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ALGEBRAIC GEOMETRY OVER COMPLETE LATTICES AND INVOLUTIVE POCRIMS

Ali Molkhasi

Farhangyian University of Iran, Tabriz-Iran

e-mail: molkhasi@gmail.com

and

Kar Ping Shum

Institute of Mathematics
Yunnan University Kunning, P.R. China

e-mail: kpshum@ynu.edu.cn

Abstract

An *involutive pocrim* is a resituated integral partially ordered commutative monoid with an involution operator, consider as an algebra. In this paper it is proved that the variety of a finitely generated by involutive pocrims of finite type has a finitely based equational theory. We also study the algebraic geometry over compete lattices and we investigate the properties of being equationally Noetherian and u_ω-compact over such lattices.

Keywords: congruence distributive, algebraically closed algebra, involutive pocrims, equationally Noetherian.

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1. Introduction

We continue our study of the algebraic structure of partially ordered commutative residuated integral monoid (pocrim) [4]. In this paper, we will introduce new algebraic results that allow us to generalize the main properties of residuated lattices. In order to make this paper as self-contained as possible, we will recall a commonly considered algebraic structure. Using a difficult 1941 result of Emil Post, in 1951 Roger Lyndon demonstrated that all 2-element algebras are finitely based. Recall that the all finite algebras with only finitely many basic operations which belong to congruence meet semidistributive residually very finite varieties are finitely based. In this note, finitely generated varieties by involutive pocrims is related to finitely based equational theory and existentially closed algebras, where by an equational theory we mean a set of equations from some fixed language which is closed with respect to logical consequences. Recall that involutive pocrims were introduced under the name pre-Boolean algebras by Wronski and krzystek in [17] and there are some related in [6, 13], and [15]. We know that pocrims include complete lattices and satisfy the infinite distributive law.

Our interest in this topic is twofold. The first part comes from studying involutive pocrims and it is shown that if a variety is a finitely generated by involutive pocrims of finite type, then it has a finitely based equational theory. The other part of our interest in the universal algebraic geometry over complete lattices i.e., different conditions relating systems of equations especially conditions about systems and sub-systems of equations over algebras and it is proved that if K is a sublattice of a complete lattice satisfies the infinite distributive law, then the lattice is not K-equationally Noetherian, where K is infinite.

2. Algebraic properties of involutive pocrims

In this section, $\mathcal{L}$ is a first order language and variety is a nonempty class of algebras of type $\mathcal{L}$ such that it is closed under subalgebras, homomorphic images, and direct products.

We recall from [2] that equations are simply ordered pairs of terms, but an equation (u, v) will be more naturally denoted by $u \approx v$ to emphasize its semantical intent and by $u \longrightarrow v$ in the context of rewriting. Also, an equational theory is a set of equations which is closed with respect to logical consequences; equivalently, an equational theory is a fully invariant congruence of the term algebra. A set B of equations is called a base for an equational theory E, provided E is the set of all equations that are consequences of B. An equational theory with a finite base is said to be finitely based. In general, a variety $\mathcal{V}$ of algebras of some type $\mathcal{L}$ is called finitely based if there is a finite set Φ of identities such that an algebra A of type $\mathcal{L}$ lies in $\mathcal{V}$ if and only if $A \models \Phi$. An algebra is called finitely based if the variety its generates is finitely based.

Now, some relationships between involutive pocrims and a finitely based equational theory are considered. Consider a commutative monoid $(A,\, \cdot,\, t)$ whose universe A is partially ordered by a relation $\leq$. Suppose that $\leq$ is compatible with $\cdot$, i.e., for all $a, b, c \in A$, if $a \leq b$, then $a \cdot c \leq b \cdot c$. The structure $(A,\, \cdot,\, t,\, \leq)$ is said to be resituated if there is a largest $c \in A$ such that $a \cdot c \leq b$ for any $a, b \in A$. The largest c with this property is denoted by $a \longrightarrow b$, so $(A,\, \cdot,\, \longrightarrow,\, t,\, \leq)$ satisfies:

$$x \cdot z \leq y \Longleftrightarrow z \leq x \longrightarrow y$$

and in particular,
$$x \leq y \Longleftrightarrow t \leq x \longrightarrow y.$$

If t is the greatest element of $(A, \leq)$, then we say that $(A, \cdot, t)$ is integral. In this case, the partial order $\leq$ is equationally definable by $x \leq y \Longleftrightarrow t \approx x \longrightarrow y$, so $(A, \cdot, t)$ is first order definitionally equivalent to the algebra $(A, \cdot, \longrightarrow, t)$. We recall form [4] that an algebra $(A, \cdot, \longrightarrow, t)$ which arises in this way is called a *pocrim* for 'partially ordered commutative residuated integral monoid'. In other words, a tuple $(A, \leq, \cdot, \rightarrow, T)$ is said to be a partially ordered commutative residuated integral monoid, briefly a pocrim, if, for every a, b, $c \in A$, the following properties hold:

1- $(A, \cdot, T)$ is a commutative monoid with neutral element,

2- $(A, \leq)$ is a partially ordered set with maximum,

the operations $\cdot$ and $\longrightarrow$ satisfy the adjointness condition, that is $a \cdot c = b$ if and only if $c \leq a \rightarrow b$. It is well known that the class $POCR$ of all pocrims is axiomatized by the identities, for all $x, y, z, t \in A$:

(M1) $(x \cdot y) \longrightarrow z \approx y \longrightarrow (x \longrightarrow z)$,

(M2) $t \longrightarrow x \approx x$,

(M3) $x \longrightarrow t \approx t$,

(M4) $(x \longrightarrow y) \longrightarrow ((y \longrightarrow z) \longrightarrow (x \longrightarrow z)) \approx t$,

together with the single quasi-identity:

(M5) $(x \longrightarrow y \approx t$ and $y \longrightarrow x \approx t) \Longrightarrow x \approx y$.

Definition. A *quasivariety* is class of algebras closed under I (isomorphisn), S (subalgebra), and P_R (close under I and S), and containing the one-element algebras.

Therefore, $POCR$ is a quasivariety. For a general study of pocrims, see [4]. An involutive pocrim is an algebra

$$(A, \cdot, \longrightarrow, \neg, t)$$

such that
$$(A, \cdot, \longrightarrow, t)$$

is a *pocrim*, $\neg$ is a unary operation on A, and $(A, \cdot, \longrightarrow, \neg, t)$ satisfies:

(M6) $\neg\neg x \approx x$,

(M7) $x \longrightarrow \neg y \approx y \longrightarrow \neg x$.

Notice that the class $IPOC$ of all involutive pocrims is therefore also a quasivariety. Recall that a variety $\mathcal{V}$ is said to be locally finite if its finitely generated members are finite algebras. In particulary, every variety generated by a finite set of finite algebras is locally finite.

Baker in [1] showed that every finite algebra of finite type that generates a congruence distributive variety is finitely based. Let A be an algebra. By a congruence relation on A we mean an equivalence relation on A that has the substitution property for A. $\mathrm{Con}(A)$ denotes the set of all congruence relations on A.

Definition. An algebra A is said congruence distributive if the lattice $\mathrm{Con}(A)$ is distributive.

Now, we can prove the following theorem.

Theorem 1. *If variety $\mathcal{V}$ is a finitely generated by involutive pocrims of finite type, then $\mathcal{V}$ has a finitely based equational theory.*

Proof. To proof we know every finitely generated variety is locally finite and also a variety generated by a class K of similar algebras, the free algebras belong to the quasivariety generated by K, hence they are involutive pocrims if $K \subseteq IPOC$. Further, since $IPOC$ satisfies $x^{n+1} \leq x^n$ for all $n \in \omega$ (where $x^0 =: t$ and $x^{n+1} = x^n \cdot x$), it follows that every finite involutive pocrims satisfies $x^n \approx x^{n+1}$ for some $n \in \omega$. So, in a variety $\mathcal{V}$ generated by involutive pocrims, if the 1-generated free algebra is finite, then $\mathcal{V}$ satisfies $x^n \approx x^{n+1}$ for some finite any n. Now any involutive pocrim satisfying $x^n \approx x^{n+1}$ also satisfies:

$$(x \longrightarrow y) \longrightarrow^n (y \longrightarrow x) \longrightarrow^n x \approx (y \longrightarrow x) \longrightarrow^n (x \longrightarrow y) \longrightarrow^n y$$

(where $x \longrightarrow^0 y : y$ and $x \longrightarrow^{n+1} y : x \longrightarrow (x \longrightarrow^n y)$ see [7]). As the 2-generated free algebra in $\mathcal{V}$ belong to $IPOC$, the above equation holds throughout $\mathcal{V}$, and it clearly entails (M5), where $\mathcal{V}$ consists of involutive pocrims. Finally, every variety of involutive pocrims, such as $\mathcal{V}$, is congruence distributive. Also, every finitely generated congruence distributive variety $\mathcal{V}$ contains only finitely many subdirectly irreducible algebras. On the other hand, if $\mathcal{V}$ is a finitely generated congruence distributive variety of finite type then $\mathcal{V}$ has a finitely based equational theory (see [1] and [2]). ∎

Now we consider a characterization of absolute retracts, and proves that, in a finitely generated by involutive pocrims of finite type, we are able to characterize the absolute retract which is a product of reduced powers of maximal subdirectly irreducibles of $\mathcal{V}$. The investigation of absolute retracts has been most fruitful when restricted to congruence distributive varieties, where the Fraser-Horn property and Jonsson's Lemma provide powerful tools for managing congruences on products.

Definition. Let V be a variety of algebras. An algebra A is said to be an absolute retract in V if and only if every embedding $A \hookrightarrow B$ in V is split (i.e., has a left inverse).

We recall form [12] that if A is an algebra, then $\mathcal{L}_A$ denotes the language $\mathcal{L}$ augmented with constant symbols $c_a : a \in A$. An algebra A is algebraically closed in $\mathcal{V}$ if and only if any finite set of equations in $\mathcal{L}_A$ which is satisfiable in some extension of A in $\mathcal{V}$ is already satisfiable in A. An embedding $A \hookrightarrow B$ is said to be existential $\exists_1-$ provided that every finite set of equations and inequations in $\mathcal{L}_A$ which is satisfiable in B is already satisfiable in A. In that case, we say that A is a $\exists_1-$ subalgebra of B. An extension $A \hookrightarrow B$ is said to be an essential extension if and only if whenever $\theta \in Con(B)$ has $\theta \uparrow_A = 0_A$, then $\theta = 0_B$. An algebra $A \in \mathcal{V}$ is an absolute retract in $\mathcal{V}$ if and only if it has no proper essential extensions in $\mathcal{V}$. An essential extension of a (finitely) subdirectly irreducible algebra is also (finitely) subdirectly irreducible. A subdirectly irreducible algebra is said to be a maximal subdirectly irreducible in $\mathcal{V}$ if and only if it has no proper essential extensions in $\mathcal{V}$. Thus each maximal subdirectly irreducible is an absolute retract.

Now, we study the class of algebraically closed members $\mathcal{V}_{AC}$ of a variety $\mathcal{V}$. Model complete theories were introduced by Robinson as a natural model-theoretic generalization of the theory of algebraically closed fields. Recall that a class of models K for $\mathcal{L}$ is called an elementary class if and only if there exists a theory T such that K is exactly the class of all models of T. Also, a theory is model complete if the class of its models is model complete.

Theorem 2. *If variety $\mathcal{V}$ is a finitely generated by involutive pocrims of finite type with the property that $\mathcal{V}_{AC}$ is closed under products and $\mathcal{V}_{AC}$ is model complete, then $\mathcal{V}_{AC}$ is existentially closed.*

Proof. We know that, if a locally finite variety is generated by involutive pocrims then it is congruence distributive variety ([7]). On the hand, every finitely generated variety is locally finite. Here, variety $\mathcal{V}$ is a finitely generated by involutive pocrims then $\mathcal{V}$ is finitely generated congruence distributive variety. Consequently, $\mathcal{V}$ is a generated congruence distributive variety of finite type. We have if A is a finite algebra, and B a Boolean algebra, then $A[B] \cong A[B]^*$ ([3]). Furthermore, if $\mathcal{F}$ is a filter on B, then $A[B]^*/\mathcal{F} \cong A[B/\mathcal{F}]^*$. It follows that if A is a finite algebra, then any reduced power $A^I/\mathcal{F}$ of A is isomorphic to a bounded Boolean power $A[\mathcal{P}(I)/\mathcal{F}]^*$. So, in a finitely generated congruence distributive variety $\mathcal{V}$, every absolute retract is a product of reduced powers of maximal subdirectly irreducibles of $\mathcal{V}$ ([12]). In this theorem, the class of algebraically closed algebras in $\mathcal{V}$ is closed under finite products, so the class of absolute retracts in $\mathcal{V}$ is closed under finite products. Suppose that M is a maximal subdirectly irreducible, then $M^I/\mathcal{F}$ is algebraically closed. Also we know an algebra of a variety of algebras is algebraically closed if and only if it can be existentially embedded into an algebraically closed. As a result, it can be existentially embedded into a product of reduced powers of maximal subdirectly irreducibles ([7] and [14]). It is well-known that a first-order theory which is preserved under updirected unions and finite direct products is axiomatized by set of universal-existential Horn sentences ([5], chapter 6). Therefore, $\mathcal{V}_{AC}$ is an elementary class. In addition, we know if $\mathcal{V}_{AC}$ is an elementary class and is model complete, then existentially closed of $\mathcal{V}_{AC}$ equivalent with $\mathcal{V}_{AC}$ and complete the proof. ∎

For an arbitrary algebra A it is possible with a use of the Compactness Theorem to construct an elementary extension A^* of A such that A^* is logically compact (see [8]). The definitions of u_ω-compactness is given in geometric form.

Corollary 3. *If variety $\mathcal{V}$ is a finitely generated by involutive pocrims of finite type with the property that $\mathcal{V}_{AC}$ is closed under products and $\mathcal{V}_{AC}$ is model complete there exists u_ω-compact algebra $\mathcal{V}^*_{AC}$ which is elementary equivalent to $\mathcal{V}_{AC}$.*

3. Algebraic geometry over complete lattices

We know that pocrims include BL-algebras (BL-logic), MV-algebras (Łukasiewicz infinite-valued logic) and, complete lattice and satisfies the infinite distributive law $x \wedge \bigvee_{i\in I} y_i = \bigvee_{i\in I}(x \wedge y_i)$.

By *System* of equations we mean an arbitrary set of equations. Let S be a system of equations in lattice A. The set of all logical consequences of S over A is denoted by $Lc_A(S)$. In other words, $Lc_A(S)$ is the set of all lattice equations $f \approx g$ such that $V_A(S) \subseteq V_A(f \approx g)$, where $V_A(S)$ is the sets of solutions of S in A. As is shown in [16], a lattice A is called equationally Noetherian, if any system of equations with coefficient in A is equivalent with a finite subsystem. Note that an algebra A is called q_ω-compact, if for any system S and any equation $p \approx q$, the condition $V_A(S) \subseteq V_A(p \approx q)$ implies that $V_A(S_0) \subseteq V_A(p \approx q)$ for some finite $S_0 \subseteq S$. Clearly, every equationally Noetherian algebra is q_ω-compact.

Theorem 4. *Let L be a complete lattice and satisfies the infinite distributive law $x \wedge \bigvee_{i\in I} y_i = \bigvee_{i\in I}(x \wedge y_i)$ and K be a sublattice, which is infinite. Then L is not K-equationally Noetherian, for all x, $y_i \in L$.*

Proof. Suppose that $x, z \in L$ and $B = \{y_i \mid i \in I\}$ is the set of elements $y_i \in L$ such that $x \wedge y_i \leq z$. We set $y = \sup_L B$ then $x \wedge y = x \wedge \bigvee_{i\in I} y_i = \bigvee_{i\in I}(x \wedge y_i) \leq z$ and consequently the pseudocomplement exists and is y.

Now, we prove that L is not K-equationally Noetherian. To see this, we focus on the case of equations with coefficients inside L (Diophantine Geometry). For simplicity we discuss Diophantine case ($K = L$). The idea of the proof is taken from a similar theorem for Boolean algebras (see [10]). Let

$$b_0, b_1, b_2, \ldots$$

be an infinite set of elements in L. Let $L_0 = \{0, 1\}$ and define L_n by inductions as follows: if $L_{n-1} = \{a_0 = 0, a_1, \ldots, a_{n-1}, a_n = 1\}$, and if $0 \leq i \leq n$, then we define

$$c_{i+1} = a_i \vee (a_{i+1} \wedge b_n).$$

For example, we have $L_0 = \{a_0 = 0, a_1 = 1\}$. Then we compute

$$c_1 = a_0 \vee (a_1 \wedge b_1) = b_1.$$

It is clear that $0 \leq b_1 \leq 1$. Let $L_1 = \{0, b_1, 1\}$ and rename its elements as $a_0 = 0, a_1 = b_1, a_2 = 1$. Now, to find L_2, we compute

$$c_1 = a_0 \vee (a_1 \wedge b_2) = b_1 \wedge b_2,$$

and

$$c_2 = a_1 \vee (a_2 \wedge b_2) = b_1 \vee b_2.$$

We have

$$0 \leq b_1 \wedge b_2 \leq b_1 \leq b_1 \vee b_2 \leq 1,$$

so L_2 consists of the above elements. Again rename $a_0 = 0, a_1 = b_1 \wedge b_2, \ldots$, and continue this process. It is clear from the construction that

$$L_0 \subset L_1 \subset L_2 \subset \cdots,$$

so the set $\mathbf{L} = \bigcup_{n \geq 0} L_n$ is an infinite chain in L.

Now, we proved that there is an infinite chain $a_0 < a_1 < a_2 < \cdots$ in H so we can consider the following system

$$S = \{x \geq a_0, x \geq a_1, x \geq a_2, \ldots\}.$$

For any finite subsystem $S_0 = \{x \geq a_0, x \geq a_1, x \geq a_2, \ldots, x \geq a_n\}$, we have $a_{n+1} \in V_H(S_0)$, while a_{n+1} does not belong to $V_L(S)$. This proves that L is not equationally Noetherian. ∎

Example 5. If L is a distributive lattice then the ideal lattice $I(L)$ is a complete Heyting algebra in which residuals are given by

$$I : J = \{x \in L \,|\, x^{\downarrow} \cap J \subseteq I\}.$$

In fact, it is clear that $I : J$ so defined is an ideal of L and is such that $J \cap (I : J) \subseteq I$. Suppose that $K \in I(L)$ is such that $x \in K$ we have $x^{\downarrow} \cap J \subseteq I$. Consequently, $x \in I : J$ and $K \subseteq I : J$.

Corollary 6. *Let L be a complete lattice and satisfies the infinite distributive law $x \bigvee_{i \in I} y_i = \bigvee_{i \in I}(x \wedge y_i)$. Then every interval of is not K-equationally Noetherian.*

Proof. Suppose that $a, b \in L$ and $c, d \in [a, b]$. Consider the element $a = (d : c) \wedge b$. Since $a \leq d \leq d : c$ and $a \leq b$ we have $\alpha \in [a, b]$. Furthermore, by ([3], Theorem 7.11), we have that $c \wedge \alpha = c \wedge (d : c) \wedge b = c \wedge d \wedge b \leq d$, and if $x \in [a, b]$ is such that $c \wedge x \leq d$ then $x \leq d : c$ and $x \leq b$, and consequently $x \leq \alpha$. Hence the residual $[d : c]_a^b$ of d by c in $[a, b]$ exists and is $\alpha = (d : c) \wedge b$. Using Theorem 4, completes the proof of the theorem. ∎

Corollary 7. *Let L be a complete lattice and satisfies the infinite distributive law $x \bigvee_{i \in I} y_i = \bigvee_{i \in I}(x \wedge y_i)$. Then the set Con L of congruence on L is not K-equationally Noetherian.*

Corollary 8. *Let L be a complete lattice and satisfies the infinite distributive law $x \bigvee_{i \in I} y_i = \bigvee_{i \in I}(x \wedge y_i)$. Then the set of multiplicative closure on L is not K-equationally Noetherian.*

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