-ideals $\langle(f_S, g_S)\,; S\rangle$ and $\langle(p_S, q_S)\,; S\rangle$ of S over U. ■

6. Conclusion

In this paper, we studied DFS generalized bi-ideals and DFS bi-ideals in intra-regular AG-groupoids. Our future work is to study these ideals in left regular AG-groupoids, right regular AG-groupoids, regular AG-groupoids, completely regular AG-groupoids, V-regular AG-groupoids. We will try to obtain some characterizations results of intra-regular, left regular, right regular, regular, V-regular, completely regular AG-groupoids using the newly defined notions.

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