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LOWER BOUND FOR THE NUMBER OF 4-ELEMENT GENERATING SETS OF DIRECT PRODUCTS OF TWO NEIGHBORING PARTITION LATTICES

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Abstract

H. Strietz proved in 1975 that the minimum size of a generating set of the partition lattice $\mathrm{Part}(n)$ on the n-element set ($n \geq 4$) equals 4. This classical result forms the foundation for this study. Strietz's results have been echoed by L. Zádori (1983), who gave a new elegant proof confirming the outcome. Several studies have indeed emerged henceforth concerning four-element generating sets of partition lattices. More recently more studies have presented the approach for the lower bounds on the number $\lambda(n)$ of the four-element generating sets of $\mathrm{Part}(n)$ and statistical approach to $\lambda(n)$ for small values of n. Also, G. Czédli and the present author have recently proved that certain direct products of partition lattices are also 4-generated. In a recent paper, G. Czédli has shown that this result has connection with information theory. On this basis, here we give a lower bound on the number $\nu(n)$ of 4-element generating sets of the direct product $\mathrm{Part}(n) \times \mathrm{Part}(n+1)$ for $n \geq 7$ using the results from previous studies. For $n = 1, \ldots, 5$, we use a computer-aided approach; it gives exact values for $n = 1, 2, 3, 4$ but we need a statistical method for $n = 5$.

Keywords: partition lattice, four-element generating set, sublattice, statistics, computer program, direct product of lattices, generating partition lattices.

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1. Introduction

Let $\mathrm{Part}(A)$ denote the lattice of all partitions of A for a non-empty set A. Also, for any positive integer n, $\mathrm{Part}(n)$ denotes $\mathrm{Part}(\{1, 2, \ldots, n\})$, which is a geometric lattice. It was established by Strietz [9, 10] that for $n \geq 3$, $\mathrm{Part}(A)$ has a four-element generating set, a fact that was given a very elegant new proof by Zádori [11]. Further to this, more studies on four-element generating sets have since been conducted by Czédli [1, 2, 3], who considered infinite partition lattices.

As usual, we denote the direct product of the partition lattices $\mathrm{Part}(n)$ and $\mathrm{Part}(n + 1)$ by $\mathrm{Part}(n) \times \mathrm{Part}(n + 1)$. The number of 4-element generating sets of $\mathrm{Part}(n)\times\mathrm{Part}(n + 1)$ will be denoted by $\nu(n)$. Since the complicated structure of $\mathrm{Part}(n) \times \mathrm{Part}(n + 1)$ prevents us from determining the exact value of $\nu(n)$ for $n > 4$, we only present a lower bound for $\nu(n)$ for $n \geq 7$. For $n \in \{1, 2, 3, 4\}$, $\nu(n)$ has been computed by computer. For $n = 5$, some computer programs have been developed and used for investigation of $\nu(n)$: two billion randomly chosen 4-element subsets of $\mathrm{Part}(5) \times \mathrm{Part}(6)$ have been tested and the paper analyses the corresponding experimental result using mathematical statistics. At present stage, the computer-assisted study for $n > 5$ does not seem to be feasible.

We follow closely the approach presented in [5], where Theorem 4.4 states that certain direct products of direct powers of partitions lattices are 4-generated. In particular, for any integer $5 \leq n$, the direct product $\mathrm{Part}(n) \times \mathrm{Part}(n + 1)$ is four-generated, i.e., 1 is a lower bound for $\nu(n)$; of course, we are going to present a much larger lower bound here for large values of n, namely, for $n \geq 7$.

It is worth noting that Czédli [4] has shown that the study of small generating sets of partitions lattices and their direct products has connection with information theory.

Outline

In Section 2, Theorem 2.1 gives a lower bound on the number $\nu(n)$ of 4-element generating sets of $\mathrm{Part}(n) \times \mathrm{Part}(n + 1)$ for $n \geq 7$ and we prove this theorem. Section 3 outlines the use of mathematical statistics in analysis of the results from the computer programs.

2. A theoretical lower bound

For elements $x_1, \ldots, x_4$ of a lattice L, we say that $\langle x_1, \ldots, x_4\rangle$ is a *generating quadruple* and $\{x_1, \ldots, x_4\}$ is a *generating set* of L if the smallest sublattice of L containing each of $x_1, \ldots, x_4$ is L itself. In this section, we prove the following theorem.

Theorem 2.1. *Let $n\geq 7$ be an integer number and define*

$$t_n := \begin{cases} \binom{n-6}{(n-5)/2}, & \text{if } n \text{ is odd, and} \\ \min\left((n-2)(n-4)/8, \binom{n-6}{n/2-3}\right), & \text{if } n \text{ is even.} \end{cases} \tag{2.1}$$

Then $\mathrm{Part}(n)\times\mathrm{Part}(n+1)$ *has at least* $t_n^2\cdot n!\cdot(n+1)!/2$ *many* 4*-element generating sets.*

Proof. To ease our forthcoming notation, we will write t instead of t_n. First, we are dealing with generating quadruples of a special kind.

For n odd, Theorem 4.4 with $n=n_1'=m_1'+4$, (4.15), (4.9) and Remark 4.3 of [5] give in a straightforward way that

$$\mathrm{Part}(n)^t\times\mathrm{Part}(n+1)^t \text{ is 4-generated.} \tag{2.2}$$

Assume that n is even, and choose the parameters in Theorem 4.4 of [5] as follows. For brevity in inline formulas, $\mathrm{binc}(x,y)$ will denote the binomial coefficient with upper and lower parameters x and y, respectively. Let $d=1$, $i=1$, $n=n_1''$, $m_1=n-5$, $m_2=n-3$, $n_2'=n+1=m_2+4$. Then $w_1=(m_1+3)(m_1+1)/8=(n-2)(n-4)/8$ by (4.14) of [4], $q_1=\mathrm{sba}(1,m_1-1)\geq\mathrm{binc}(m_1-2,0)=1$ by (4.7) and (4.15), and $p_2\geq\mathrm{sba}(m_2-d,m_2-d-1)=\mathrm{sba}(n-4,n-5)\geq\mathrm{binc}(n-6,(n-6)/2)$ by (4.8) and (4.16). Since (4.20) and Remark 3 of [4] allow $\min(w_1q_1,p_2)$ and the computation above shows that $w_1q_1\geq(n-2)(n-4)/8$ and $p_2\geq\mathrm{binc}(n-6,n/2-3)$, it follows that with $t=t_n$ defined in (2.1), (2.2) also holds for n even. That is, in other words,

$$\underbrace{\mathrm{Part}(n)\times\cdots\times\mathrm{Part}(n)}_{t\text{ times}}\times\underbrace{\mathrm{Part}(n+1)\times\cdots\times\mathrm{Part}(n+1)}_{t\text{ times}} \tag{2.3}$$

is 4-generated for all $n\geq 7$.

Note that for an *odd* number n, elementary properties of Pascal's triangle show that $t=t_n$ given in (2.1) is the best lower bound one can extract from Theorem 4.4 of [4], and this is so even if n is large. Although Theorem 4.4 of [4] would allow a larger t_n for a *large* even n, we have no explicit formula for this larger t_n and the t_n given in (2.1) is the only possibility that Theorem 4.4 of [4] yields for $n=8$.

Let us fix a quadruple $\langle\vec\alpha,\vec\beta,\vec\gamma,\vec\delta\rangle$ such that $\{\vec\alpha,\vec\beta,\vec\gamma,\vec\delta\}$ generates the direct product (2.3). With more details, this quadruple consists of

$$\begin{aligned}
\vec{\alpha} &= \langle \alpha'_1,\ \alpha'_2, \dots, \alpha'_t, \quad \alpha''_{t+1}, \alpha''_{t+2}, \dots, \alpha''_{2t}\rangle,\\
\vec{\beta} &= \langle \beta'_1,\ \beta'_2, \dots, \beta'_t, \quad \beta''_{t+1}, \beta''_{t+2}, \dots, \beta''_{2t}\rangle,\\
\vec{\gamma} &= \langle \gamma'_1,\ \gamma'_2, \dots, \gamma'_t, \quad \gamma''_{t+1}, \gamma''_{t+2}, \dots, \gamma''_{2t}\rangle,\\
\vec{\delta} &= \langle \delta'_1,\ \delta'_2, \dots, \delta'_t, \quad \delta''_{t+1}, \delta''_{t+1}, \dots, \delta''_{2t}\rangle.
\end{aligned} \tag{2.4}$$

We also need the "columns" of (2.4), which we write in row vectors as follows:

$$\vec{g}^{(1)} = \langle \alpha'_1, \beta'_1, \gamma'_1, \delta'_1\rangle, \dots, \vec{g}^{(t)} = \langle \alpha'_t, \beta'_t, \gamma'_t, \delta'_t\rangle, \tag{2.5}$$

$$\vec{h}^{(t+1)} = \langle \alpha''_{t+1}, \beta''_{t+1}, \gamma''_{t+1}, \delta''_{t+1}\rangle, \dots,\ \vec{h}^{(2t)} = \langle \alpha''_{2t}, \beta''_{2t}, \gamma''_{2t}, \delta''_{2t}\rangle. \tag{2.6}$$

It would not be too hard to observe that the quadruples in (2.5) are pairwise different and the same holds for (2.6), but actually we are going to prove even more. But first, we need to fix some notation. The set of all permutations of $\{1, 2, \dots, n\}$ will be denoted by S_n; the meaning of S_{n+1} is analogous. Each $\pi \in S_n$ induces an automorphism $\widehat{\pi}$ of $\mathrm{Part}(n)$ in the natural way. That is, for $\varepsilon \in \mathrm{Part}(n)$, a pair $\langle i, j\rangle$ is collapsed by ε if and only if $\langle \pi(i), \pi(j)\rangle$ is collapsed by $\widehat{\pi}(\varepsilon)$. Let $\widehat{\pi}^*$ denote the componentwise action of $\widehat{\pi}$ on quadruples. In particular, $\widehat{\pi}^*(g^{(i)})$ is $\langle \widehat{\pi}(\alpha'_i), \widehat{\pi}(\beta'_i), \widehat{\pi}(\gamma'_i), \widehat{\pi}(\delta'_i)\rangle$ by the definition of $\widehat{\pi}^*$. Note that $\widehat{\pi}^*$ is an automorphism of the direct power $\mathrm{Part}(n)^4$. We claim that

$$\begin{cases} \text{for any } i, i' \in \{1, \dots, t\} \text{ and } \pi_1, \pi_2 \in S_n, \text{ if} \\ \langle i, \pi_1\rangle \neq \langle i', \pi_2\rangle, \text{ then } \widehat{\pi}_1^*(g^{(i)}) \neq \widehat{\pi}_2^*(g^{(i')}), \text{ and} \end{cases} \tag{2.7}$$

$$\begin{cases} \text{for any } j, j' \in \{t+1, \dots, 2t\} \text{ and } \sigma_1, \sigma_2 \in S_{n+1}, \\ \text{if } \langle j, \sigma_1\rangle \neq \langle j', \sigma_2\rangle, \text{ then } \widehat{\sigma}_1^*(h^{(j)}) \neq \widehat{\sigma}_2^*(h^{(j')}). \end{cases} \tag{2.8}$$

It suffices to deal with (2.7), because the argument for (2.8) is similar. Suppose that (2.7) fails and pick $i, i' \in \{1, \dots, t\}$ and $\pi_1, \pi_2 \in S_n$ such that

$$\langle i, \pi_1\rangle \neq \langle i', \pi_2\rangle \tag{2.9}$$

but $\widehat{\pi}_1^*(g^{(i)}) = \widehat{\pi}_2^*(g^{(i')})$. This equality means that

$$\langle \widehat{\pi}_1(\alpha'_i), \widehat{\pi}_1(\beta'_i), \widehat{\pi}_1(\gamma'_i), \widehat{\pi}_1(\delta'_i)\rangle = \langle \widehat{\pi}_2(\alpha'_{i'}), \widehat{\pi}_2(\beta'_{i'}), \widehat{\pi}_2(\gamma'_{i'}), \widehat{\pi}_2(\delta'_{i'})\rangle. \tag{2.10}$$

We let $\pi := \pi_2^{-1} \circ \pi_1$; note that we compose permutations from right to left, that is, $(\pi_2^{-1} \circ \pi_1)(x) = \pi_2^{-1}(\pi_1(x))$. Note also that $\widehat{\pi} = \widehat{\pi}_2^{-1} \circ \widehat{\pi}_1$. Hence, (2.10) yields that

$$\widehat{\pi}(\alpha'_i) = (\widehat{\pi}_2^{-1} \circ \widehat{\pi}_1)(\alpha'_i)) = \widehat{\pi}_2^{-1}(\widehat{\pi}_1(\alpha'_i)) = \widehat{\pi}_2^{-1}(\widehat{\pi}_2(\alpha'_{i'})) = (\widehat{\pi}_2^{-1} \circ \widehat{\pi}_2)(\alpha'_{i'}) = \alpha'_{i'}.$$

Similarly for the rest of components. So

$$\widehat{\pi}^*(\vec{g}^{(i)}) = \langle \widehat{\pi}(\alpha'_i), \widehat{\pi}(\beta'_i), \widehat{\pi}(\gamma'_i), \widehat{\pi}(\delta'_i)\rangle = \langle \alpha'_{i'}, \beta'_{i'}, \gamma'_{i'}, \delta'_{i'}\rangle = \vec{g}^{(i')}. \tag{2.11}$$

Now let f be a quaternary lattice term. Using that $\widehat{\pi}$ is a lattice automorphism and thus it commutes with f, let us compute:

$$\widehat{\pi}(f(\alpha_i, \beta_i, \gamma_i, \delta_i)) = f(\widehat{\pi}(\alpha_i), \widehat{\pi}(\beta_i), \widehat{\pi}(\gamma_i), \widehat{\pi}(\delta_i))) \overset{(2.11)}{=} f(\alpha'_{i'}, \beta'_{i'}, \gamma'_{i'}, \delta'_{i'}). \tag{2.12}$$

Since $\{\vec{\alpha}, \vec{\beta}, \vec{\gamma}, \vec{\delta}\}$ generates the direct product (2.3), for each

$$\vec{\mu} = \big(\mu'_1, \mu'_2, \ldots, \mu'_i, \ldots, \mu'_{i'}, \ldots, \mu'_t, \mu''_{t+1}, \ldots, \mu''_j, \ldots, \mu''_{2t}\big) \tag{2.13}$$

of the direct product (2.3), there is a quaternary lattice term f such that $\vec{\mu}$ is of the form

$$\begin{aligned} \vec{\mu} &= f\big(\vec{\alpha},\ \vec{\beta},\ \vec{\gamma},\ \vec{\delta}\big) \\ &= \langle \ldots, \underbrace{f(\alpha'_i, \beta'_i, \gamma'_i, \delta'_i)}_{\mu'_i}, \ldots, \underbrace{f(\alpha'_{i'}, \beta'_{i'}, \gamma'_{i'}, \delta'_{i'})}_{\mu'_{i'}}, \ldots, \underbrace{f(\alpha''_j, \beta''_j, \gamma''_j, \delta''_j)}_{\mu''_j}, \ldots\rangle \end{aligned} \tag{2.14}$$

where $j \in \{t+1, \ldots, 2t\}$. (Note that j and μ''_j will only be needed later, not here.) Combining (2.12), (2.13) and (2.14), it follows that

$$\widehat{\pi}(\mu'_i) = \widehat{\pi}(f(\alpha'_i, \beta'_i, \gamma'_i, \delta'_i)) = f(\alpha'_{i'}, \beta'_{i'}, \gamma'_{i'}, \delta'_{i'}) = \mu'_{i'}. \tag{2.15}$$

Now if $\pi_1 = \pi_2$, then π and $\widehat{\pi}$ are the identity permutations and (2.15) turns into $\mu'_i = \mu'_{i'}$. But this is a contradiction since $i \neq i'$ by (2.9) and so the fact that $\vec{\mu}$ is (2.13) is an arbitrary $(2t)$-tuple of (2.3) allows us to choose μ'_i and $\mu'_{i'}$ such that $\mu'_i \neq \mu'_{i'}$. Thus $\pi_1 \neq \pi_2$ and the automorphism $\widehat{\pi}$ is not identity map of $\mathrm{Part}(n)$. However, in the arbitrary $(2t)$-tuple (2.13), we can pick $\mu'_i \in \mathrm{Part}(n)$ arbitrarily, and we can let $\mu'_{i'} := \mu'_i$ regardless if $i' = i$ or $i' \neq i$. With this choice of $\mu'_{i'}$, we obtain from (2.15) that $\widehat{\pi}(\mu'_i) = \mu'_i$ for all $\mu'_i \in \mathrm{Part}(n)$, which contradicts the fact that now $\widehat{\pi}$ is not the identity map. The argument proving (2.7) is complete. Then, as we have already mentioned, (2.8) is also true.

Next, we claim that, for all $i \in \{1, \ldots, t\}$ and $j \in \{t+1, \ldots, 2t\}$,

$$\{\langle \alpha'_i, \alpha''_j\rangle, \langle \beta'_i, \beta''_j\rangle, \langle \gamma'_i, \gamma''_j\rangle, \langle \delta'_i, \delta''_j\rangle\} \text{ generates } \mathrm{Part}(n) \times \mathrm{Part}(n+1). \tag{2.16}$$

Let $\langle \mu'_i, \mu''_j\rangle$ be an arbitrary element of $\mathrm{Part}(n) \times \mathrm{Part}(n+1)$. We can extend the pair $\langle \mu'_i, \mu''_j\rangle$ to a $(2t)$-component vector $\vec{\mu}$ as in (2.13). As (2.14), shows, $\langle \mu'_i, \mu''_j\rangle$ is of the form

$$\begin{aligned} \langle \mu'_i, \mu''_j\rangle &= \langle f(\alpha'_i, \beta'_i, \gamma'_i, \delta'_i), f(\alpha''_j, \beta''_j, \gamma''_j, \delta'_j)\rangle \\ &= f(\langle \alpha'_i, \alpha''_j\rangle, \langle \beta'_i, \beta''_j\rangle, \langle \gamma'_i, \gamma''_j\rangle, \langle \delta'_i, \delta''_j\rangle). \end{aligned} \tag{2.17}$$

with some quaternary lattice term f. Hence, $\langle \mu'_i, \mu''_j\rangle$ belongs to the sublattice generated by $\{\langle \alpha'_i, \alpha''_j\rangle, \langle \beta'_i, \beta''_j\rangle, \langle \gamma'_i, \gamma''_j\rangle, \langle \delta'_i, \delta''_j\rangle\}$, proving (2.16).

Next, based on (2.16), we state even more than (2.16). Namely, we state that

$$\begin{array}{l}\text{for every } i \in \{1, \dots, t\}, \text{ for every } j \in \{t+1, \dots, 2t\},\\ \text{and for arbitrary permutations } \pi \in S_n \text{ and } \sigma \in S_{n+1},\\ \langle\langle \widehat{\pi}(\alpha'_i), \widehat{\sigma}(\alpha''_j)\rangle, \langle \widehat{\pi}(\beta'_i), \widehat{\sigma}(\beta''_j)\rangle, \langle \widehat{\pi}(\gamma'_i), \widehat{\sigma}(\gamma''_j\rangle, \langle \widehat{\pi}(\delta'_i), \sigma(\widehat{\delta''_j}\rangle)\rangle\\ \text{is a generating quadruple of } \mathrm{Part}(n) \times \mathrm{Part}(n+1).\end{array} \tag{2.18}$$

Clearly, the map $\kappa\colon \mathrm{Part}(n) \times \mathrm{Part}(n+1) \to \mathrm{Part}(n) \times \mathrm{Part}(n+1)$, defined by $\langle \mu'_i, \mu''_j\rangle \mapsto \langle \widehat{\pi}(\mu'_i), \widehat{\sigma}(\mu''_j)\rangle$, is bijective. Since lattice operations are computed componentwise and since both $\widehat{\pi}$ and $\widehat{\sigma}$ are automorphisms, it follows that κ is an automorphism of the direct product $\mathrm{Part}(n) \times \mathrm{Part}(n+1)$. Therefore, the elementwise κ-image of a generating set is again a generating set and (2.16) implies (2.18).

Next, we count how many generating quadruples occur in (2.18). Each of the parameters i and j can be chosen in t ways. Hence, the pair of subscripts $\langle i, j\rangle$ can be chosen in t^2 ways. There are $n! = |S_n|$ ways to chose the parameter π and, similarly, $(n+1)!$ ways to pick a permutation σ. Therefore,

$$\begin{array}{l}\text{there are } t^2 \cdot n! \cdot (n+1)! \text{ ways to chose a quadruple}\\ \langle i, j, \pi, \sigma\rangle \text{ with components occurring in (2.18).}\end{array} \tag{2.19}$$

We need to show that whenever a meaningful quadruple $\langle i', j', \pi', \sigma'\rangle$ of parameters is different from the quadruple occurring in (2.19) then, for the corresponding generating quadruple of $\mathrm{Part}(n) \times \mathrm{Part}(n+1)$,

$$\begin{aligned}&\langle\langle \widehat{\pi}(\alpha'_i), \widehat{\sigma}(\alpha''_j)\rangle, \langle \widehat{\pi}(\beta'_i), \widehat{\sigma}(\beta''_j)\rangle, \langle \widehat{\pi}(\gamma'_i), \widehat{\sigma}(\gamma''_j)\rangle, \langle \widehat{\pi}(\delta'_i), \widehat{\sigma}(\delta''_j)\rangle\rangle\\ &\neq \langle\langle \widehat{\pi}'(\alpha'_{i'}), \widehat{\sigma}'(\alpha''_{j'})\rangle, \langle \widehat{\pi}'(\beta'_{i'}), \widehat{\sigma}'(\beta''_{j'})\rangle, \langle \widehat{\pi}'(\gamma'_{i'}), \widehat{\sigma}'(\gamma''_{j'})\rangle, \langle \widehat{\pi}'(\delta'_{i'}), \widehat{\sigma}'(\delta''_{j'})\rangle\rangle.\end{aligned} \tag{2.20}$$

Here $\widehat{\pi}'$ denotes $\widehat{\pi'}$ and similarly for $\widehat{\sigma}'$, of course. So assume that $\langle i, j, \pi, \sigma\rangle \neq \langle i', j', \pi', \sigma'\rangle$. Then $\langle i, \pi\rangle \neq \langle i', \pi'\rangle$ or $\langle j, \sigma\rangle \neq \langle j', \sigma'\rangle$. Since the first t components of (2.3) and the last t components play a similar role, we can assume that $\langle i, \pi\rangle \neq \langle i', \pi'\rangle$. Then, applying (2.7) with $\langle \pi, \pi'\rangle$ playing the role of $\langle \pi_1, \pi_2\rangle$ and taking (2.5) account, we obtain that

$$\begin{aligned}&\langle \widehat{\pi}(\alpha'_i), \widehat{\pi}(\beta'_i), \widehat{\pi}(\gamma'_i), \widehat{\pi}(\delta'_i)\rangle = \widehat{\pi}^*(\vec{g}^{(i)}) \neq \widehat{\pi}'^*(\vec{g}^{(i')})\\ &= \langle \widehat{\pi}'(\alpha'_{i'}), \widehat{\pi}'(\beta'_{i'}), \widehat{\pi}'(\gamma'_{i'}), \widehat{\pi}'(\delta'_{i'})\rangle.\end{aligned} \tag{2.21}$$

Thinking of the first components of the pairs occurring in (2.20), we obtain that (2.21) implies (2.20). This shows the validity of (2.20).

Now, (2.18), (2.19) and (2.20) together imply that

(2.22) the number of generating quadruples we have considered is $t^2 \cdot n! \cdot (n+1)!$.

Next, consider a generating quadruple

$$\langle\langle\widehat{\pi}(\alpha'_i), \widehat{\sigma}(\alpha''_j)\rangle, \langle\widehat{\pi}(\beta'_i), \widehat{\sigma}(\beta''_j)\rangle, \langle\widehat{\pi}(\gamma'_i), \widehat{\sigma}(\gamma''_j\rangle, \langle\widehat{\pi}(\delta'_i), \sigma(\widehat{\delta}''_j\rangle)\rangle \tag{2.23}$$

from (2.18). It determines a generating set

$$\{\langle\widehat{\pi}(\alpha'_i), \widehat{\sigma}(\alpha''_j)\rangle, \langle\widehat{\pi}(\beta'_i), \widehat{\sigma}(\beta''_j)\rangle, \langle\widehat{\pi}(\gamma'_i), \widehat{\sigma}(\gamma''_j\rangle, \langle\widehat{\pi}(\delta'_i), \sigma(\widehat{\delta}''_j\rangle)\}. \tag{2.24}$$

Using the same technique with quaternary lattice terms as in the neighborhood of (2.14), it is straightforward to see that the first components of the pairs in (2.24) generate $\mathrm{Part}(n)$. We know from Zádori [11] that $\mathrm{Part}(n)$ cannot be generated with fewer than four elements. Hence, there are four different first components in (2.24), implying that (2.24) is a 4-element set, so a 4-element generating set.

Assume that a generating quadruple

$$\langle\langle\widehat{\pi}'(\alpha'_{i'}), \widehat{\sigma}'(\alpha''_{j'})\rangle, \langle\widehat{\pi}'(\beta'_{i'}), \widehat{\sigma}'(\beta''_{j'})\rangle, \langle\widehat{\pi}'(\gamma'_{i'}), \widehat{\sigma}'(\gamma''_{j'})\rangle, \langle\widehat{\pi}'(\delta'_{i'}), \widehat{\sigma}'(\delta''_{j'})\rangle\rangle \tag{2.25}$$

different from (2.23) gives the same generating set (2.24) as (2.23). In the worst case, there could be $4! = 24$ different generating quadruples giving the same set (2.24); if this was the case then the denominator in the theorem would be 24 rather than 2. But in [5], α_i and δ_i were constructed in a way that each of them has some specific property that distinguish it from the rest of the four partitions. These specific properties are explicitly described in page 422 and (the beginning of) page 423 in [5]. We do not give the exact details of these properties here; we only mention that for a large odd n, α_i is the only partition out of α_i, β_i, γ_i and δ_i that has an $(n+1)/2$-element block and has exactly two blocks. The specific properties described in [5] are clearly preserved by automorphisms. This implies that $\widehat{\pi}(\alpha'_i) = \widehat{\pi}'(\alpha'_{i'})$ and $\widehat{\pi}(\delta'_i) = \widehat{\pi}'(\delta'_{i'})$. Although we did not characterize β_i and γ_i by individual properties among the four partitions constructed in [5], we did characterize the *set* $\{\beta_i, \gamma_i\}$ by such a property in pages 422–423 of [5]; this property again is preserved by automorphisms. Hence, $\{\widehat{\pi}(\beta'_i), \widehat{\pi}(\gamma'_i)\}$ is necessarily the same as the set $\{\widehat{\pi}'(\beta'_{i'}), \widehat{\pi}'(\gamma'_{i'})\}$. This implies that there are only at most two ways to choose the quadruple (2.25): either it is the same as (2.23), or we get it from (2.23) by interchanging the middle two pairs of partitions. Now we are in the position to conclude that the number of 4-element generating sets is at least half of the number of generating quadruples given in (2.22). This completes the proof of Theorem 2.1. ∎

3. Computer assisted results and statistical analysis

3.1. Confidence interval estimation

This subsection outlines the theoretical background of extracting the most information from experimental data by means of Statistical analysis. In the subsequent subsection, we apply this theory to the experimental data obtained by computer programs.

We explore the estimation of the unknown parameters of a probability model. In our case, the probability model that was used is known as the binomial distribution which has exactly two outcomes of an experiment "success" and "failure". These outcomes have fixed probabilities, which are denoted by p and q, respectively. Here $p, q \in [0,1] \subset \mathbb{R}$ are our unknown parameters that we would like to determine with certain confidence. Of course, $p + q = 1$, so it suffices to determine p. To achieve our goal, we take a (big) natural number N and we perform N independent trials, and count the number of successes; this number is usually denoted by X. Considering N fixed, X (the number of successes) is a random variable. This random variable X is called the *binomial distribution with parameters N and p.* As one would certainly expect, we estimate the unknown parameter p by

$$\widehat{p} = \frac{X}{N}. \tag{3.1}$$

The real statistics enters the scene with our intention to establish how reliable this estimation is. We therefore, going after standard books like Hodges and Lehmann [6], Lefebvre [7] and Mendenhall [8], adopt the construction of confidence interval as a way of achieving the desired estimation, as it proposes plausible values for the unknown parameter p. Hence, let $\widehat{q} := 1 - \widehat{p}$, and with a "confidence level" of $1 - \alpha \in (0,1) \subset \mathbb{R}$, we define the so-called unbiased estimator of the variance as

$$\widehat{\sigma} := \sqrt{\frac{\widehat{p} \cdot \widehat{q}}{N-1}}. \tag{3.2}$$

We also need the positive real number z that is implicitly defined by the equation

$$1 - \alpha = \int_{-z}^{z} \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2} dx. \tag{3.3}$$

Notably, the function to be integrated in (3.3) is the density function of the *standard normal distribution.* It is worth noting that the value of z for the most common values of α are given in any statistics book, including the already-mentioned Hodges and Lehmann [6], Lefebvre [7] and Mendenhall [8]. However, for this study, we shall use the value given in the last column of Table 1, which has been obtained by computer algebra (Maple V).

α	0.100	0.050	0.010	0.001	0.000 001
$1-\alpha$	0.900	0.950	0.990	0.999	0.999 999
$z = z_{1-\alpha}$	1.644 85	1.959 96	2.575 83	3.290 53	4.891 638 475

Table 1. Except for the last column with $\alpha = 0.000\,001$, $z = z_{1-\alpha}$ is taken from `https://mathworld.wolfram.com/ConfidenceInterval.html`

We are going to use the acronym $\mathrm{CI}_{1-\alpha}$ for Confidence Intervals at confidence level $1-\alpha$. Depending on the number N and N samples, it is defined as

$$\mathrm{CI}_{1-\alpha} = \left[\widehat{p} - z_{1-\alpha}\sqrt{\frac{\widehat{p}\cdot\widehat{q}}{N-1}};\ \widehat{p} + z_{1-\alpha}\sqrt{\frac{\widehat{p}\cdot\widehat{q}}{N-1}}\,\right]. \tag{3.4}$$

Since the samples are randomly chosen, the confidence intervals are random, while p is a concrete real number. The connection between p and $\mathrm{CI}_{1-\alpha}$ is that the probability of $p \in \mathrm{CI}_{1-\alpha}$ tends to $1-\alpha$ as $N \to \infty$.

3.2. Data from computer programs

Most of the data to be reported in this section were achieved by programs which are available from G. Czédli's website [1]. However and importantly, two other independent programs were developed by the present authors on different computers and with different attitudes to computer programming in which the same results were obtained. This is a good indicator for the accuracy and reliability of the results. These programs can be found on the first author's website [2]. The programs due to G. Czédli were written in Bloodshed Dev-Pascal v1.9.2 (Freepascal) under Windows 10 and also in Maple V. Release 5 (1997). Meanwhile, the programs due to the present authors were written in Python 3.8 which works well with any operating system. *Sympy* which is an external library with functions for computing partitions was imported into Python to compute *partitions*. Additionally, *itertool*, a built-in function in Python was crucial for calculating combinations for the four partitions from which meets and joins were evaluated. Using these programs, we are going to give our results on the number $\nu(n)$ of the 4-element generating sets of the direct product $\mathrm{Part}(n) \times \mathrm{Part}(n+1)$ for some values of n. Specifically, we give $\nu(n)$ for $n \in \{2, 3, 4\}$ in Table 2.

As it is usual in Combinatorics, the number of partitions of an n-element set, that is, the size of $\mathrm{Part}(n)$ is denoted by $\mathrm{Bell}(n)$. Therefore, $\mathrm{Part}(n) \times \mathrm{Part}(n+1)$ consists of $\mathrm{Bell}(n) \cdot \mathrm{Bell}(n+1)$ many elements. This allows us to compute the

[1] http://www.math.u-szeged.hu/~czedli/
[2] http://www.math.u-szeged.hu/~oluoch/

number of its four-elements subsets, and we obtain that the exact "theoretical" probability that a randomly chosen four-element subset is a generating one is

$$p = p(n) := \nu(n) \cdot \binom{\mathrm{Bell}(n) \cdot \mathrm{Bell}(n+1)}{4}^{-1}. \tag{3.5}$$

Note that $100 \cdot p(n)$ is also given in Table 2.

$\mathrm{Part}(n) \times \mathrm{Part}(n+1)$	$\mathrm{Part}(2) \times \mathrm{Part}(3)$	$\mathrm{Part}(3) \times \mathrm{Part}(4)$	$\mathrm{Part}(4) \times \mathrm{Part}(5)$
$\binom{\mathrm{Bell}(n) \cdot \mathrm{Bell}(n+1)}{4}$	210	1 215 450	15 304 580 655
$\nu(n)$	14	600	2 049 960
%, i.e. $100 \cdot p(n)$	6.666 666 667	0.031 593 237	0.013 394 421
Computer time	0.3 sec	2.8 sec	1 day + 17 hours

Table 2. The number $\nu(n)$ of the four-element generating sets of $\mathrm{Part}(n) \times \mathrm{Part}(n+1)$ for $n \in \{2, 3, 4\}$.

It would be interesting to add a column for $\mathrm{Part}(5) \times \mathrm{Part}(6)$ to Table 2 but this is not feasible with our programs and computers. This is why we need a statistical approach to $\mathrm{Part}(5) \times \mathrm{Part}(6)$.

Experiment 3.1. Out of 2 billion $= 2 \times 10^9$ experiments, we encountered a total of 182 107 successes which translates to 0.018 210 7% of success which consumed a total of 124 hours + 617 minutes computer time.

In order to estimate $p = p(5)$, first we choose the confidence level to be 0.999. Then $z_{0.999} \approx 3.290\,526\,731$, obtained by computer algebra, is more exact than Table 1. By (3.4), we obtain that

$$\begin{aligned} &\mathrm{CI}_{0.999} = [0.000\,090\,351\,432\,33\ ;\ \ 0.000\,091\,755\,567\,67] \\ &\text{for } p = p(5) \text{ with estimated confidence level } 0.999\,. \end{aligned} \tag{3.6}$$

With percentage rather than portion, this means that

$$\begin{aligned} &\text{the interval } [0.009\,035\,143\,233\ ;\ \ 0.009\,175\,556\,767] \\ &\text{contains } 100 \cdot p(5) \text{ with estimated probability } 0.999\,. \end{aligned} \tag{3.7}$$

Taking the number of all four-element subsets into account, (3.7) leads to

$$\begin{aligned} &\nu(5) \in [46\,716\,946\,330\ ;\ \ 47\,442\,965\,990] \text{ with} \\ &\text{estimated confidence level } 1 - \alpha = 0.999. \end{aligned} \tag{3.8}$$

Similarly, going after the last column of Table 1, we obtain that

$$\begin{aligned} &\nu(5) \in [46\,540\,312\,210\ ;\ \ 47\,619\,600\,110] \text{ with es-} \\ &\text{timated confidence level } 1 - \alpha = 0.999\,999. \end{aligned} \tag{3.9}$$

The word "estimated" in (3.6)–(3.9) indicates that the containment tends to have the given probability. That is, the containment "$\in$" in (3.8) and (3.9) holds with approximate probabilities 0.999 and 0.999 999, respectively. Fortunately, $N = 2 \cdot 10^9$ is big enough to say that 0.999 and 0.999 999 are close to the "real" confidence levels. Even if it is not absolutely sure that $\nu(5)$ belongs the interval mentioned in, say, (3.9), the probability that our 2 billion experiments have led to an interval *not* containing $\nu(5)$ is at most 0.000 001.

Finally, for some values of n, Table 3 below shows $t_n^2 \cdot n! \cdot (n+1)!/2$; remember that the number $t_n^2 \cdot n! \cdot (n+1)!/2$ defined in (2.1) is only a lower estimate of $\nu(n)$. The "percentage" in Table 3 gives

$$100 \cdot t_n^2 \cdot n! \cdot (n+1)!/2 \cdot \binom{\mathrm{Bell}(n) \cdot \mathrm{Bell}(n+1)}{4}^{-1}.$$

n	$t_n^2 \cdot n! \cdot (n+1)!/2$	percentage
7	$1.016\,064\,000 \cdot 10^{8}$	$0.140\,324\,430 \cdot 10^{-14}$
8	$2.926\,264\,320 \cdot 10^{10}$	$0.119\,544\,169 \cdot 10^{-17}$
9	$5.925\,685\,248 \cdot 10^{12}$	$0.393\,094\,745 \cdot 10^{-21}$
10	$2.607\,301\,509 \cdot 10^{15}$	$0.163\,142\,214 \cdot 10^{-24}$
49	$1.023\,816\,392 \cdot 10^{151}$	$0.156\,021\,809 \cdot 10^{-218}$
50	$1.796\,834\,549 \cdot 10^{135}$	$0.319\,339\,534 \cdot 10^{-244}$
99	$2.877\,390\,087 \cdot 10^{367}$	$0.196\,174\,914 \cdot 10^{-548}$
100	$6.082\,937\,175 \cdot 10^{323}$	$0.720\,917\,306 \cdot 10^{-604.}$

Table 3. The lower estimates given by Theorem 2.1 for some n.

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