Discussiones Mathematicae
General Algebra and Applications 42 (2022) 363–381
https://doi.org/10.7151/dmgaa.1396

n-FOLD FANTASTIC AND n-FOLD INVOLUTIVE IDEALS IN BOUNDED COMMUTATIVE RESIDUATED LATTICES

Fabrice Tchoua Yinga

Blaise B. Koguep Njionou[1]

Department of Mathematics and Computer Science
Faculty of Science, University of Dschang
P.O. Box 67, Dschang, Cameroon

e-mail: tchoufab@gmail.com
koguep@yahoo.com
blaise.koguep@univ-dschang.org

and

Etienne R. Temgoua Alomo

Department of Mathematics
École Normale Supérieure de Yaoundé, University of Yaoundé 1
P.O. Box 47, Yaoundé, Cameroon

e-mail: retemgoua@yahoo.fr

Abstract

In this paper, we introduce the concepts of n-fold obstinate ideals, n-fold normal ideals, n-fold fantastic ideals and n-fold involutive ideals in residuated lattices, state and prove some of their properties. Several characterizations of these notions are derived and the relations between those notions are investigated. Also, we construct the correspondence between the notions of n-fold ideal and n-fold filter in residuated lattices.

Keywords: residuated lattice, ideal, n-fold ideal, n-fold filter.

2010 Mathematics Subject Classification: 06B10, 06B99, 03G10.

In the theory of MV-algebras, the notion of ideal is at the heart, while in residuated lattice, the main focus has been on deductive systems (or equivalently filters). The study of residuated lattices has experienced a tremendous growth over recent years, and due to the fact that MV-algebras are BL-algebras and

[1]Corresponding author.

BL-algebras are residuated lattices, it is natural to generalize some notions of MV-algebras to residuated lattices. In MV-algebras, filters and ideals are dual notions. Some authors have claimed that the notion of ideals is missing in BL-algebras, since BL-algebra lacks a suitable algebraic addition (see [20]). To fill this gap, Lele and Nganou [11] introduced the notions of ideals, prime ideals and Boolean ideals in BL-algebras and derived some characterizations of them. They also investigated the relationship between ideals and filters by exploiting the set of complement elements. Furthermore, they have used their notion of ideal in the characterization of BL-algebras.

In the theory of residuated lattices, filter theory plays an important role. From logical point of view, various filters have natural interpretation as sets of provable formulas. At present, the filter theory of residuated lattice has been widely studied and some important results are obtained. The foldness theory for filters has been studied in BL-algebras, MTL-algebras, RL-monoids and residuated lattices. Although these structures are ordered by inclusion, the foldness theory for filters has not been studied on these structures in a gradual way. For example, the foldness theory for filters is studied by Haveshki *et al.* [5]–[6] in the BL-algebras in 2006 and 2008, [7] in the RL-monoids in 2010; Motamed *et al.* [14] in the BL-algebras in 2011; Borzooei *et al.* [2] in the BL-algebras in 2013; Zahiri *et al.* [21] in the MTL-algebras in 2014; Kadji *et al.* [8] in the residuated lattices in 2014; Paad *et al.* [15] in the BL-algebras in 2015. In particular, Kadji *et al.* [8] defined the notion of n-fold boolean filters, n-fold implicative filters, n-fold positive implicative filters, n-fold integral filters, n-fold fantastic filters, n-fold obstinate filters, n-fold normal filters and n-fold involutive filters in residuated lattices and studied the relation among many of them (see for example Ref. [8]).

By the fact that ideals and filters are not dual notions in residuated lattices, it appears natural to built up the foldness theory of ideals in the framework of residuated lattices. In [19], authors introduce and investigate the concepts of n-fold boolean ideals, n-fold implicative ideals and n-fold integral ideals in residuated lattices; but they have not studied the relations between these different n-folds. In this paper, we introduce the concepts of n-fold obstinate ideals, n-fold normal ideals, n-fold fantastic ideals and n-fold involutive ideals in residuated lattices. Moreover, using the complement elements, we study relationship between n-fold obstinate (respectively normal, fantastic or involutive) ideals and n-fold obstinate (respectively normal, fantastic or involutive) filters in residuated lattices. Moreover, we built the diagram summarizing all the relationships between these types of ideals as in the case of filters [8]. This paper is organized as follows. In Section 1, we recall some basic properties on residuated lattices. In Section 2, we define new classes of ideals (n-fold ideals) and investigate their properties. In Section 3, we study relation between n-fold ideals and n-fold filters by using the set of complement element. Section 4 studies relationship between n-fold ideals in a residuated lattice and we have built the diagram summarizing all the relationships between these types of ideals.

1. Preliminaries

In this section, we review some of the standard facts on residuated lattices. For more complete theory, we refer the reader to [3, 13].

Definition [3]**.** *A bounded commutative residuated lattice* or simply *residuated lattice* is an algebra $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ such that:

RL1 $(L, \wedge, \vee, 0, 1)$ is a bounded lattice;

RL2 $(L, \odot, 1)$ is a commutative monoid;

RL3 for all x, y, $z \in L$, $x \odot y \leq z$ if and only if $x \leq y \rightarrow z$.

In what follows, L will denote a residuated lattice and for $x \in L$, $x' := x \rightarrow 0$ and $x'' = (x')'$. In addition, $x^0 = 1$ and for every integer $n \geq 1$, $x^n = (x^{n-1}) \odot x$.

Definition [3]**.** (i) A residuated lattice satisfying the *divisibility* condition (div) : $x \wedge y = x \odot (x \rightarrow y)$ is called a RL-*monoid.*

(ii) A residuated lattice satisfying the *prelinearity* condition (pre) : $(x \rightarrow y) \vee (y \rightarrow x) = 1$ is called a MTL-*algebra.*

(iii) A residuated lattice satisfying *prelinearity* and *divisibility* conditions is called a BL-*algebra.*

(iv) A BL-algebra satisfying the *double negation* condition (dn) : $x'' = x$ is called a MV-*algebra.*

We have the following well known properties in residuated lattices.

Proposition 1 [13]**.** *In any residuated lattice* $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$*, the following relations hold for any* $x, y, z \in L$ *and* $n \geq 1$*:*

(P1) $x \odot y \leq x \wedge y$, $x \leq y \rightarrow x$;

(P2) $0' = 1$, $1' = 0$, $1 \rightarrow x = x$, $x \rightarrow 1 = 1$, $x \odot x' = 0$, $x \leq x''$, $x' = x'''$;

(P3) *if* $x \leq y$, *then* $y \rightarrow z \leq x \rightarrow z$, $z \rightarrow x \leq z \rightarrow y$, $x \odot z \leq y \odot z$, $x^n \leq y^n$;

(P4) $(x \odot y)'' = x'' \odot y''$.

We recall here some definitions of the foldness theory of residuated lattices.

Definition [8]**.** A residuated lattice is said to be:

(i) *locally finite* if for every $x \neq 1$, there exists an integer $n \geq 1$ such that $x^n = 0$;

(ii) *n-fold obstinate* if, for all $x \in L$, $x \neq 1$, $x^n = 0$;

(iii) *n-fold normal* if, for all $x, y \in L$, $y^n \to x \leq x$ implies $x \to y \leq y$;

(iv) *n-fold fantastic* if, for all $x, y \in L$, $(x^n \to y) \to y \leq (y \to x) \to x$;

(v) *n-fold involutive* if, for all $x \in L$, $(x^n)'' \to x = 1$;

(vi) *n-fold extended involutive* if, $(x^n)'' = 1$ implies $x = 1$, for all $x \in L$.

Recall now some definitions about the notion of filter.

Definition [3]**.** Let $F \subseteq L$ be a nonempty subset of L.

(i) F is called *a filter* if it satisfies the following two conditions:

(F_1) for every $x, y \in F$, $x \odot y \in F$;

(F_2) for every $x, y \in L$, if $x \in F$ and $x \leq y$, then $y \in F$.

(ii) F is called *a deductive system* if $1 \in F$, and for all $x, y \in L$, $x \to y \in F$ and $x \in F$ implies $y \in F$.

It is known that in a residuated lattice, filters and deductive systems coincide [3].

Definition [8]**.** A proper filter F is said to be:

(i) *prime*, if for all $x, y \in L$ $x \to y \in F$ or $y \to x \in F$;

(ii) *prime of the second kind*, if for all $x, y \in L$ $x \vee y \in F$ implies $x \in F$ or $y \in F$.

In the following, we recall some definitions of the foldness theory of filters in residuated lattices and we give some related results.

Definition [8]**.** Let F be a *filter* of L,

(i) F is said to be *n-fold boolean*, if for all $x \in L$, $x \vee (x^n)' \in F$.

(ii) F is said to be *n-fold implicative*, if for all $x, y, z \in L$, if $x^n \to (y \to z) \in F$ and $x^n \to y \in F$, then $x^n \to z \in F$.

(iii) F is said to be *n-fold integral*, if for all $x, y \in L$, $(x \odot y)' \in F$ implies $(x^n)' \in F$ or $(y^n)' \in F$.

(iv) F is said to be *n-fold normal*, if for all $x, y \in L$, $(y^n \to x) \to x \in F$ implies $(x \to y) \to y \in F$.

(v) F is said to be *n-fold fantastic*, if, $0 \in F$ and, for all $x, y \in L$, $y \to x \in F$ implies $((x^n \to y) \to y) \to x \in F$.

(vi) F is said to be *n-fold obstinate*, if for all $x \in L$, $x \notin F$ implies $(x^n)' \in F$.

(vii) F is said to be *n-fold involutive* of L, if $(x^n)'' \to x \in F$, for all $x \in L$.

(viii) F is said to be *n-fold extended involutive* of L, if $(x^n)'' \in F$ implies $x \in F$, for all $x \in L$.

We recall here some relations about n-fold filter in residuated lattices.

Proposition 2 [8]. *Let $n \geq 1$.*

1. *Any n-fold boolean filter is n-fold fantastic and n-fold implicative;*
2. *Any proper filter of L which is n-fold obstinate filter is also n-fold integral.*
3. *Any n-fold fantastic filter is n-fold normal.*

Proposition 3 [8]. *The following conditions are equivalent for any proper filter F and any $n \geq 1$:*

1. *F is n-fold obstinate;*
2. *F is maximal and n-fold boolean;*
3. *F is maximal and n-fold implicative;*
4. *F is prime of the second kind and n-fold boolean.*

Remark 4 [8]. Prime filters are prime filters of the second kind. The converse is true if L is a MTL-algebra.

In order to introduce the notion of ideals in residuated lattices as a generalization of the existing notion in MV-algebras, Lele and Nganou introduced the pseudo addition operation (in a BL-algebra L): $x \oslash y := x' \to y$, for any $x, y \in L$, and defined the concept of ideals as follows.

Definition [13]. Let $(L, \wedge, \vee, \odot, \to, 0, 1)$ be a residuated lattice and I be a non-empty subset of L. I is called *an ideal* if it satisfies for any $x, y \in L$:

(I_1) $x, y \in I$ implies $x \oslash y \in I$;

(I_2) if $x \leq y$ and $y \in I$, then $x \in I$.

From the above definition, it is easy to see that $0 \in I$, and $x \in I$ if and only if $x'' \in I$ for any $x \in L$. It is easy to prove that $\{0\}$ is an ideal of L.

Definition [13]. Let I be an ideal of residuated lattice L.

(i) I is said to be *a prime ideal*, if for any $x, y \in L$, $(x \to y)' \in I$ or $(y \to x)' \in I$;

(ii) I is said to be *a prime ideal of the second kind*, if for any $x, y \in L$, $x \wedge y \in I$ implies $x \in I$ or $y \in I$.

(iii) I is said to be *a maximal ideal*, if it is not properly contained in any other proper ideal of L.

In [19], we introduced a new operation and gave another description of ideals in residuated lattice.

Let L be a residuated lattice, according to [20], for any $n \in \mathbb{N}^*$, we define the pseudo implication operation $\rightharpoonup_n$ by $x \rightharpoonup_n y := x \odot (y')^n$ for any $x, y \in L$. In particular, $x \rightharpoonup_1 y = x \odot y' := x \rightharpoonup y$. It is easy to see that $x \leq y \oslash z$ if and only if $x \rightharpoonup y \leq z$.

Proposition 5 [19]**.** *Let L be a residuated lattice, for any $x, y, z \in L$ and $n, m \in \mathbb{N}^*$, we have:*

(PS1) *$x \leq y$ implies $z \rightharpoonup_n y \leq z \rightharpoonup_n x$, $x \rightharpoonup_n z \leq y \rightharpoonup_n z$ and $x \rightharpoonup_n y = 0$;*

We recall here some definitions foldness theory of ideals in residuated lattices.

Definition [19]**.** (i) An ideal I of L is called an *n-fold boolean ideal*, if for all $x \in L$, $x^n \wedge (x^n)' \in I$.

(ii) An ideal I of L is called an *n-fold boolean ideal of second kind*, if for all $x \in L$, $x \wedge (x^n)' \in I$.

(iii) A non-empty subset I of a residuated lattice L is called *n-fold implicative ideal* if it satisfies:

- $0 \in I$;
- $(x \rightharpoonup y) \rightharpoonup z^n \in I$ and $y \rightharpoonup z^n \in I$ imply $x \rightharpoonup z^n \in I$, for any $x, y, z \in L$.

(iv) An ideal I of L is called an *n-fold integral ideal*, if for all $x, y \in L$, $x \odot y \in I$ implies $x^n \in I$ or $y^n \in I$.

Now, we recall definition and some properties of operator N. It establish a correspondence between the notions of filters and ideals. Later, we can use it to establish a relationship between the notions of n-fold filters and n-fold ideals.

Definition [11]**.** Let $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ be a residuated lattice and X any subset of L. *The set of complement elements* (with respect to X) is denoted by $N(X)$ and is defined by $N(X) := \{x \in L | x' \in X\}$.

If X, Y are two nonempty subsets of L such that $X \subseteq Y$ then $N(X) \subseteq N(Y)$.

Let X be a nonempty subset of L. We denote by X' the set $X' := \{x' \in L : x \in X\}$. We establish some properties of the operator N.

Theorem 6 [19]**.** *Let F be a filter and I an ideal of L, we have the following results:*

N1 *The set of complement elements $N(I)$ is a filter and $I' \subseteq N(I)$;*

N2 *The set of complement elements $N(F)$ is the ideal generated by F';*

N3 *$I = N(N(I))$;*

N4 *$F \subseteq N(N(F))$;*

N5 *$N(F) = N(N(N(F)))$.*

Theorem 7 [19]**.** *Let F be a filter and I an ideal of L, we have the following results.*

1. *If F is an n-fold implicative filter, then $N(F)$ is an n-fold implicative ideal.*

2. *If I is an n-fold boolean ideal, then $N(I)$ is an n-fold boolean filter.*
3. *I is an n-fold integral ideal if and only if $N(I)$ is an n-fold integral filter of L.*
4. *F is an n-fold integral filter if and only if $N(F)$ is an n-fold integral ideal of L.*

The following theorem defines congruence and quotient structure in residuated lattices using ideals.

Theorem 8 [13]. *Let I be an ideal of a residuated lattice L. Define the relation $\sim_I$ on L by: for any $x, y \in L$, $(x \sim_I y)$ if and only if $(x \to y)' \in I$ and $(y \to x)' \in I$. The relation $\sim_I$ is a congruence relation on L.*

Let L be a residuated lattice and I an ideal of L, the set of all congruence classes of $\sim_I$ is denoted by L/I, that is $L/I := \{[x] : x \in L\}$ where $[x] = \{y \in L : x \sim_I y\}$.

In [13] it is prove that, the operations $\wedge, \vee, \odot, \to$ defined on L/I for any $x, y \in L$ as follows:

$$[x] \wedge [y] = [x \wedge y]; \quad [x] \odot [y] = [x \odot y]; \quad [x] \vee [y] = [x \vee y]; \quad [x] \to [y] = [x \to y],$$

make $(L/I, \wedge, \vee, \odot, \to, [0], [1])$ a residuated lattice which is called quotient residuated lattice with respect to I.

Remark 9 [19]. For any ideal I of a residuated lattice L, $[0] = I$ and $L/I = L/N(I)$.

2. Some n-fold ideals

2.1. n-fold obstinate ideal

The notion of n-fold obstinate ideal ($n \in \mathbb{N}^*$) have been studied by Forouzesh [4] in the framework of MV-algebras, by Motamed [14] and Paad [17] in the framework of BL-algebras. In this section, we introduce this notion in residuated lattices and we give some related results.

Definition. Let $n \geq 1$. An ideal I of L, $I \neq L$ is said to be *n-fold obstinate* if for all $x, y \in L$, $x, y \notin I$ implies $x \rightharpoonup_n y \in I$ and $y \rightharpoonup_n x \in I$.

In particular an 1-fold obstinate ideal is called an *obstinate ideal.*

The following result gives a characterization of n-fold obstinate ideal.

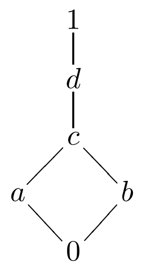
Proposition 10. *Let $n \geq 1$. An ideal I is n-fold obstinate if and only if, for all $x \in L$, $x \in I$ or $(x')^n \in I$.*

Proof. ($\Longrightarrow$) Let $x \in L$. We have two cases: $x \in I$ or $x \notin I$. Assume that $x \notin I$. By setting $y = 1$ in the definition, we have $1 \rightharpoonup_n x = 1 \odot (x')^n = (x')^n \in I$.

($\Longleftarrow$) Conversely, let $x, y \notin I$; then $(x')^n$, $(y')^n \in I$. Since $y \leq 1$, by using (PS1), we have $y \rightharpoonup_n x \leq 1 \rightharpoonup_n x = (x')^n \in I$. Therefore, $y \rightharpoonup_n x \in I$. Similarly, by using $x \leq 1$ and (PS1), we have $x \rightharpoonup_n y \in I$. ∎

The following examples illustrate the above definition.

Example 11. Let $L = \{0, a, b, c, d, 1\}$ be a set with Hasse diagram and Cayley tables of $\odot$ and $\rightarrow$ the following:



$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	a	a
b	0	0	b	b	b	b
c	0	a	b	c	c	c
d	0	a	b	c	c	d
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	b	1	b	1	1	1
b	a	a	1	1	1	1
c	0	a	b	1	1	1
d	0	a	b	d	1	1
1	0	a	b	c	d	1

Then $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a residuated lattice and it is not a BL-algebra because $(a \rightarrow b) \vee (b \rightarrow a) = b \vee a = c \neq 1$. L, $\{0\}$, $\{0, a\}$ and $\{0, b\}$ are the only ideals of L.

The ideals L, $\{0, a\}$ and $\{0, b\}$ are n-fold obstinate and the ideal $\{0\}$ is not n-fold obstinate since $a \neq 0$ and $(a')^n = b \neq 0$.

Example 12. Let L be the unit interval; the operations $\odot$ and $\rightarrow$ defined by: for all $x, y \in L$, such that

$$x \odot y = \begin{cases} 0, & \text{if } x + y \leq \frac{1}{2}; \\ x \wedge y, & \text{elsewhere}. \end{cases} \qquad x \rightarrow y = \begin{cases} 1, & \text{if } x \leq y; \\ \max\{\frac{1}{2} - x; y\}, & \text{elsewhere}. \end{cases}$$

Then $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a residuated lattice and it is not a BL-algebra because if $x, y \in L$ are such that $0 < y < x$ and $x + y < \frac{1}{2}$. Then $y < \frac{1}{2} - x$ and $0 \neq y = x \wedge y$; but $x \odot (x \rightarrow y) = x \odot (\frac{1}{2} - x) = 0$.

The subsets L, $J_a = [0; a]$, $I_a = [0; a)$, and $I = \left[0; \frac{1}{4}\right)$ $\left(a \in \left[0; \frac{1}{4}\right)\right)$ are the only ideals of L.

The ideal I and L are n-fold obstinate and the others are not n-fold obstinate since for $x \in \left(a; \frac{1}{4}\right)$ and $(x')^n = \left(\frac{1}{2} - x\right)^n = \frac{1}{2} - x > \frac{1}{4}$.

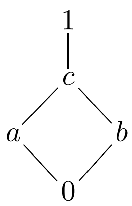
Using Proposition 10, the extension property for n-fold obstinate ideals is as follow.

Remark 13. Let $n \geq 1$ and I and J be two ideals of L such that $I \subseteq J$. If I is an n-fold obstinate ideal of L, then J is also an n-fold obstinate ideal.

Particularly, if $\{0\}$ is an n-fold obstinate ideal, then every ideal of L is n-fold obstinate ideal.

It is easy to see that n-fold obstinate ideal I is $(n+1)$-fold obstinate ideal. But the converse is not true as the following example show.

Example 14. Let $L = \{0, a, b, c, 1\}$ be a set with Hasse diagram and Cayley tables of $\odot$ and $\to$ the following:



$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	0	a
b	0	0	0	0	b
c	0	0	0	0	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	c	1	1
b	c	c	1	1	1
c	c	c	c	1	1
1	0	a	b	c	1

Then $(L, \wedge, \vee, \odot, \to, 0, 1)$ is a residuated lattice and it is not a BL-algebra because $(a \to b) \vee (b \to a) = c \vee c = c \neq 1$. L and $\{0\}$ are the only ideals of L.

The ideal $\{0\}$ is a 2-fold obstinate but it is not a 1-fold obstinate ideal because $a \neq 0$ and $a' = c \neq 0$.

2.2. n-fold normal ideal

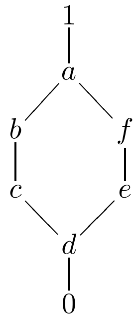
In the framework of residuated lattices, Kadji *et al.* [8] have studied the notion of n-fold normal filter ($n \in \mathbb{N}^*$), and Ahadpanah *et al.* [1] have introduced the notion of normal filter in the same framework. This section is devoted to the study of n-fold normal ideal in residuated lattices.

Let $n \geq 1$ be an integer.

Definition. An ideal I is *n-fold normal* if, for all $x, y \in L$, $((y^n \to x) \to x)' \in I$ implies $((x \to y) \to y)' \in I$.

In particular a 1-fold normal ideal is called a *normal ideal*.

Example 15. Let $L = \{0, a, b, c, d, e, f, 1\}$ be a set with Hasse diagram and Cayley tables of $\odot$ and $\to$ the following:



$\odot$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	c	c	c	0	d	d	a
b	0	c	c	c	0	0	d	b
c	0	c	c	c	0	0	0	c
d	0	0	0	0	0	0	0	d
e	0	d	0	0	0	d	d	e
f	0	d	d	0	0	d	d	f
1	0	a	b	c	d	e	f	1

$\to$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	d	1	a	a	f	f	f	1
b	e	1	1	a	f	f	f	1
c	f	1	1	1	f	f	f	1
d	a	1	1	1	1	1	1	1
e	b	1	a	a	a	1	1	1
f	c	1	a	a	a	a	1	1
1	0	a	b	c	d	e	f	1

Then $(L, \wedge, \vee, \odot, \to, 0, 1)$ is a residuated lattice and it is not a BL-algebra because $(c \to f) \vee (f \to c) = f \vee a = a \neq 1$. One can observe that the only ideals are L, $\{0\}$ and $I = \{0, d, e, f\}$.

For any $n \geq 1$, simple computations prove that I is a n-fold normal ideal, but the ideal $\{0\}$ is not n-fold normal, since $((a^n \to f) \to f)' = 0$ and $((f \to a) \to a)' = d \neq 0$.

Proposition 16. *Any n-fold normal ideal I is a $(n+1)$-fold normal, for all $n \geq 1$.*

Proof. Assume that I is a n-fold normal ideal. Let $x, y \in L$ such that $((y^{n+1} \to x) \to x)' \in I$. By (P1) and (P3), we have $((y^n \to x) \to x)' \leq ((y^{n+1} \to x) \to x)'$. Since I is an ideal, then $((y^n \to x) \to x)' \in I$. Hence, $((x \to y) \to y)' \in I$ because I is a n-fold normal ideal. Therefore, I is a $(n+1)$-fold normal ideal. ■

2.3. n-fold fantastic ideal

The notion of n-fold fantastic filter ($n \in \mathbb{N}^*$) has been studied by Kadji *et al.* [8] in the framework of residuated lattices. In this section, we introduce the notion of n-fold fantastic ideal in residuated lattices and we give some related results.

Definition. Let $n \geq 1$. A subset I of L is a *n-fold fantastic ideal* if $0 \in I$ and, for all $x, y \in L$, $(y \to x)' \in I$ implies $(((x^n \to y) \to y) \to x)' \in I$.

In particular a 1-fold fantastic ideal is a *fantastic ideal.*

Example 17. Let $n \geq 2$.

1. Let L be the residuated lattice of Example 14. It is easy to check that $\{0\}$ is a n-fold fantastic ideal. $\{0\}$ is not a fantastic ideal since $(0 \to a)' = 1' = 0 \in \{0\}$, but $(((a \to 0) \to 0) \to a)' = ((c \to 0) \to a)' = (c \to a)' = c' = c \notin \{0\}$.
2. For the residuated lattice of Example 15, $\{0\}$ is not a n-fold fantastic ideal since $(f \to a)' = 1' = 0 \in \{0\}$, but $(((a^n \to f) \to f) \to a)' = (((c \to f) \to f) \to a)' = ((f \to f) \to a)' = (1 \to a)' = a' = d \notin \{0\}$.

Remark 18. Let $n \geq 1$ and I and J be two ideals of L such that $I \subseteq J$. If I is a n-fold fantastic ideal of L, then J is also a n-fold fantastic ideal.

Particularly, if $\{0\}$ is a n-fold fantastic ideal, then every ideal of L is n-fold fantastic ideal.

Remark 19.
1. A filter F of L is n-fold fantastic if and only if L/F is a n-fold fantastic residuated lattice [8].
2. One can deduce from the above propositions that an ideal I of L is n-fold fantastic if and only if L/I is a n-fold fantastic residuated lattice.

2.4. n-fold involutive ideal

In [21], Zahiri and Farahani introduced the notion of n-fold involutive filter of MTL-algebra. In [8], Kadji *et al.* introduced the notion of n-fold involutive filter in residuated lattices. In this section, we follow their idea and give definition of n-fold involutive ideal in residuated lattices.

Definition. Let I be an ideal of the residuated lattice L. I is called an *n-fold involutive ideal* of L if $((x^n)'' \to x)' \in I$, for all $x \in L$.

Proposition 20. *Any n-fold involutive ideal I is a $(n+1)$-fold involutive, for all $n \geq 1$.*

Proof. Let I be an ideal of the residuated lattice L. Assume that I is a n-fold involutive ideal. Let $x \in L$. We have $((x^n)'' \to x)' \in I$. By (P1) and (P3), we establish that $((x^{n+1})'' \to x)' \leq ((x^n)'' \to x)'$. Since I is an ideal, then $((x^{n+1})'' \to x)' \in I$. Therefore, I is a $(n+1)$-fold involutive ideal. ■

Remark 21. Let $n \geq 1$ and I and J be two ideals of L such that $I \subseteq J$. If I is an n-fold involutive ideal of L, then J is also an n-fold involutive ideal.

Particularly, if $\{0\}$ is an n-fold involutive ideal, then every ideal of L is n-fold involutive ideal.

Definition. Let I be an ideal of the residuated lattice L. I is called an *n-fold extended involutive ideal* of the residuated lattice L if $(x^n)' \in I$ implies $x' \in I$, for all $x \in L$.

Proposition 22. *Any n-fold extended involutive ideal I is a $(n+1)$-fold extended involutive, for all $n \geq 1$.*

Proof. Let I be an ideal of the residuated lattice L. Assume that I is a n-fold extended involutive ideal. Let $x \in L$ such that $(x^{n+1})' \in I$. By (P1), we have $(x^n)' \leq (x^{n+1})'$. Since I is an ideal, then $(x^n)' \in I$. Hence, $x' \in I$ because I is a n-fold extended involutive ideal. Therefore, I is a $(n+1)$-fold extended involutive ideal. ■

3. Relations between n-fold ideals and n-fold filters

Although the notions of filters and ideals are not dual notions, we examine in this section, the relations between n-fold ideals and n-fold filters.

Proposition 23. *Let $n \geq 1$ be an integer and I be an ideal of L. Then I is an n-fold obstinate ideal if and only if $N(I)$ is an n-fold obstinate filter.*

Proof. Let I be an ideal of L.

($\Rightarrow$) Assume that I is an n-fold obstinate ideal. Let $x \in L$ such that $x \notin N(I)$. $x \notin N(I) \Longrightarrow x' \notin I \Longrightarrow (x'')^n \in I$. We show by induction on n using (P4) that $(x'')^n = (x^n)''$; then $(x^n)' \in N(I)$. Therefore $N(I)$ is a n-fold obstinate filter.

($\Leftarrow$) Assume that $N(I)$ is an n-fold obstinate filter. Let $x \in L$ such that $x \notin I$. $x \notin I \Longrightarrow x'' \notin I \Longrightarrow x' \notin N(I) \Longrightarrow ((x')^n)' \in N(I) \Longrightarrow ((x')^n)'' \in I \overset{(P4)}{\Longrightarrow} (x''')^n \in I \Longrightarrow (x')^n \in I$. Therefore, I is an n-fold obstinate ideal. ■

Proposition 24 [8]**.** *Let $n \geq 1$ be an integer and F be a filter of L. Then F is an n-fold obstinate filter if and only if L/F is an n-fold obstinate residuated lattice.*

Remark 25. By Proposition 23, we can see that I is an n-fold obstinate ideal if and only if L/I is an n-fold obstinate residuated lattice.

Proposition 26. *Let L be a residuated lattice.*

(1) *If I is a maximal ideal of L, then $N(I)$ is a maximal filter.*

(2) *If F is a maximal filter of L, then $N(F)$ is a maximal ideal.*

Proof. (1) Let I be a maximal ideal of L. Then, $N(I)$ is a filter of L. In addition, suppose that there exist a filter F such that $N(I) \subseteq F$, we need to show that $N(I) = F$ or $F = L$. Since $N(I) \subseteq F$, we have $I = N(N(I)) \subseteq N(F)$. We apply the fact that I is a maximal ideal of L and obtain $N(F) = I$ or $N(F) = L$. If $N(F) = I$, then $F \subseteq N(N(F)) = N(I) \subseteq F$, and it follows that $N(I) = F$. On the other hand, if $N(F) = L$, then $1 \in N(F)$. That is, $0 \in F$, and therefore $F = L$. Hence, $N(I)$ is a maximal filter of L.

(2) This can be proved similarly to (1). ∎

Proposition 27 [8]**.** *For any filter F of a residuated lattice L, the following conditions are equivalent:*

(1) *F is a maximal filter of L;*

(2) *for any $x \in L$, $x \notin F$ if and only if $(x^n)' \in F$ for some $n \in \mathbb{N}^*$;*

(3) *L/M is a locally finite residuated lattice.*

Proposition 28. *Let M be a proper ideal of L. Then the following conditions are equivalent:*

(1) *M is a maximal ideal of L;*

(2) *for any $x \in L$, $x \notin M$ implies $(x')^n \in M$ for some $n \in \mathbb{N}^*$;*

(3) *L/M is a locally finite residuated lattice.*

Proof. (1)⇒(2) Let M be a maximal ideal of L and $x \in L$. Suppose that $x \notin M$. Then $x'' \notin M$ and $x' \notin N(M)$. Since $N(M)$ is a maximal filter (see Proposition 26), $((x')^n)' \in N(M)$, for some $n \in \mathbb{N}^*$ (see Proposition 27). Hence, $x \notin M$ implies $(x')^n \in M$, for some $n \in \mathbb{N}^*$.

(2)⇒(3) Let $x \in L$ such that $[1] \neq [x] \in L/M$. Then $(x \to 1)' \notin M$ or $(1 \to x)' \notin M$. Because $(x \to 1)' = 0 \in M$, $(1 \to x)' = x' \notin M$. Therefore, $((x')')^n \in M$, for some $n \in \mathbb{N}^*$. By (P2) and (P3), $x^n \leq (x'')^n$. Hence, $x^n \in M$, and $[x]^n = [0]$. Therefore, L/M is a locally finite residuated lattice.

(3)⇒(1) Suppose that L/M is a locally finite residuated lattice. Then $L/N(M)$ is locally finite residuated lattice because $L/N(M) = L/M$. By Proposition 27, $N(M)$ is a maximal filter and by Proposition 26, $N(N(M)) = M$ is a maximal ideal of L. ∎

Proposition 29. *Let $n \geq 1$ be an integer and I be an ideal of L. Then I is a n-fold normal ideal if and only if $N(I)$ is a n-fold normal filter.*

Proof. Let I be an ideal of L. Assume that I is a n-fold normal ideal. Let $x, y \in L$ such that $(y^n \to x) \to x \in N(I)$.

$(y^n \to x) \to x \in N(I) \Longrightarrow ((y^n \to x) \to x)' \in I \Longrightarrow ((x \to y) \to y)' \in I \Longrightarrow (x \to y) \to y \in N(I)$. Hence, $N(I)$ is a n-fold normal filter.

Conversely, assume that $N(I)$ is a n-fold normal filter. Let $x, y \in L$ such that $((y^n \to x) \to x)' \in I$.

$((y^n \to x) \to x)' \in I \Longrightarrow (y^n \to x) \to x \in N(I) \Longrightarrow (x \to y) \to y \in N(I) \Longrightarrow ((x \to y) \to y)' \in I$. Hence, I is a n-fold normal ideal.

Therefore, $N(I)$ is a n-fold normal filter if and only if I is a n-fold normal ideal. ∎

Proposition 30. *Let L be a residuated lattice which satisfies for all $x, y \in L$, $((x \to y) \to y)' = ((y \to x) \to x)'$. Let I be an ideal of L. The following conditions are equivalent:*

(1) *I is n-fold normal;*

(2) *For every $x \in L$, if $(x^n)' \in I$, then $x' \in I$;*

(3) *$D(\{0\}) \subseteq N(I)$, where $D : \mathcal{P}(L) \longrightarrow \mathcal{P}(L)$ is the operator defined by $D(X) := \{x \in L : (x^n)' \in X\}$.*

Proof. (1)⇒(2) Suppose that I is n-fold normal, and let $x \in L$ such that $(x^n)' \in I$. Then $(x^n)' \overset{(P2)}{=} (x^n)''' = ((x^n \to 0) \to 0)' \in I$, and since I is n-fold normal, then $((0 \to x) \to x)' \in I$; that is, $x' \in I$.

(2)⇒(1) Suppose that (2) holds. Let $x, y \in L$ such that $((x^n \to y) \to y)' \in I$. Since $x^n \leq x$, by (P3) we have $((x \to y) \to y)' \leq ((x^n \to y) \to y)'$. Hence, $((x \to y) \to y)' \in I$. Thus, $((y \to x) \to x)' \in I$.

(2)⇒(3) Suppose that (2) holds, and let $x \in D(\{0\})$. Then, $(x^n)' = 0 \in I$. Hence, by (2), $x' \in I$. Thus, $D(\{0\}) \subseteq N(I)$ as needed.

(3)⇒(2) Suppose $D(\{0\}) \subseteq N(I)$, and let $x \in L$ such that $(x^n)' \in I$. By (P1) and (P3), $x' \leq (x^n)' \in I$. That is, $x' \in I$. Therefore, (2) holds. ∎

Remark 31. As Proposition 29, we can also prove that an ideal I is a n-fold fantastic ideal (respectively an n-fold involutive ideal or an n-fold extended involutive ideal) if and only if $N(I)$ is a n-fold fantastic filter (respectively an n-fold involutive filter or an n-fold extended involutive filter).

We give a simple characterization of n-fold fantastic ideals but we first remember a characterization of n-fold fantastic filter.

Proposition 32 [8]**.** *Let $n \geq 1$ and let F be a filter. F is a n-fold fantastic filter if and only if $((x^n \to y) \to y) \to (x \vee y) \in F$, for all $x, y \in L$.*

By applying Remark 31 and Proposition 32, we can see that I is a n-fold fantastic ideal if and only if $(((x^n \to y) \to y) \to (x \vee y))' \in I$, for all $x, y \in L$.

Proposition 33 [8]**.** *L is n-fold fantastic if and only if Every Filter F of L is n-fold fantastic.*

Combining Remark 31 and Proposition 33, we can observe that L is n-fold fantastic if and only if Every ideal I of L is n-fold fantastic if and only if $\{0\}$ is a n-fold fantastic ideal of L.

Proposition 34 [8]**.** *Let $n \geq 1$ be an integer and F be a proper filter of L. Then F is an n-fold involutive (respectively extended involutive) filter of L if and only if L/F is an n-fold involutive (respectively extended involutive) residuated lattice.*

Remark 35. By combining Remark 31, Remark 9 and Proposition 34 we deduce that I is an n-fold involutive ideal (respectively extended involutive ideal) if and only if L/I is an n-fold involutive residuated lattice (respectively extended involutive residuated lattice).

4. Relation between different types of n-fold ideals

In this section, we study the relations between different types of n-fold ideals and draw a diagram summarizing those relations.

Proposition 36. *If I is prime and n-fold boolean ideal, then I is n-fold integral ideal.*

Proof. Assume that I is prime and n-fold boolean ideal, then for all $x, y \in L$, $(x \to y)' \in I$ or $(y \to x)' \in I$. Hence $x \to y \in N(I)$ or $y \to x \in N(I)$, for all $x, y \in L$. Therefore $N(I)$ is a prime filter. By Remark 4, Proposition 3 and Theorem 7, $N(I)$ is an n-fold obstinate filter. By Proposition 2(2), $N(I)$ is an n-fold integral filter. Finally by Theorem 7, I is an n-fold integral ideal. ∎

From Proposition 2 (1), Theorem 7 and Theorem 6, we can obtain that if I is n-fold boolean ideal, then I is n-fold implicative ideal.

By Proposition 10, we can establish that, for all $n \in \mathbb{N}^*$, any n-fold obstinate ideal is a maximal ideal.

Similarly, from Remark 31 and Proposition 2 (3), we can prove that any n-fold fantastic ideal is n-fold normal.

Remark 37 [19]**.** Any n-fold boolean ideal of the second kind is n-fold boolean ideal.

By applying Theorem 7 and Proposition 2(1), we can easily obtain that if an ideal I of L is n-fold Boolean then, it is n-fold fantastic and n-fold implicative.

Remark 38. By setting $y = 0$ in Definitions 2.3, we deduce that n-fold fantastic ideals are n-fold involutive.

Proposition 39. *An n-fold involutive ideal (respectively a n-fold normal ideal) is n-fold extended involutive.*

Proof. Let I an n-fold involutive ideal and $x \in L$ then, $((x^n)'' \to x)' \in I$. Let $(x^n)' = (x^n)''' \in I$. We have $(x^n)'' \to x, (x^n)'' \in N(I)$. Since $N(I)$ is a deductive system then, $x \in N(I)$. Therefore $x' \in I$ and I is n-fold extended involutive ideal.

Let I a n-fold normal ideal then, for all $x, y \in L$, $((x^n \to y) \to y)' \in I$ implies $((y \to x) \to x)' \in I$. By setting $y = 0$, we obtain that $(x^n)' \in I$ implies $x' \in I$, for all $x \in L$. Therefore I is n-fold extended involutive ideal. ■

Combining Proposition 2(2) (respectively Proposition 3) and Proposition 7, we can observe that any proper ideal of L which is n-fold obstinate is also n-fold integral (respectively n-fold boolean).

The previous results can be summarized as a diagram describing the relations between the different n-fold ideals. This diagram is given in the appendix.

CONCLUSION

Filter theory (or equivalently deductive system), which plays an important role in studying algebraic structures related to logical systems, have been widely studied. Particularly, the foldness theory for filters has been studied in BL-algebras, MTL-algebras and RL-monoids. In MV-algebras, filters and ideals are dual notions, some authors have claimed that the notion of ideals is missing in BL-algebras due to the lack of a suitable algebraic addition. Based on the definition of an ideal of BL-algebras given by Lele and Nganou [11] and since all the algebraic structures mention above are some particular types of residuated lattice, the aim of this paper was to develop the concept of foldness theory for ideals in the framework of residuated lattices. We study new classes of ideals such as n-fold obstinate ideals, n-fold normal ideals, n-fold fantastic ideals and n-fold involutive ideals and derive several important properties and some of their characterizations. Also, we prove that a residuated lattice is n-fold fantastic if and only if its trivial ideal $\{0\}$ is a n-fold fantastic ideal. Moreover, we study relationship between those classes of ideals and the corresponding classes of filters by using the set of complement elements. Since the notion of ideal is not the dual of the notion of filter in residuated lattices, we have built the diagram summarizing all the relationships between these types of ideals as in the case of filters [8].

Based on the work of Celestin Lele and Salissou Moutari [10] some computational algorithms can be built for these concepts of foldness for ideals of bounded commutative residuated latttices.

Acknowledgements

The authors thank the anonymous reviewers for their insights and their comments that greatly improved the manuscript.

References

[1] A. Ahadpanah and L. Torkzadeh, *Normal filter in residuated lattices*, Le Matematiche **70** (2015) 81–92.
https://doi.org/10.4418/2015.70.1.6

[2] R.A. Borzooei and A. Paad, *Integral filters and integral BL-algebras*, Italian J. Pure Appl. Math. **30** (2013) 303–316.

[3] R. Cretan and A. Jeflea, *On the lattice of congruence filters of a residuated lattice*, Annals of University of Craiova, Mathematics and Computer Science Series. **33** (2006) 174–188.

[4] F. Forouzesh, *n-fold obstinate ideal in MV-algebras*, New Math. Natural Comput. **12** (2016) 265–275.
https://doi.org/10.1142/S1793005716500186

[5] M. Haveshki, B. Saeid and E. Eslami, *Some types of filters in BL-algebras*, Soft Comput. **10** (2006) 657–664.
https://doi.org/10.1007/s00500-005-0534-4

[6] M. Haveshki and E. Eslami, *n-fold filters in BL-algebra*, Math. Log. Quart. **54** (2008) 176–186.
https://doi.org/10.1002/malq.200710029

[7] M. Haveshki and M. Mohamadhasani, *Folding theory applied to Rl-monoid*, Annals of the University of Craiova, Mathematics and Computer Science Series **37** (2010) 9–17.

[8] A. Kadji, C. Lele, J.B. Nganou and M. Tonga, *Folding theory applied to residuated lattices*, Int. J. Math. and Math. Sci. **4** (2014) 1–12.
https://doi.org/10.1155/2014/428940

[9] M. Kondo and W. Dudek, *Filter theory of BL algebras*, Soft Computing **12** (2009) 419–423.
https://doi.org/10.1007/s00500-007-0178-7

[10] C. Lele and S. Moutari, *On some computational algorithms for n-fold ideals in BCK-algebras*, J. Appl. Math. Comput. **23** (2007) 369–383.
https://doi.org/10.1007/BF02831984

[11] C. Lele and J.B. Nganou, *MV-algebras derived from ideals in BL-algebras*, Fuzzy Sets and Systems **218** (2013) 103–113.
https://doi.org/10.1016/j.fss.2012.09.014

[12] C. Lele and J.B. Nganou, *Pseudo-addidion and fuzzy ideal in BL-algebras*, Annals Math. Inform. **8** (2014) 193–207.

[13] Y. Liu, Y. Qin, X. Qin and Y. Xu, *Ideals and fuzzy ideals on residuated lattices*, Int. J. Machine Learning and Cybernetics, **8** (2014) 239–253. https://doi.org/10.1007/s13042-014-0317-2

[14] S. Motamed and A.B. Saeid, *n-fold obstinate filters in BL-algebras*, Neural Comput. Appl. **20** (2011) 461–472. https://doi.org/10.1007/s00521-011-0548-z

[15] A. Paad and R.A. Borzooei, *Generalization of integral filters in BL-algebras and n-fold integral BL-algebras*, Africa Matematika **26** (2015) 1299–1311. https://doi.org/10.1007/s13370-014-0275-6

[16] A. Paad, *n-Fold integral ideals and n-fold Boolean ideals in BL-algebras*, Africa Matematika **28** (2017) 971–984. https://doi.org/10.1007/s13370-017-0497-5

[17] A. Paad, *Folding theory of implicative and obstinate ideals in BL-algebras*, Discuss. Math. Gen. Alg. and Appl. **38** (2018) 255–271. https://doi.org/10.7151/dmgaa.1295

[18] A.B. Saeid and S. Motamed, *Some results in BL-algebras*, Math. Logic Quarterly **55** (2009) 649–658. https://doi.org/10.1002/Malq.200910025

[19] Y.F. Tchoua, N.B.B. Koguep, A.E.R. Temgoua and C. Lele, *n-fold boolean, implicative and integral ideals on bounded commutative residuated lattices*, New Math. and Natural Comput. **15** (2019) 427–445. https://doi.org/10.1142/S1793005719500248

[20] Y. Yang and X. Xin, *On chracterizations of BL-algebras via implicative ideals*, Italian J. Pure and Appl. Math. **37** (2017) 493–506.

[21] O. Zahiri and H. Farahani, *n-Fold filters of MTL-algebras*, Africa Matematika **25** (2014) 1165–1178. https://doi.org/10.1007/s13370-013-0184-0

Received 30 October 2020
Revised 10 May 2021
Accepted 31 May 2022

Appendix

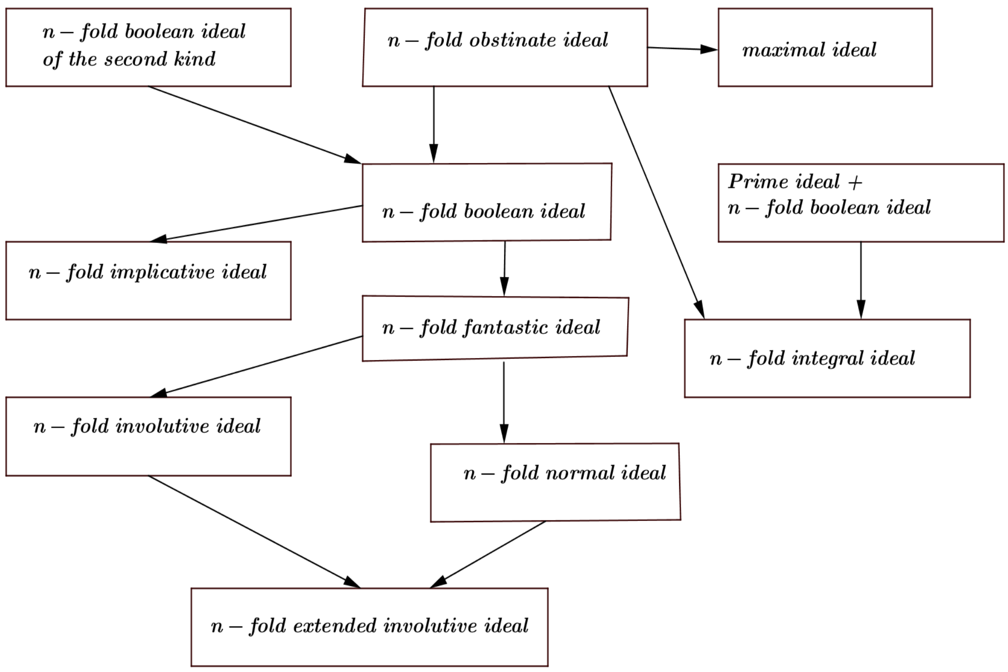


Figure 1. Diagram summarizing the relations between the different n-fold ideals.