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A STUDY ON FIBONACCI AND LUCAS BIHYPERNOMIALS

Anetta Szynal-Liana and Iwona Włoch

Rzeszow University of Technology
Faculty of Mathematics and Applied Physics
al. Powstańców Warszawy 12, 35–959 Rzeszów, Poland

e-mail: aszynal@prz.edu.pl
iwloch@prz.edu.pl

Abstract

The bihyperbolic numbers are extension of hyperbolic numbers to four dimensions. In this paper we introduce and study the Fibonacci and Lucas bihypernomials, i.e., polynomials, which are a generalization of the bihyperbolic Fibonacci numbers and the bihyperbolic Lucas numbers, respectively.

Keywords: Fibonacci numbers, recurrence relations, hyperbolic numbers, bihyperbolic numbers, polynomials.

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1. Introduction

Let F_n be the nth Fibonacci number defined recursively by $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with $F_0 = 0$, $F_1 = 1$. The nth Lucas number L_n is defined recursively by $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$ with $L_0 = 2$, $L_1 = 1$. The direct formulas for the nth Fibonacci number and the nth Lucas number are named as Binet formulas and have the form

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$
$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

For any variable quantity x, the Fibonacci polynomial $F_n(x)$ is defined as $F_n(x) = x \cdot F_{n-1}(x) + F_{n-2}(x)$ for $n \geq 2$ with $F_0(x) = 0$, $F_1(x) = 1$. The Lucas polynomial

$L_n(x)$ is defined as $L_n(x) = x \cdot L_{n-1}(x) + L_{n-2}(x)$ for $n \geq 2$ with the initial terms $L_0(x) = 2$, $L_1(x) = x$. For $x = 1$ the Fibonacci and Lucas polynomials give the Fibonacci and Lucas numbers, respectively.

Based on the properties of sequences defined by the second-order linear recurrence relations we can give Binet formulas for $F_n(x)$ and $L_n(x)$. Then

$$F_n(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)} \tag{1}$$

and

$$L_n(x) = \alpha^n(x) + \beta^n(x), \tag{2}$$

where $\alpha(x) = \frac{1}{2}\left(x + \sqrt{x^2+4}\right)$ and $\beta(x) = \frac{1}{2}\left(x - \sqrt{x^2+4}\right)$.

Fibonacci polynomials were first studied by Bicknell, see for details [1] and next the theory of Fibonacci type polynomials were developed among others in [2, 6, 11, 14, 19, 20]. The interest in investigations of Fibonacci polynomials properties follows from their connections with the Chebyshev polynomials and from their applications in distinct branches of science, see [12, 13, 18].

Hyperbolic numbers are two dimensional number system. Hyperbolic imaginary unit, so-called *unipotent*, introduced in 1848 by James Cockle (see [7–10]), is an element $\mathbf{h}$ such that $\mathbf{h}^2 = 1$ and $\mathbf{h} \neq \pm 1$. The set of hyperbolic numbers is defined as

$$\mathbb{H} = \left\{x + y\mathbf{h} : x, y \in \mathbb{R}, \mathbf{h}^2 = 1\right\}.$$

Hyperbolic numbers can be used for describing the bidimensional space-time setting of the theory of relativity, details of this applications can be found in [15]. Some algebraic properties of hyperbolic numbers were given among others in [16, 17].

Bihyperbolic numbers are a generalization of hyperbolic numbers. Let $\mathbb{H}_2$ be the set of bihyperbolic numbers ζ of the form

$$\zeta = x_0 + j_1x_1 + j_2x_2 + j_3x_3,$$

where $x_0, x_1, x_2, x_3 \in \mathbb{R}$ and j_1, j_2, $j_3 \notin \mathbb{R}$ are operators such that

$$j_1^2 = j_2^2 = j_3^2 = 1,\ j_1j_2 = j_2j_1 = j_3,\ j_1j_3 = j_3j_1 = j_2,\ j_2j_3 = j_3j_2 = j_1. \tag{3}$$

From the above rules the multiplication of bihyperbolic numbers can be made analogously to the multiplication of algebraic expressions. The addition and the subtraction of bihyperbolic numbers is done by adding and subtracting corresponding terms and hence their coefficients.

The addition and multiplication on $\mathbb{H}_2$ are commutative and associative. Moreover, $(\mathbb{H}_2, +, \cdot)$ is a commutative ring.

For the algebraic properties of bihyperbolic numbers, see [3].

A special kind of bihyperbolic numbers, namely bihyperbolic Fibonacci numbers, were introduced in [4] as follows.

The nth bihyperbolic Fibonacci number BhF_n is defined as

$$BhF_n = F_n + j_1 F_{n+1} + j_2 F_{n+2} + j_3 F_{n+3}. \tag{4}$$

In the same way we define bihyperbolic Lucas numbers. The nth bihyperbolic Lucas number BhL_n is defined as

$$BhL_n = L_n + j_1 L_{n+1} + j_2 L_{n+2} + j_3 L_{n+3}. \tag{5}$$

Note that some combinatorial properties of bihyperbolic numbers of the Fibonacci type we can find in [5].

In this paper we introduce the Fibonacci and Lucas bihypernomials, i.e., polynomials, which can be considered as a generalization of the bihyperbolic Fibonacci numbers and the bihyperbolic Lucas numbers.

For $n \geq 0$ the Fibonacci and Lucas bihypernomials are defined by

$$BhF_n(x) = F_n(x) + j_1 F_{n+1}(x) + j_2 F_{n+2}(x) + j_3 F_{n+3}(x) \tag{6}$$

and

$$BhL_n(x) = L_n(x) + j_1 L_{n+1}(x) + j_2 L_{n+2}(x) + j_3 L_{n+3}(x), \tag{7}$$

where $F_n(x)$ is the nth Fibonacci polynomial, $L_n(x)$ is the the n-th Lucas polynomial and j_1, j_2, j_3 are bihyperbolic units satisfy (3).

For $x = 1$ we obtain the bihyperbolic Fibonacci numbers and the bihyperbolic Lucas numbers, respectively.

We recall selected theorems related to Fibonacci and Lucas polynomials, which will be used in the next part of this paper.

Theorem 1 [1]. *Let $m \geq 2$, $n \geq 1$ be integers. Then*

$$F_{m-1}(x)F_n(x) + F_m(x)F_{n+1}(x) = F_{m+n}(x). \tag{8}$$

Theorem 2 [11]. *Let $n \geq 2$ be an integer. Then*

$$\sum_{l=1}^{n-1} F_l(x) = \frac{F_n(x) + F_{n-1}(x) - 1}{x}. \tag{9}$$

Theorem 3 [11]. *Let $n \geq 2$ be an integer. Then*

$$\sum_{l=1}^{n-1} L_l(x) = \frac{L_n(x) + L_{n-1}(x) - 2 - x}{x}. \tag{10}$$

2. Main results

Theorem 4. *For any variable quantity x, we have*

$$BhF_n(x) = x \cdot BhF_{n-1}(x) + BhF_{n-2}(x) \text{ for } n \geq 2 \tag{11}$$

with $BhF_0(x) = j_1 + j_2 \cdot x + j_3 \cdot (x^2 + 1)$
and $BhF_1(x) = 1 + j_1 \cdot x + j_2 \cdot (x^2 + 1) + j_3 \cdot (x^3 + 2x)$.

Proof. If $n = 2$ we have

$$\begin{aligned} BhF_2(x) &= x \cdot BhF_1(x) + BhF_0(x) \\ &= x \cdot (1+j_1 \cdot x + j_2 \cdot (x^2+1) + j_3 \cdot (x^3+2x)) + j_1 + j_2 \cdot x + j_3 \cdot (x^2+1) \\ &= x + j_1 \cdot (x^2+1) + j_2 \cdot (x^3+2x) + j_3 \cdot (x^4+3x^2+1) \\ &= F_2(x) + j_1 F_3(x) + j_2 F_4(x) + j_3 F_5(x). \end{aligned}$$

If $n \geq 3$ then using the definition of the Fibonacci polynomials we have

$$\begin{aligned} BhF_n(x) &= F_n(x) + j_1 F_{n+1}(x) + j_2 F_{n+2}(x) + j_3 F_{n+3}(x) \\ &= (x \cdot F_{n-1}(x) + F_{n-2}(x)) + j_1(x \cdot F_n(x) + F_{n-1}(x)) \\ &+ j_2(x \cdot F_{n+1}(x) + F_n(x)) + j_3(x \cdot F_{n+2}(x) + F_{n+1}(x)) \\ &= x\,(F_{n-1}(x) + j_1 \cdot F_n(x) + j_2 \cdot F_{n+1}(x) + j_3 \cdot F_{n+2}(x)) \\ &+ F_{n-2}(x) + j_1 \cdot F_{n-1}(x) + j_2 \cdot F_n(x) + j_3 \cdot F_{n+1}(x) \\ &= x \cdot BhF_{n-1}(x) + BhF_{n-2}(x), \end{aligned}$$

which ends the proof. ■

In the same way we obtain the next result for Lucas bihypernomials.

Theorem 5. *For any variable quantity x, we have*

$$BhL_n(x) = x \cdot BhL_{n-1}(x) + BhL_{n-2}(x) \text{ for } n \geq 2$$

with $BhL_0(x) = 2 + j_1 \cdot x + j_2 \cdot (x^2 + 2) + j_3 \cdot (x^3 + 3x)$
and $BhL_1(x) = x + j_1 \cdot (x^2 + 2) + j_2 \cdot (x^3 + 3x) + j_3 \cdot (x^4 + 4x^2 + 2)$.

Theorem 6. *Let $n \geq 0$ be an integer. Then*

$$\begin{aligned} &BhF_n(x) - j_1 BhF_{n+1}(x) - j_2 BhF_{n+2}(x) + j_3 BhF_{n+3}(x) \\ &= F_n(x) - F_{n+2}(x) - F_{n+4}(x) + F_{n+6}(x), \end{aligned} \tag{12}$$

$$\begin{aligned} &BhL_n(x) - j_1 BhL_{n+1}(x) - j_2 BhL_{n+2}(x) + j_3 BhL_{n+3}(x) \\ &= L_n(x) - L_{n+2}(x) - L_{n+4}(x) + L_{n+6}(x), \end{aligned} \tag{13}$$

$$
\begin{aligned}
&BhF_n(x) - j_1 BhF_{n+1}(x) - j_2 BhF_{n+2}(x) - j_3 BhF_{n+3}(x) \\
&= F_n(x) - F_{n+2}(x) - F_{n+4}(x) + F_{n+6}(x) - 2j_3 BhF_{n+3}(x),
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
&BhL_n(x) - j_1 BhL_{n+1}(x) - j_2 BhL_{n+2}(x) - j_3 BhL_{n+3}(x) \\
&= L_n(x) - L_{n+2}(x) - L_{n+4}(x) + L_{n+6}(x) - 2j_3 BhL_{n+3}(x).
\end{aligned} \tag{15}
$$

Proof. By formula (6) we get

$$
\begin{aligned}
&BhF_n(x) - j_1 BhF_{n_1}(x) - j_2 BhF_{n+2}(x) + j_3 BhF_{n+3}(x) \\
&= F_n(x) + j_1 F_{n+1}(x) + j_2 F_{n+2}(x) + j_3 F_{n+3}(x) \\
&- j_1 F_{n+1}(x) - F_{n+2}(x) - j_3 F_{n+3}(x) - j_2 F_{n+4}(x) \\
&- j_2 F_{n+2}(x) - j_3 F_{n+3}(x) - F_{n+4}(x) - j_1 F_{n+5}(x) \\
&+ j_3 F_{n+3}(x) + j_2 F_{n+4}(x) + j_1 F_{n+5}(x) + F_{n+6}(x) \\
&= F_n(x) - F_{n+2}(x) - F_{n+4}(x) + F_{n+6}(x)
\end{aligned}
$$

and we obtain (12). By the same method we can prove formulas (13)–(15). ∎

Now we give the Binet formulas for the Fibonacci and Lucas bihypernomials.

Theorem 7 (Binet formulas for Fibonacci and Lucas bihypernomials)**.** *Let* $n \geq 0$ *be an integer. Then*

$$
\begin{aligned}
BhF_n(x) &= \frac{\alpha^n(x)}{\alpha(x) - \beta(x)} \left(1 + j_1\alpha(x) + j_2\alpha^2(x) + j_3\alpha^3(x)\right) \\
&- \frac{\beta^n(x)}{\alpha(x) - \beta(x)} \left(1 + j_1\beta(x) + j_2\beta^2(x) + j_3\beta^3(x)\right),
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
BhL_n(x) &= \alpha^n(x) \left(1 + j_1\alpha(x) + j_2\alpha^2(x) + j_3\alpha^3(x)\right) \\
&+ \beta^n(x) \left(1 + j_1\beta(x) + j_2\beta^2(x) + j_3\beta^3(x)\right),
\end{aligned} \tag{17}
$$

where $\alpha(x) = \frac{1}{2}\left(x + \sqrt{x^2+4}\right)$, $\beta(x) = \frac{1}{2}\left(x - \sqrt{x^2+4}\right)$.

Proof. Using (1), (4) and (6) we have

$$
\begin{aligned}
BhF_n(x) &= F_n(x) + j_1 F_{n+1}(x) + j_2 F_{n+2}(x) + j_3 F_{n+3}(x) \\
&= \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)} + j_1 \frac{\alpha^{n+1}(x) - \beta^{n+1}(x)}{\alpha(x) - \beta(x)} \\
&+ j_2 \frac{\alpha^{n+2}(x) - \beta^{n+2}(x)}{\alpha(x) - \beta(x)} + j_3 \frac{\alpha^{n+3}(x) - \beta^{n+3}(x)}{\alpha(x) - \beta(x)}
\end{aligned}
$$

and after calculations the result (16) follows. In the same way, using (2), (5) and (7), we obtain the Binet formula (17) for the Lucas bihypernomials. ∎

Fibonacci identities are the bases in finding Fibonacci sequence properties. Among them we can distinguish the well-known identities such as Catalan identity, Cassini identity and d'Ocagne identity and to give corresponding identities for Fibonacci and Lucas bihypernomials.

For simplicity of notation let

$$\hat{\alpha}(x) = 1 + j_1\alpha(x) + j_2\alpha^2(x) + j_3\alpha^3(x),$$
$$\hat{\beta}(x) = 1 + j_1\beta(x) + j_2\beta^2(x) + j_3\beta^3(x).$$

Then we can write (16) and (17) as

$$BhF_n(x) = \frac{\alpha^n(x)}{\alpha(x)-\beta(x)}\hat{\alpha}(x) - \frac{\beta^n(x)}{\alpha(x)-\beta(x)}\hat{\beta}(x)$$

and

$$BhL_n(x) = \alpha^n(x)\hat{\alpha}(x) + \beta^n(x)\hat{\beta}(x),$$

respectively.

Moreover,

$$\alpha(x)\cdot\beta(x) = -1,$$
$$\alpha(x)+\beta(x) = x,$$
$$\alpha(x)-\beta(x) = \sqrt{x^2+4}$$

and

$$\begin{aligned}&\hat{\alpha}(x)\cdot\hat{\beta}(x) = \hat{\beta}(x)\cdot\hat{\alpha}(x)\\ &= 1 + j_1\beta(x) + j_2\beta^2(x) + j_3\beta^3(x) + j_1\alpha(x) - 1 - j_3\beta(x) - j_2\beta^2(x)\\ &+ j_2\alpha^2(x) - j_3\alpha(x) + 1 + j_1\beta(x) + j_3\alpha^3(x) - j_2\alpha^2(x) + j_1\alpha(x) - 1\\ &= j_1\left(2\alpha(x) + 2\beta(x)\right) + j_3\left(\alpha^3(x) - \alpha(x) + \beta^3(x) - \beta(x)\right)\\ &= j_1\cdot 2x + j_3\cdot\left(x^3+2x\right).\end{aligned}$$

Theorem 8 (Catalan type identity for Fibonacci bihypernomials)**.** *Let $n \geq 0$, $r \geq 0$ be integers such that $n \geq r$. Then*

$$\begin{aligned}&BhF_{n-r}(x)\cdot BhF_{n+r}(x) - (BhF_n(x))^2\\ &= \frac{(-1)^{n-r+1}(\alpha^r(x)-\beta^r(x))^2}{(\alpha(x)-\beta(x))^2}\hat{\alpha}(x)\hat{\beta}(x)\end{aligned}$$

$$= \frac{(-1)^{n-r+1} 2x \left(\left(\frac{1}{2} \left(x + \sqrt{x^2+4} \right) \right)^r - \left(\frac{1}{2} \left(x - \sqrt{x^2+4} \right) \right)^r \right)^2}{x^2+4} j_1$$
$$+ \frac{(-1)^{n-r+1} \left(x^3 + 2x \right) \left(\left(\frac{1}{2} \left(x + \sqrt{x^2+4} \right) \right)^r - \left(\frac{1}{2} \left(x - \sqrt{x^2+4} \right) \right)^r \right)^2}{x^2+4} j_3.$$

Proof. Applying Theorem 7 we have

$$\begin{aligned}
&BhF_{n-r}(x) \cdot BhF_{n+r}(x) - (BhF_n(x))^2 \\
&= -\frac{\alpha^{n-r}(x)}{\alpha(x)-\beta(x)} \hat{\alpha}(x) \frac{\beta^{n+r}(x)}{\alpha(x)-\beta(x)} \hat{\beta}(x) - \frac{\beta^{n-r}(x)}{\alpha(x)-\beta(x)} \hat{\beta}(x) \frac{\alpha^{n+r}(x)}{\alpha(x)-\beta(x)} \hat{\alpha}(x) \\
&+ \frac{\alpha^{n}(x)}{\alpha(x)-\beta(x)} \hat{\alpha}(x) \frac{\beta^{n}(x)}{\alpha(x)-\beta(x)} \hat{\beta}(x) + \frac{\beta^{n}(x)}{\alpha(x)-\beta(x)} \hat{\beta}(x) \frac{\alpha^{n}(x)}{\alpha(x)-\beta(x)} \hat{\alpha}(x) \\
&= -\frac{\alpha^{n-r}(x)\beta^{n+r}(x)}{(\alpha(x)-\beta(x))^2} \hat{\alpha}(x)\hat{\beta}(x) - \frac{\beta^{n-r}(x)\alpha^{n+r}(x)}{(\alpha(x)-\beta(x))^2} \hat{\beta}(x)\hat{\alpha}(x) \\
&+ \frac{\alpha^{n}(x)\beta^{n}(x)}{(\alpha(x)-\beta(x))^2} \hat{\alpha}(x)\hat{\beta}(x) + \frac{\beta^{n}(x)\alpha^{n}(x)}{(\alpha(x)-\beta(x))^2} \hat{\beta}(x)\hat{\alpha}(x) \\
&= \frac{\alpha^{n}(x)\beta^{n}(x)}{(\alpha(x)-\beta(x))^2} \hat{\alpha}(x)\hat{\beta}(x) \frac{2\alpha^r(x)\beta^r(x) - (\beta^r(x))^2 - (\alpha^r(x))^2}{(\alpha(x)\beta(x))^r} \\
&= \frac{(\alpha(x)\beta(x))^n}{(\alpha(x)-\beta(x))^2} \hat{\alpha}(x)\hat{\beta}(x)(-1) \frac{(\alpha^r(x) - \beta^r(x))^2}{(\alpha(x)\beta(x))^r} \\
&= \frac{(-1)^{n-r+1}(\alpha^r(x) - \beta^r(x))^2}{(\alpha(x)-\beta(x))^2} \hat{\alpha}(x)\hat{\beta}(x),
\end{aligned}$$

so the result follows. ∎

In the same way one can easily prove the next theorem, which gives Catalan type identity for the Lucas bihypernomials.

Theorem 9 (Catalan type identity for Lucas bihypernomials)**.** *Let $n \geq 0$, $r \geq 0$ be integers such that $n \geq r$. Then*

$$\begin{aligned}
&BhL_{n-r}(x) \cdot BhL_{n+r}(x) - (BhL_n(x))^2 \\
&= (-1)^{n-r}(\alpha^r(x) - \beta^r(x))^2 \cdot \hat{\alpha}(x)\hat{\beta}(x).
\end{aligned}$$

Note that for $r = 1$ we get the Cassini type identities for the Fibonacci and Lucas bihypernomials.

Corollary 10 (Cassini type identities for Fibonacci and Lucas bihypernomials)**.** *Let $n \geq 1$ be an integer. Then*

(18)
$$\begin{aligned}&BhF_{n-1}(x)\cdot BhF_{n+1}(x)-(BhF_n(x))^2\\&=(-1)^n\left(j_1\cdot 2x+j_3\cdot\left(x^3+2x\right)\right),\end{aligned}$$

(19)
$$\begin{aligned}&BhL_{n-1}(x)\cdot BhL_{n+1}(x)-(BhL_n(x))^2\\&=(-1)^{n-1}(x^2+4)\left(j_1\cdot 2x+j_3\cdot\left(x^3+2x\right)\right).\end{aligned}$$

Theorem 11 (d'Ocagne type identity for Fibonacci bihypernomials)**.** *Let $m\geq 0$, $n\geq 0$ be integers such that $m\geq n$. Then*

$$\begin{aligned}&BhF_m(x)\cdot BhF_{n+1}(x)-BhF_{m+1}(x)\cdot BhF_n(x)\\&=\frac{(-1)^n\left(\alpha^{m-n}(x)-\beta^{m-n}(x)\right)}{\alpha(x)-\beta(x)}\hat{\alpha}(x)\hat{\beta}(x).\end{aligned}$$

Proof. Applying Theorem 7 we have

$$\begin{aligned}&BhF_m(x)\cdot BhF_{n+1}(x)-BhF_{m+1}(x)\cdot BhF_n(x)\\&=\frac{\alpha^{m+n+1}(x)}{(\alpha(x)-\beta(x))^2}\hat{\alpha}^2(x)-\frac{\alpha^m(x)\beta^{n+1}(x)}{(\alpha(x)-\beta(x))^2}\hat{\alpha}(x)\hat{\beta}(x)\\&-\frac{\alpha^{n+1}(x)\beta^m(x)}{(\alpha(x)-\beta(x))^2}\hat{\beta}(x)\hat{\alpha}(x)+\frac{\beta^{m+n+1}(x)}{(\alpha(x)-\beta(x))^2}\hat{\beta}^2(x)\\&-\frac{\alpha^{m+1+n}(x)}{(\alpha(x)-\beta(x))^2}\hat{\alpha}^2(x)+\frac{\alpha^{m+1}(x)\beta^n(x)}{(\alpha(x)-\beta(x))^2}\hat{\alpha}(x)\hat{\beta}(x)\\&+\frac{\alpha^n(x)\beta^{m+1}(x)}{(\alpha(x)-\beta(x))^2}\hat{\beta}(x)\hat{\alpha}(x)-\frac{\beta^{m+1+n}(x)}{(\alpha(x)-\beta(x))^2}\hat{\beta}^2(x)\\&=\left(\frac{\alpha^m(x)\beta^n(x)(\alpha(x)-\beta(x))}{(\alpha(x)-\beta(x))^2}-\frac{\alpha^n(x)\beta^m(x)(\alpha(x)-\beta(x))}{(\alpha(x)-\beta(x))^2}\right)\hat{\alpha}(x)\hat{\beta}(x)\\&=\frac{(\alpha(x)\beta(x))^n}{\alpha(x)-\beta(x)}\left(\alpha^{m-n}(x)-\beta^{m-n}(x)\right)\hat{\alpha}(x)\hat{\beta}(x)\\&=\frac{(-1)^n\left(\alpha^{m-n}(x)-\beta^{m-n}(x)\right)}{\alpha(x)-\beta(x)}\hat{\alpha}(x)\hat{\beta}(x),\end{aligned}$$

so the result follows. ∎

In the same way we can prove the next theorem.

Theorem 12 (d'Ocagne type identity for Lucas bihypernomials)**.** *Let $m\geq 0$, $n\geq 0$ be integers such that $m\geq n$. Then*

$$\begin{aligned}&BhL_m(x)\cdot BhL_{n+1}(x)-BhL_{m+1}(x)\cdot BhL_n(x)\\&=(-1)^n(\alpha(x)-\beta(x))\left(\beta^{m-n}(x)-\alpha^{m-n}(x)\right)\hat{\alpha}(x)\hat{\beta}(x).\end{aligned}$$

Theorem 13. *Let $m \geq 0$, $n \geq 0$ be integers. Then*

$$\begin{aligned}&BhF_m(x) \cdot BhL_n(x) - BhL_m(x) \cdot BhF_n(x)\\&= \frac{2(-1)^n(\alpha^{m-n}(x) - \beta^{m-n}(x))}{\alpha(x) - \beta(x)}\hat{\alpha}(x)\hat{\beta}(x).\end{aligned}$$

Proof. Applying Theorem 7 we have

$$\begin{aligned}&BhF_m(x) \cdot BhL_n(x) - BhL_m(x) \cdot BhF_n(x)\\&= \frac{\alpha^m(x)\alpha^n(x)}{\alpha(x) - \beta(x)}\hat{\alpha}^2(x) + \frac{\alpha^m(x)\beta^n(x)}{\alpha(x) - \beta(x)}\hat{\alpha}(x)\hat{\beta}(x)\\&- \frac{\beta^m(x)\alpha^n(x)}{\alpha(x) - \beta(x)}\hat{\beta}(x)\hat{\alpha}(x) - \frac{\beta^m(x)\beta^n(x)}{\alpha(x) - \beta(x)}\hat{\beta}^2(x)\\&- \frac{\alpha^m(x)\alpha^n(x)}{\alpha(x) - \beta(x)}\hat{\alpha}^2(x) + \frac{\alpha^m(x)\beta^n(x)}{\alpha(x) - \beta(x)}\hat{\alpha}(x)\hat{\beta}(x)\\&- \frac{\beta^m(x)\alpha^n(x)}{\alpha(x) - \beta(x)}\hat{\beta}(x)\hat{\alpha}(x) + \frac{\beta^n(x)\beta^m(x)}{\alpha(x) - \beta(x)}\hat{\beta}^2(x)\\&= \frac{2\alpha^m(x)\beta^n(x) - 2\beta^m(x)\alpha^n(x)}{\alpha(x) - \beta(x)}\hat{\alpha}(x)\hat{\beta}(x)\\&= \frac{2(-1)^n(\alpha^{m-n}(x) - \beta^{m-n}(x))}{\alpha(x) - \beta(x)}\hat{\alpha}(x)\hat{\beta}(x),\end{aligned}$$

so the result follows. ■

Theorem 14 (Convolution identity for Fibonacci bihypernomials)**.** *Let $m \geq 2$, $n \geq 1$ be integers. Then*

$$\begin{aligned}&BhF_{m-1}(x) \cdot BhF_n(x) + BhF_m(x) \cdot BhF_{n+1}(x)\\&= 2BhF_{m+n}(x) + 2j_3BhF_{m+n+3}(x)\\&- F_{m+n}(x) + F_{m+n+2}(x) + F_{m+n+4}(x) - F_{m+n+6}(x).\end{aligned}$$

Proof. Using (4) we have

$$\begin{aligned}&BhF_{m-1}(x) \cdot BhF_n(x) + BhF_m(x) \cdot BhF_{n+1}(x)\\&= F_{m-1}(x)F_n(x) + j_1F_{m-1}(x)F_{n+1}(x) + j_2F_{m-1}(x)F_{n+2}(x)\\&+ j_3F_{m-1}(x)F_{n+3}(x) + j_1F_m(x)F_n(x) + F_m(x)F_{n+1}(x)\\&+ j_3F_m(x)F_{n+2}(x) + j_2F_m(x)F_{n+3}(x) + j_2F_{m+1}(x)F_n(x)\\&+ j_3F_{m+1}(x)F_{n+1}(x) + F_{m+1}(x)F_{n+2}(x) + j_1F_{m+1}(x)F_{n+3}(x)\\&+ j_3F_{m+2}(x)F_n(x) + j_2F_{m+2}(x)F_{n+1}(x) + j_1F_{m+2}(x)F_{n+2}(x)\\&+ F_{m+2}(x)F_{n+3}(x) + F_m(x)F_{n+1}(x) + j_1F_m(x)F_{n+2}(x)\\&+ j_2F_m(x)F_{n+3}(x) + j_3F_m(x)F_{n+4}(x) + j_1F_{m+1}(x)F_{n+1}(x)\end{aligned}$$

$$
\begin{aligned}
&+F_{m+1}(x)F_{n+2}(x)+j_3F_{m+1}(x)F_{n+3}(x)+j_2F_{m+1}(x)F_{n+4}(x)\\
&+j_2F_{m+2}(x)F_{n+1}(x)+j_3F_{m+2}(x)F_{n+2}(x)+F_{m+2}(x)F_{n+3}(x)\\
&+j_1F_{m+2}(x)F_{n+4}(x)+j_3F_{m+3}(x)F_{n+1}(x)+j_2F_{m+3}(x)F_{n+2}(x)\\
&+j_1F_{m+3}(x)F_{n+3}(x)+F_{m+3}(x)F_{n+4}(x)\\
&=F_{m+n}(x)+F_{m+n+2}(x)+F_{m+n+4}(x)+F_{m+n+6}(x)\\
&+2j_1F_{m+n+1}(x)+2j_1F_{m+n+5}(x)+2j_2F_{m+n+2}(x)+2j_2F_{m+n+4}(x)\\
&+2j_3F_{m+n+3}(x)+2j_3F_{m+n+3}(x).
\end{aligned}
$$

Using convolution identity for Fibonacci polynomials (8) we obtain

$$
\begin{aligned}
&BhF_{m-1}(x)\cdot BhF_n(x)+BhF_m(x)\cdot BhF_{n+1}(x)\\
&=2\left(F_{m+n}(x)+j_1F_{m+n+1}(x)+j_2F_{m+n+2}(x)+j_3F_{m+n+3}(x)\right)\\
&+2j_3\left(F_{m+n+3}(x)+j_1F_{m+n+4}(x)+j_2F_{m+n+5}(x)+j_3F_{m+n+6}(x)\right)\\
&-F_{m+n}(x)+F_{m+n+2}(x)+F_{m+n+4}(x)-F_{m+n+6}(x)\\
&=2BhF_{m+n}(x)+2j_3BhF_{m+n+3}(x)\\
&-F_{m+n}(x)+F_{m+n+2}(x)+F_{m+n+4}(x)-F_{m+n+6}(x),
\end{aligned}
$$

which ends the proof. ■

Corollary 15. *Let $n \geq 2$ be an integer. Then*

$$
\begin{aligned}
&(BhF_n(x))^2+(BhF_{n+1}(x))^2\\
&=2BhF_{2n+1}(x)+2j_3BhF_{2n+4}(x)\\
&-F_{2n+1}(x)+F_{2n+3}(x)+F_{2n+5}(x)-F_{2n+7}(x).
\end{aligned}
$$

Next we shall give the generating function for the Fibonacci bihypernomials.

Theorem 16. *The generating function for the Fibonacci bihypernomial sequence $\{BhF_n(x)\}$ is*

$$
G(t)=\frac{j_1+j_2\cdot x+j_3\cdot(x^2+1)+(1+j_2+j_3\cdot x)t}{1-xt-t^2}.
$$

Proof. Assume that the generating function of the Fibonacci bihypernomial sequence $\{BhF_n(x)\}$ has the form $G(t)=\sum_{n=0}^{\infty} BhF_n(x)t^n$. Then

$$
G(t)=BhF_0(x)+BhF_1(x)t+BhF_2(x)t^2+\dots
$$

Multiply the above equality on both sides by $-xt$ and then by $-t^2$ we obtain

$$-G(t)xt = -BhF_0(x)xt - BhF_1(x)xt^2 - BhF_2(x)xt^3 - \dots$$
$$-G(t)t^2 = -BhF_0(x)t^2 - BhF_1(x)t^3 - BhF_2(x)t^4 - \dots$$

By adding the three equalities above, we will get

$$G(t)(1 - xt - t^2) = BhF_0(x) + (BhF_1(x) - BhF_0(x)x)t$$

since $BhF_n(x) = x \cdot BhF_{n-1}(x) + BhF_{n-2}(x)$ (see (11)) and the coefficients of t^n for $n \geq 2$ are equal to zero. Moreover, $BhF_0(x) = j_1 + j_2 \cdot x + j_3 \cdot (x^2 + 1)$, $BhF_1(x) - BhF_0(x)x = 1 + j_2 + j_3 \cdot x$, and the result follows. ■

In the same way we obtain the next theorem.

Theorem 17. *The generating function for the Lucas bihypernomial sequence* $\{BhF_n(x)\}$ *is*

$$g(t) = \frac{BhL_0(x) + (BhL_1(x) - BhL_0(x)x)t}{1 - xt - t^2},$$

where $BhL_0(x) = 2 + j_1 \cdot x + j_2 \cdot (x^2 + 2) + j_3 \cdot (x^3 + 3x)$
and $BhL_1(x) - BhL_0(x)x = -x + 2j_1 + j_2 \cdot x + j_3 \cdot (-3x^2 + 4x + 2)$.

Theorem 18. *Let* $n \geq 2$ *be an integer. Then*

$$\sum_{l=1}^{n-1} BhF_l(x) = \frac{BhF_n(x) + BhF_{n-1}(x) - BhF_0(x) - BhF_1(x)}{x}.$$

Proof. Let consider the sum $\sum_{l=1}^{n-1} BhF_l(x)$. Then

$$\begin{aligned}
\sum_{l=1}^{n-1} BhF_l(x) &= BhF_1(x) + BhF_2(x) + \cdots + BhF_{n-1}(x) \\
&= F_1(x) + j_1F_2(x) + j_2F_3(x) + j_3F_4(x) \\
&+ F_2(x) + j_1F_3(x) + j_2F_4(x) + j_3F_5(x) + \cdots \\
&+ F_{n-1}(x) + j_1F_n(x) + j_2F_{n+1}(x) + j_3F_{n+2}(x) \\
&= F_1(x) + F_2(x) + \cdots + F_{n-1}(x) \\
&+ j_1(F_2(x) + F_3(x) + \cdots + F_n(x) + F_1(x) - F_1(x)) \\
&+ j_2(F_3(x) + F_4(x) + \cdots + F_{n+1}(x) + F_1(x) + F_2(x) - F_1(x) - F_2(x)) \\
&+ j_3(F_4(x) + F_5(x) + \cdots + F_{n+2}(x) + F_1(x) + F_2(x) + F_3(x) \\
&- F_1(x) - F_2(x) - F_3(x)).
\end{aligned}$$

Using (9) we have

$$\sum_{l=1}^{n-1} BhF_l(x) = \frac{F_n(x)+F_{n-1}(x)-1}{x}$$
$$+j_1\left(\frac{F_{n+1}(x)+F_n(x)-1-x}{x}\right)$$
$$+j_2\left(\frac{F_{n+2}(x)+F_{n+1}(x)-1-x-x^2}{x}\right)$$
$$+j_3\left(\frac{F_{n+3}(x)+F_{n+2}(x)-1-x-x^2-x(x^2+1)}{x}\right)$$
$$=\frac{F_n(x)+j_1F_{n+1}(x)+j_2F_{n+2}(x)+j_3F_{n+3}(x)}{x}$$
$$+\frac{F_{n-1}(x)+j_1F_n(x)+j_2F_{n+1}(x)+j_3F_{n+2}(x)}{x}$$
$$+\frac{-(0+1)-j_1(1+x)-j_2(x+(x^2+1))-j_3((x^2+1)+(x^3+2x))}{x}$$
$$=\frac{BhF_n(x)+BhF_{n-1}(x)-BhF_0(x)-BhF_1(x)}{x},$$

thus the Theorem is proved. ■

Theorem 19. *Let $n \geq 2$ be an integer. Then*

$$\sum_{l=1}^{n-1} BhL_l(x) = \frac{BhL_n(x)+BhL_{n-1}(x)-BhL_0(x)-BhL_1(x)}{x}.$$

Proof. Using (10) and proceeding in the same way as in the Theorem 18 the result follows. ■

Let consider the matrix generator of the Fibonacci bihypernomials.

Theorem 20. *Let $n \geq 0$ be an integer. Then*

$$\begin{bmatrix} BhF_{n+2}(x) & BhF_{n+1}(x) \\ BhF_{n+1}(x) & BhF_n(x) \end{bmatrix} = \begin{bmatrix} BhF_2(x) & BhF_1(x) \\ BhF_1(x) & BhF_0(x) \end{bmatrix} \cdot \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}^n.$$

Proof. (by induction on n)
If $n = 0$ then assuming that the matrix to the power 0 is the identity matrix the result is obvious. Now assume that for any $n \geq 0$ holds

$$\begin{bmatrix} BhF_{n+2}(x) & BhF_{n+1}(x) \\ BhF_{n+1}(x) & BhF_n(x) \end{bmatrix} = \begin{bmatrix} BhF_2(x) & BhF_1(x) \\ BhF_1(x) & BhF_0(x) \end{bmatrix} \cdot \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}^n.$$

We shall show that

$$\begin{bmatrix} BhF_{n+3}(x) & BhF_{n+2}(x) \\ BhF_{n+2}(x) & BhF_{n+1}(x) \end{bmatrix} = \begin{bmatrix} BhF_2(x) & BhF_1(x) \\ BhF_1(x) & BhF_0(x) \end{bmatrix} \cdot \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}^{n+1}.$$

By simple calculation using induction's hypothesis we have

$$\begin{aligned} &\begin{bmatrix} BhF_2(x) & BhF_1(x) \\ BhF_1(x) & BhF_0(x) \end{bmatrix} \cdot \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}^{n} \cdot \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} BhF_{n+2}(x) & BhF_{n+1}(x) \\ BhF_{n+1}(x) & BhF_{n}(x) \end{bmatrix} \cdot \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} x \cdot BhF_{n+2}(x) + BhF_{n+1}(x) & BhF_{n+2}(x) \\ x \cdot BhF_{n+1}(x) + BhF_{n}(x) & BhF_{n+1}(x) \end{bmatrix} \\ &= \begin{bmatrix} BhF_{n+3}(x) & BhF_{n+2}(x) \\ BhF_{n+2}(x) & BhF_{n+1}(x) \end{bmatrix}, \end{aligned}$$

which ends the proof. ■

In the same way we obtain the matrix generator for the Lucas bihypernomials.

Theorem 21. *Let $n \geq 0$ be an integer. Then*

$$\begin{bmatrix} BhL_{n+2}(x) & BhL_{n+1}(x) \\ BhL_{n+1}(x) & BhL_{n}(x) \end{bmatrix} = \begin{bmatrix} BhL_2(x) & BhL_1(x) \\ BhL_1(x) & BhL_0(x) \end{bmatrix} \cdot \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}^{n}.$$

Due to the commutation of multiplication determinant properties can be used in the set of bihypernomials the Cassini type formula (18) for Fibonacci bihypernomials follows.

For integer $n \geq 1$ we have

$$\begin{aligned} &BhF_{n+1}(x) \cdot BhF_{n-1}(x) - (BhF_n(x))^2 \\ &= \left(BhF_2(x) \cdot BhF_0(x) - (BhF_1(x))^2\right) \cdot (-1)^{n-1} \\ &= \Big[\big(x + j_1 \cdot (x^2+1) + j_2 \cdot (x^3+2x) + j_3 \cdot (x^4+3x^2+1)\big) \\ &\quad \cdot \big(j_1 + j_2 \cdot x + j_3 \cdot (x^2+1)\big) \\ &- \big(1 + j_1 \cdot x + j_2 \cdot (x^2+1) + j_3 \cdot (x^3+2x)\big)^2\Big] \cdot (-1)^{n-1} \\ &= \big(j_1 \cdot (-2x) + j_3 \cdot \left(-x^3 - 2x\right)\big) \cdot (-1)^{n-1} \\ &= (-1)^n \left(j_1 \cdot 2x + j_3 \cdot \left(x^3 + 2x\right)\right). \end{aligned}$$

In the same way we can obtain the Cassini type formula (19) for the Lucas bihypernomials.

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