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RADICALS OF GENERALIZED PRIME IDEALS IN TERNARY SEMIGROUPS

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Abstract

In this paper, the concepts of f-prime ideals and f-semiprime ideals on a ternary semigroup are considered as a generalization of pseudo prime ideals and pseudo semiprime ideals, respectively. Then such ideals introduced are used to describe left (respectively, right) f-primary ideals on a ternary semigroup.

Keywords: ternary semigroup, pseudo prime ideal, good mapping, f-prime ideal, left f-primary decomposition.

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1. Introduction

A *ternary semigroup* is a nonempty set T together with a ternary operation $[\ ] : T \times T \times T \to T$ written as $(a, b, c) \to [abc]$ satisfying the associative law of the first kind:

$$[[abc]uv] = [a[bcu]v] = [ab[cuv]]$$

for all $a, b, c, u, v \in T$. Yet prior, the structures was contemplated by Kasner [9] who give the possibility of n-ary algebras. In [6], Hewitt and Zuckerman studied the concept of ternary semigroups and showed that every semigroup can be considered as a ternary semigroup but there is an example of a ternary semigroup which does not reduce to a semigroup. Recently, ternary semigroups have been widely studied (see, [1, 2, 3, 4, 11, 13]).

Let $(T, [\,])$ be a ternary semigroup. For nonempty subsets A, B, C of T, the set product $[ABC]$ is defined by

$$[ABC] = \{[abc] \mid a \in A,\ b \in B,\ c \in C\}.$$

If $A = \{a\}$, we write $[ABC]$ as $[aBC]$. The cases $[A\{b\}C]$ and $[AB\{c\}]$ can be written similarly.

For convenience, we shall write T for a ternary semigroup $(T, [\,])$, and write ABC for the product $[ABC]$.

Let T be a ternary semigroup. A nonempty subset A of T is said to be

(1) a *left ideal* of T if $TTA \subseteq A$;

(2) a *right ideal* of T if $ATT \subseteq A$;

(3) a *lateral ideal* (or *middle ideal*) of T if $TAT \subseteq A$;

(4) a *two-sided ideal* of T if it is both a left and a right ideal of T;

(5) an *ideal* of T if it is a left, a right and a lateral ideal of T.

For any $a \in T$, it is clear that the intersection of all ideals of T containing a is an ideal of T. It is called the *principal ideal* of T generated by a and denoted by (a).

There are many authors considered m-systems and n-systems on semigroups being related to pseudo prime ideals and pseudo semiprime ideals of the semigroups. By amplifying these concepts, Rao and Sarala [10] introduced an m-system and an n-system on ternary semigroups and then presented properties of pseudo prime ideals and pseudo semiprime ideals on ternary semigroups by these systems. The following definitions we refer to [10] (see also [7]).

Definition. Let T be a ternary semigroup and P a proper ideal of T. Then P is said to be a *pseudo prime ideal* of T if for any ideals A, B, C of T,

$$ABC \subseteq P \text{ implies } A \subseteq P, B \subseteq P \text{ or } C \subseteq P.$$

Lemma 1 [7]. *Let P be a proper ideal of a ternary semigroup T. Then P is pseudo prime if and only if for any $a, b, c \in T$, $aTbTc \subseteq P$, $aTTbTTc \subseteq P$, $TaTbTTc \subseteq P$ and $aTTbTcT \subseteq P$ imply $a \in P, b \in P$ or $c \in P$.*

Definition. Let T be a ternary semigroup and P a proper ideal of T. Then P is said to be a *pseudo semiprime* ideal of T if for any ideal A of T,

$$AAA \subseteq P \text{ implies } A \subseteq P.$$

Remark 2. Let P be a proper ideal of a ternary semigroup T. Then P is pseudo semiprime if and only if for any $a \in T$, $aTaTa \subseteq P$, $aTTaTTa \subseteq P$, $TaTaTTa \subseteq P$ and $aTTaTaT \subseteq P$ imply $a \in P$.

Definition. Let T be a ternary semigroup. Then a subset A of T is called an *m-system* of T if for any $a, b, c \in A$,

$$(aTbTc \cup aTTbTTc \cup TaTbTTc \cup aTTbTcT) \cap A \neq \emptyset.$$

Definition. Let T be a ternary semigroup. Then a subset A of T is called an *n-system* of T if for any $a \in A$,

$$(aTaTa \cup aTTaTTa \cup TaTaTTa \cup aTTaTaT) \cap A \neq \emptyset.$$

2. f-PRIME IDEALS AND f-PRIME RADICALS

The notion of f-prime ideals on a ring was introduced to be a generalization of prime ideals by Murata, Kurata and Marubayashi [8]. Then such concept was extended to semirings by Sardar and Goswami [12]. After that, it appear on ordered algebraic structures having one operation, called ordered semigroups, by Gu [5]. In this section, we extend this concept to ternary semigroups.

Definition. Let $A(T)$ denote the set of all ideals of a ternary semigroup T. A mapping $f : T \longrightarrow A(T)$ is called a *good mapping* on T if

(i) $\forall a \in T, a \in f(a)$;

(ii) $\forall a \in T \ \forall A \in A(T), x \in f(a) \cup A \Rightarrow f(x) \subseteq f(a) \cup A$.

Definition. Let f be a good mapping on a ternary semigroup T. A subset F of T is called an *f-system* of T if F contains an m-system F^* of T such that

$$\forall t \in F, f(t) \cap F^* \neq \emptyset.$$

Here, F^* is called the *kernel* of F.

Remark 3. Every m-system of a ternary semigroup T is an f-system of T with kernel itself.

Definition. Let T be a ternary semigroup and f a good mapping on T. An ideal A of T is called an *f-prime ideal* of T if its complement $C(A)$ is an f-system.

Remark 4. Let T be a ternary semigroup. Then the following statements hold:

(1) if P is a pseudo prime ideal of T, then $C(P)$ is an m-system of T;

(2) if P is a pseudo prime ideal of T, then P is an f-prime ideal of T.

Proof. (1) Assume that P is a pseudo prime ideal of T and let $a, b, c \in C(P)$. Since P is pseudo prime,

$$aTbTc \nsubseteq P,\ aTTbTTc \nsubseteq P,\ TaTbTTc \nsubseteq P \text{ or } aTTbTcT \nsubseteq P.$$

Thus

$$(aTbTc \cup aTTbTTc \cup TaTbTTc \cup aTTbTcT) \cap C(P) \neq \emptyset.$$

Therefore $C(P)$ is an m-system of T.

(2) It is obtained directly from (1) and Remark 3. ■

Lemma 5. *Let P be an f-prime ideal of a ternary semigroup T. For any $a, b, c \in T$, if*

$$f(a)Tf(b)Tf(c) \subseteq P,\ Tf(a)Tf(b)TTf(c) \subseteq P \text{ and } f(a)TTf(b)Tf(c)T \subseteq P,$$

then $a \in P, b \in P$ or $c \in P$.

Proof. Let $a, b, c \in T$. Suppose contrary that $a, b, c \in C(P)$. Since $C(P)$ is an f-system of T,

$$f(a) \cap (C(P))^* \neq \emptyset, f(b) \cap (C(P))^* \neq \emptyset \text{ and } f(c) \cap (C(P))^* \neq \emptyset.$$

Let $x_1 \in f(a) \cap (C(P))^*, x_2 \in f(b) \cap (C(P))^*$ and $x_3 \in f(c) \cap (C(P))^*$. Since $(C(P))^*$ is an m-system of T,

$$(x_1Tx_2Tx_3 \cup x_1TTx_2TTx_3 \cup Tx_1Tx_2TTx_3 \cup x_1TTx_2Tx_3T) \cap (C(P))^* \neq \emptyset.$$

Consequently,

$$x_1Tx_2Tx_3 \nsubseteq P, x_1TTx_2TTx_3 \nsubseteq P, Tx_1Tx_2TTx_3 \nsubseteq P, \text{ or } x_1TTx_2Tx_3T \nsubseteq P.$$ ■

Definition. Let T be a ternary semigroup and

$$fS(T) = \text{the set of all } f\text{-systems of } T;$$
$$fPI(T) = \text{the set of all } f\text{-prime ideals of } T.$$

For any ideal A of T, the *f-prime radicals* of A is defined by

$$r_f(A) = \{a \in T \mid \forall F \in fS(T), a \in F \Rightarrow F \cap A \neq \emptyset\}.$$

Theorem 6. *Let T be a ternary semigroup and A be an ideal of T. Then*

$$r_f(A) = \bigcap \{P \mid A \subseteq P \in fPI(T)\}.$$

Proof. Suppose that $x \notin \bigcap\{P \mid A \subseteq P \in fPI(T)\}$. Then $x \notin P$ for some $P \in fPI(T)$ with $A \subseteq P$. Since $C(P)$ is an f-system of T and $C(P) \cap A = \emptyset$, then $x \notin r_f(A)$. Hence $r_f(A) \subseteq \bigcap\{P \mid A \subseteq P \in fPI(T)\}$. On the other hand, suppose that $x \notin r_f(A)$. Then there exists $F \in fS(T)$ such that $x \in F$ and $F \cap A = \emptyset$. Thus $C(F)$ is an f-prime ideal of T containing A. Then $x \notin C(F)$ implies $x \notin \bigcap\{P \mid A \subseteq P \in fPI(T)\}$. Therefore

$$\bigcap\{P \mid A \subseteq P \in fPI(T)\} \subseteq r_f(A).$$

■

3. f-SEMIPRIME IDEALS

Definition. Let T be a ternary semigroup. A subset E of T is said to be an *fn-system* of T if

$$E = \bigcup_{i \in \Gamma} F_i$$

where $\{F_i \mid i \in \Gamma\} \subseteq fS(T)$.

Definition. An ideal P of a ternary semigroup T is said to be an *f-semiprime ideal* of T if its complement $C(P)$ in T is an fn-system of T.

Remark 7. Let T be a ternary semigroup. Then the following statements hold:

(1) every f-system of T is an fn-system of T;

(2) every f-prime ideal of T is an f-semiprime ideal of T.

The following example shows that f-prime need not be pseudo prime and f-semiprime need not be pseudo semiprime.

Example 8. Consider the ternary semigroup $T = \{a, b, c\}$ defined by:

a	a	b	c
a	a	a	a
b	a	a	a
c	a	a	a

b	a	b	c
a	a	a	a
b	a	a	a
c	a	a	b

c	a	b	c
a	a	a	a
b	a	a	b
c	a	b	c

The ideals of T are $\{a\}, \{a, b\}$ and T.

Define $f(a) = \{a\}, f(b) = f(c) = T$; then f is a good mapping. Let $A = \{a\}$. Then $C(A) = \{b, c\}$ is an f-system with kernel $F^* = \{c\}$. Hence, A is f-prime and f-semiprime. However, A is not pseudo prime and not pseudo semiprime. Indeed $\{a, b\}\{a, b\}\{a, b\} \subseteq A$, but $\{a, b\} \nsubseteq A$.

Lemma 9. *Let T be a ternary semigroup. Then the following statements hold:*

(1) *if P is a pseudo semiprime ideal of T, then $C(P)$ is an n-system of T;*

(2) *if N is an n-system of T, then N is the union of some m-systems of T.*

Proof. (1) Assume that P is a pseudo semiprime ideal of T. Let $a \in C(P)$. Then $aTaTa \nsubseteq P$, $aTTaTTa \nsubseteq P$, $TaTaTTa \nsubseteq P$ or $aTTaTaT \nsubseteq P$. Thus,

$$(aTaTa \cup aTTaTTa \cup TaTaTTa \cup aTTaTaT) \cap C(P) \neq \emptyset.$$

(2) Assume that N is an n-system of T. Let $a \in N$. Then

$$(aTaTa \cup aTTaTTa \cup TaTaTTa \cup aTTaTaT) \cap N \neq \emptyset.$$

Given

$$a_1 \in (aTaTa \cup aTTaTTa \cup TaTaTTa \cup aTTaTaT) \cap N.$$

We have that

$$(a_1Ta_1Ta_1 \cup a_1TTa_1TTa_1 \cup Ta_1Ta_1TTa_1 \cup a_1TTa_1Ta_1T) \cap N \neq \emptyset.$$

Then there exists

$$a_2 \in (a_1Ta_1Ta_1 \cup a_1TTa_1TTa_1 \cup Ta_1Ta_1TTa_1 \cup a_1TTa_1Ta_1T) \cap N.$$

Continue in the same manner, we obtain the set

$$M_a = \{a_0 = a, a_1, a_2, \ldots\}.$$

Next, we will show that M_a is an m-system of T. Let $a_i, a_j, a_k \in M_a$.

If $i = \max\{i, j, k\}$, we have

$$\begin{aligned}
&a_{i+1} \in a_iTa_iTa_i \cup a_iTTa_iTTa_i \cup Ta_iTa_iTTa_i \cup a_iTTa_iTa_iT \\
&\subseteq a_iTa_{i-1}Ta_{i-1} \cup a_iTTa_{i-1}TTa_{i-1} \cup Ta_iTa_{i-1}TTa_{i-1} \cup a_iTTa_{i-1}Ta_{i-1}T \\
&\vdots \\
&\subseteq a_iTa_jTa_k \cup a_iTTa_jTTa_k \cup Ta_iTa_jTTa_k \cup a_iTTa_jTa_kT.
\end{aligned}$$

If $j = \max\{i, j, k\}$ or $k = \max\{i, j, k\}$, we obtain that

$$a_{j+1} \in a_iTa_jTa_k \cup a_iTTa_jTTa_k \cup Ta_iTa_jTTa_k \cup a_iTTa_jTa_kT$$

and

$$a_{k+1} \in a_iTa_jTa_k \cup a_iTTa_jTTa_k \cup Ta_iTa_jTTa_k \cup a_iTTa_jTa_kT,$$

respectively. Thus

$$(a_iTa_jTa_k \cup a_iTTa_jTTa_k \cup Ta_iTa_jTTa_k \cup a_iTTa_jTa_kT) \cap M_a \neq \emptyset.$$

Therefore M_a is an m-system of T. Hence

$$N = \bigcup_{a \in N} M_a$$

is the union of m-systems of T. ∎

Lemma 10. *Let T be a ternary semigroup. If A is a pseudo semiprime ideal of T, then A is an f-semiprime ideal of T.*

Proof. Assume that A is a pseudo semiprime ideal of T. By Lemma 9(1) and (2), $C(A)$ is the union of some m-systems of T. By Remark 3, $C(A)$ is the union of some f-systems of T. Hence $C(A)$ is an fn-system of T. Therefore A is an f-semiprime ideal of T. ■

Lemma 11. *Let P be an f-semiprime ideal of a ternary semigroup T and $a \in T$. If $f(a)Tf(a)Tf(a) \subseteq P$, $Tf(a)Tf(a)TTf(a) \subseteq P$ and $f(a)TTf(a)Tf(a)T \subseteq P$, then $a \in P$.*

Proof. Given

$$C(P) = \bigcup_{i \in \Gamma} F_i,$$

where $\{F_i \mid i \in \Gamma\} \subseteq fS(T)$. Let $a \in C(P)$. Then $a \in F_i$ for some $i \in \Gamma$. Since F_i is an f-system of T, it follows that $f(a) \cap F_i^* \neq \emptyset$. Let $x \in f(a) \cap F_i^*$. Since F_i^* is an m-system of T, we obtain

$$\emptyset \neq (xTxTx \cup xTTxTTx \cup TxTxTTx \cup xTTxTxT) \cap F_i^* \subseteq F_i^* \subseteq F_i \subseteq C(P).$$

These imply that $xTxTx \nsubseteq P, xTTxTTx \nsubseteq P, TxTxTTx \nsubseteq P$ or $xTTxTxT \nsubseteq P$. This proof is complete. ■

Lemma 12. *Let P be an f-semiprime ideal of a ternary semigroup T. Then $r_f(P) = P$.*

Proof. It is clear that $P \subseteq r_f(P)$. We will show that $r_f(P) \subseteq P$, by proving that $C(P) \subseteq C(r_f(P))$. Let $x \in C(P)$. Since $C(P)$ is an fn-system of T,

$$C(P) = \bigcup_{i \in \Gamma} F_i,$$

where $\{F_i \mid i \in \Gamma\} \subseteq fS(T)$. This implies $x \in F_i$ for some $i \in \Gamma$. Since $x \notin P$ and F_i is an f-system of T such that $F_i \cap P = \emptyset$, then $x \notin r_f(P)$. Thus $x \in C(r_f(P))$. The proof is complete. ■

Theorem 13. *Let A be an ideal of a ternary semigroup T. Then $r_f(A)$ is the smallest f-semiprime ideal of T containing A.*

Proof. It is clear that $A \subseteq r_f(A)$. By Theorem 6,

$$r_f(A) = \bigcap \{P \mid A \subseteq P \in fPI(T)\}.$$

Then

$$C(r_f(A)) = \bigcup \{C(P) \mid A \subseteq P \in fPI(T)\}.$$

Thus $C(r_f(A))$ is an fn-system of T. Therefore $r_f(A)$ is an f-semiprime ideal of T. Let P be an f-semiprime ideal of T containing A. Then

$$r_f(A) \subseteq r_f(P) = P.$$

Thus $r_f(A)$ is the least f-semiprime ideal of T containing A. ■

Definition. Let f be a good mapping of a ternary semigroup T. For $a \in T$ and $A \in A(T)$, define the set $A : a$ to be

$$A : a := \{x \in T \mid f(a)Tf(a)Tf(x) \subseteq A, Tf(a)Tf(a)TTf(x) \text{ and } f(a)TTf(a)Tf(x)T \subseteq A\}.$$

This set is called the *left f-quotient* of A by a. Moreover, for any $B \in A(T)$, the *left f-quotient* of A by B is defined by

$$A : B = \bigcap_{b \in B} (A : b).$$

We note that $A : a$ may be empty. See the following example.

Example 14. We consider the ternary semigroup T of Example 8. If we define $f(a) = f(b) = f(c) = T$, then the mapping f is a good mapping. Let $A = \{a\}$. Then $A : a, A : b$ and $A : c$ are all empty.

Lemma 15. *Let f be a good mapping on a ternary semigroup T. Let $A, A_1, A_2, B, B_1, B_2 \in A(T)$ and $a \in T$.*

(1) $A_1 \subseteq A_2 \Rightarrow A_1 : a \subseteq A_2 : a$.

(2) $A_1 \subseteq A_2 \Rightarrow A_1 : B \subseteq A_2 : B$.

(3) $B_1 \subseteq B_2 \Rightarrow A : B_1 \supseteq A : B_2$.

(4) $(A_1 \cap A_2) : a = (A_1 : a) \cap (A_2 : a)$.

(5) $(A_1 \cap A_2) : B = (A_1 : B) \cap (A_2 : B)$.

Lemma 16. *Let f be a good mapping on a ternary semigroup T. For any $A \in A(T)$ and $a \in T$, $A : a$ is either empty or an ideal of T containing A.*

Proof. Let A be an ideal of T and $a \in T$. Suppose that $A : a \neq \emptyset$. We will show that $A : a$ is an ideal of T. Let $x \in A : a$ and $r_1, r_2 \in T$. Then the following inclusions hold:

$$f(a)Tf(a)Tf(xr_1r_2) \subseteq f(a)Tf(a)Tf(x) \subseteq A;$$
$$Tf(a)Tf(a)TTf(xr_1r_2) \subseteq Tf(a)Tf(a)TTf(x) \subseteq A;$$
$$f(a)TTf(a)Tf(xr_1r_2)T \subseteq f(a)TTf(a)Tf(x)T \subseteq A.$$

Thus $xr_1r_2 \in A : a$. We can show on the same way that $r_1xr_2 \in A : a$ and $r_1r_2x \in A : a$. Let $y \in A : a$ and $a' \in A$. Then $f(a)Tf(a)Tf(a') \subseteq f(a)Tf(a)T(f(y) \cup A) = f(a)Tf(a)Tf(y) \cup f(a)Tf(a)TA \subseteq A$.

Similarly,

$$Tf(a)Tf(a)TTf(a') \subseteq A$$

and

$$f(a)TTf(a)Tf(a')T \subseteq A.$$

Thus $a' \in A : a$. Therefore $A : a$ is an ideal of T containing A. ■

Let f be a good mapping on a ternary semigroup T. Denote the following condition by (α):

$$\forall F \in fS(T) \forall A \in A(T), F \cap A \neq \emptyset \Rightarrow F^* \cap A \neq \emptyset.$$

Remark 17. If $f(a) = (a)$ for every $a \in T$, then T satisfies the condition (α).

Proof. Assume $f(a) = (a)$ for every $a \in T$. Let $F \in fS(T)$ and $A \in A(T)$ such that $F \cap A \neq \emptyset$. Let $a \in F \cap A$. Since F is an f-system, $F^* \cap f(a) \neq \emptyset$. By assumption,

$$\emptyset \neq F^* \cap f(a) = F^* \cap (a) \subseteq F^* \cap A.$$ ■

Lemma 18. *Let f be a good mapping on a ternary semigroup T and $A, B \in A(T)$. Then the following statements hold:*

(1) $A \subseteq B \Rightarrow r_f(A) \subseteq r_f(B)$;

(2) $r_f(r_f(A)) = r_f(A)$;

(3) *if T satisfies the condition (α), then* $r_f(A \cap B) = r_f(A) \cap r_f(B)$.

Proof. (1) Assume that $A \subseteq B$. Then

$$r_f(A) = \bigcap \{P \mid A \subseteq P \in fPI(T)\} \subseteq \bigcap \{P \mid B \subseteq P \in fPI(T)\} = r_f(B).$$

(2) It is clear that $r_f(A) \subseteq r_f(r_f(A))$. Let $a \in r_f(r_f(A))$ and F be an f-system of T containing a. Then

$$F \cap r_f(A) \neq \emptyset.$$

This implies

$$F \cap A \neq \emptyset.$$

Thus $a \in r_f(A)$.

(3) Assume that T satisfies the condition (α). It is clear by (1) that $r_f(A \cap B) \subseteq r_f(A) \cap r_f(B)$. To prove the composite inclusion, let $x \in r_f(A) \cap r_f(B)$. For any f-system F of T containing x, we have that $F \cap A \neq \emptyset$ and $F \cap B \neq \emptyset$. By the condition (α),

$$F^* \cap A \neq \emptyset \text{ and } F^* \cap B \neq \emptyset.$$

Let $a \in F^* \cap A$ and $b \in F^* \cap B$. Since F^* is an m-system of T,

$$\begin{aligned} \emptyset \neq (aTF^*Tb \cup aTTF^*TTb \cup TaTF^*TTb \cup aTTF^*TbT) \cap F^* \\ \subseteq (A \cap B) \cap F. \end{aligned}$$

Thus $x \in r_f(A \cap B)$. ∎

4. f-left primary decompositions

Definition. Let f be a good mapping on a ternary semigroup T. An ideal P of T is called *left f-primary* if, for $a, b, c \in T$,

$$f(a)Tf(b)Tf(c) \subseteq P,\ Tf(a)Tf(b)TTf(c) \subseteq P \text{ and } f(a)TTf(b)Tf(c)T \subseteq P$$

imply

$$a \in r_f(P), b \in r_f(P) \text{ or } c \in P.$$

Remark 19. Every f-prime ideal of a ternary semigroup T is a left f-primary ideal of T.

Theorem 20. *Let T be a ternary semigroup satisfying the condition (α). If P_1 and P_2 are left f-primary ideals of T such that $r_f(P_1) = r_f(P_2)$, then $P = P_1 \cap P_2$ is also a left f-primary ideal of T such that $r_f(P) = r_f(P_1) = r_f(P_2)$.*

Proof. Assume that P_1 and P_2 are left f-primary ideals of T such that $r_f(P_1) = r_f(P_2)$. Let $P = P_1 \cap P_2$. Then $\emptyset \neq P_1TP_2 \subseteq P_1 \cap P_2$ and

$$r_f(P) = r_f(P_1 \cap P_2) = r_f(P_1) \cap r_f(P_2) = r_f(P_1) \cap r_f(P_1) = r_f(P_1).$$

Let $a, b, c \in T$ be such that $f(a)Tf(b)Tf(c) \subseteq P$, $Tf(a)Tf(b)TTf(c) \subseteq P$ and $f(a)TTf(b)Tf(c) \subseteq P$. Since P_1 and P_2 are left f-primary, $a \in r_f(P), b \in r_f(P)$ or $c \in P_1 \cap P_2 = P$. Thus P is a left f-primary ideal of T. ∎

Let T be a ternary semigroup and f a good mapping on T. Denote the following condition by (β):

$$\forall A, B \in A(T), B \not\subseteq r_f(A) \Rightarrow A : B \neq \emptyset.$$

Theorem 21. *Let T be a ternary semigroup satisfying the condition (β). If an ideal A of T is left f-primary, then $A : B = A$ for every ideal $B \not\subseteq r_f(A)$.*

Proof. Assume that A is a left f-primary ideal of T. Let B be an ideal of T not contained in $r_f(A)$. Since $A : B \neq \emptyset$,

$$A : b \neq \emptyset$$

for all $b \in B$. Therefore $A \subseteq A : b$ for all $b \in B$. Hence $A \subseteq A : B$. To show the opposite inclusion, let $a \in A : B$ and $c \in B \setminus r_f(A)$. Then $A : c \neq \emptyset$ and

$$f(c)Tf(c)Tf(a) \subseteq A,\ Tf(c)Tf(c)TTf(a) \subseteq A \text{ and } f(c)TTf(c)Tf(a)T \subseteq A.$$

Since A is left f-primary and $c \notin r_f(A)$, then $a \in A$. Thus $A : B \subseteq A$. ∎

Definition. If an ideal P of a ternary semigroup T can be written as

$$P = P_1 \cap P_2 \cap \cdots \cap P_n$$

where each P_i is a left f-primary ideal, then this is called a *left f-primary decomposition* of T and each P_i is called the *left f-primary component* of the decomposition.

Definition. Let $P = \bigcap_{i \in \mathcal{I}} P_i$ be a left f-primary decomposition of a ternary semigroup T. Then P is called *irredundant* if

$$\bigcap_{i \in \mathcal{I} \setminus \{j\}} P_i \not\subseteq P_j$$

for all $j \in \mathcal{I}$. Moreover, an irredundant left f-primary decomposition is called a *normal decomposition* if

$$r_f(P_i) \neq r_f(P_j)$$

for all $i, j \in \mathcal{I}$ such that $i \neq j$.

Let f be a good mapping on a ternary semigroup T. Denote the following condition by (γ) :

for any left f-primary ideal P of T, we have $P : P = T$.

Remark 22. Let T be a ternary semigroup. If $f(a) = (a)$ for all $a \in T$, then T satisfies the condition (γ).

Proof. Let P be a left f-primary ideal of T. For any $x \in T$ and $a \in P$,

$$f(a)Tf(a)Tf(x) = (a)T(a)T(x) \subseteq P;$$
$$Tf(a)Tf(a)TTf(x) = T(a)T(a)TT(x) \subseteq P;$$
$$f(a)TTf(a)Tf(x)T = (a)TT(a)T(x)T \subseteq P.$$

Hence $T \subseteq P : P$. ∎

Theorem 23. *Let T be a ternary semigroup satisfying the conditions $(\alpha), (\beta)$ and (γ). If an ideal K of T has two normal left f-primary decompositions*

$$K = \bigcap_{i=1}^{n} P_i = \bigcap_{i=1}^{m} Q_i,$$

then $n = m$ and $r_f(P_i) = r_f(Q_i)$ for $1 \leq i \leq n = m$ by a suitable ordering.

Proof. This proof is a modification of the proof of Theorem 4.7 in [5]. It is easy to see that the result holds in the case $K = T$. Next we assume that $K \neq T$, where all left f-primary components $P_1, \ldots, P_n, Q_1, \ldots, Q_m$ are proper ideals of T. We may assume that $r_f(P_1)$ is maximal in the set

$$\{r_f(P_1), \ldots, r_f(P_n), r_f(Q_1), \ldots, r_f(Q_m)\}.$$

Now we prove that $r_f(P_1) = r_f(Q_i)$ for some $1 \leq i \leq m$. It is enough to show that $P_1 \subseteq r_f(Q_i)$. Suppose that $P_1 \not\subseteq r_f(Q_i)$ for all $1 \leq i \leq m$. Then, by Theorem 21, we have

$$Q_i : P_1 = Q_i$$

for all $1 \leq i \leq m$. Then $K : P_1 = (Q_1 \cap Q_2 \cap \cdots \cap Q_m) : P_1 = (Q_1 : P_1) \cap (Q_2 : P_1) \cap \cdots \cap (Q_m : P_1) = Q_1 \cap Q_2 \cap \cdots \cap Q_m = K$.

Case 1. $n = 1$. By the condition (γ), we obtain

$$T = P_1 : P_1 = K : P_1 = K,$$

which is a contradiction.

Case 2. $n > 1$. By the condition (γ) and the fact that $P_1 \not\subseteq r_f(P_i)$ for all $2 \leq i \leq n$, we have

$K = K : P_1 = (P_1 \cap P_2 \cap \cdots \cap P_n) : P_1 = (P_1 : P_1) \cap (P_2 : P_1) \cap \cdots \cap (P_n : P_1) = T \cap (P_2 : P_1) \cap \cdots \cap (P_n : P_1) = (P_2 : P_1) \cap \cdots \cap (P_n : P_1) = P_2 \cap P_3 \cap \cdots \cap P_n$.

This is also a contradiction. Thus, $r_f(P_1) \subseteq r_f(Q_i)$ for some $1 \leq i \leq m$. By a suitable ordering, we assume $r_f(P_1) = r_f(Q_1)$.

We use an induction on the number n of left f-primary components. For $n = 1$, we have

$$K = P_1 = \bigcap_{j=1}^{m} Q_j.$$

Suppose that $m > 1$. Then $P_1 \not\subseteq r_f(Q_j)$ for all $2 \leq j \leq m$. It follows that

$$T = P_1 : P_1 = (Q_1 : P_1) \cap (Q_2 : P_1) \cap \cdots \cap (Q_m : P_1) \subseteq Q_m : P_1 = Q_m.$$

This is a contradiction. Thus $m = 1 = n$. Now let us suppose that the conclusion hold for the ideals which are represented by fewer than n of f-primary components. Let $P = P_1 \cap Q_1$. Then P is a left f-primary ideal such that

$$r_f(P) = r_f(P_1) = r_f(Q_1).$$

By the condition (γ),

$$T = P_1 : P_1 \subseteq P_1 : P$$

and thus $P_1 : P = T$. From the fact that $P \nsubseteq r_f(P_i)$ for all $2 \leq i \leq n$, we obtain $P_i : P = P_i$ for all $2 \leq i \leq n$. Hence $K : P = \bigcap_{i=1}^{n}(P_i : P)= (P_1 : P) \cap (P_2 : P) \cap \cdots \cap (P_n : P)= T \cap P_2 \cap P_3 \cap \cdots \cap P_n = \bigcap_{i=2}^{n} P_i$.

Similarly, we can show that

$$K : P = \bigcap_{j=2}^{m} Q_j.$$

Since both decompositions are normal, $n - 1 = m - 1$ implies $n = m$. Moreover, by a suitable ordering, we have $r_f(P_i) = r_f(Q_i)$ for all $2 \leq i \leq n = m$. ■

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