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# ON THE NON-INVERSE GRAPH OF A GROUP

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**Abstract**

Let $(G, *)$ be a finite group and $S = \{u \in G | u \neq u^{-1}\}$, then the inverse graph is defined as a graph whose vertices coincide with $G$ such that two distinct vertices $u$ and $v$ are adjacent if and only if either $u * v \in S$ or $v * u \in S$. In this paper, we introduce a modified version of the inverse graph, called $i^*$-graph associated with a group $G$. The $i^*$-graph is a simple graph with vertex set consisting of elements of $G$ and two vertices $x, y \in \Gamma$ are adjacent if $x$ and $y$ are not inverses of each other. We study certain properties and characteristics of this graph. Some parameters of the $i^*$-graph are also determined.

**Keywords:** inverse graph, non-inverse graph.

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## 1. Introduction

For the terms and definitions in graph theory, refer to [14] and for those in group theory, refer to [6]. To avoid confusions regarding the terminology of groups and graphs, we represent the identity element of a group $G$ by $i_G$ and a graph by $\Gamma$.

Graph constructions using various concepts of group theory have been extensively studied in the literature. In [2], zero-divisor graph of a commutative ring and its properties were studied and the same were investigated for semigroups in [7]. Recently, some studies on a new family of graphs, as a generalization of zero-divisor graphs have been introduced in [11] and determined an upper-bound for the diameter of those graphs.

In [5], the intersection graph of non-trivial left ideals of a ring are studied and the rings for which the intersection graph is connected are characterized. In [3], the power graph of a finite group in which two vertices are adjacent if one is a power of the other was introduced and it was eshtablished that the only finite group whose automorphism group is the same as that of its power graph is the Klein 4-group.

In [1], a new graph construction called *inverse graph* associated with finite groups was introduced as follows. If $(G, *)$ is a finite group $S = \{u \in G | u \neq u^{-1}\}$, then inverse graph is defined as a graph whose vertices coincide with $G$ such that two distinct vertices $u$ and $v$ are adjacent if and only if either $u*v \in S$ or $v*u \in S$. Motivated by the studies mentioned above, we introduce a modified version of inverse graphs in the following section.

## 2. THE NON-INVERSE GRAPH OF A GROUP

**Definition 1.** Let $G$ be a group with binary operation $*$. The *non-inverse graph* (in short, *$i^*$-graph*) of $G$, denoted by $\Gamma$, is a simple graph with vertex set consisting of elements of $G$ and two vertices $x, y \in \Gamma$ are adjacent if $x$ and $y$ are not inverses of each other. That is, $x \sim y \iff x * y \neq i_G \neq y * x$, where $i_G$ is the identity element of $G$.

First, recall the definition of the direct product of groups as given below.

**Definition 2** [12]**.** A group $G$ decomposes into a direct product of subgroups $G_1, \dots, G_k$ if

(i) every element $g \in G$ decomposes uniquely as $g = g_1 \cdots g_k$, $g_i \in G_i$;

(ii) $g_i g_j = g_j g_i$ for $g_j \in G_j, i \neq j$.

The result given below discusses the non-inverse graph of the direct product of $n$ cyclic groups and we make use of the lemma stated below.

**Lemma 2.1** [8]**.** *If $\{G_i | i = 1, 2, \dots, n\}$ is a family of groups, then the direct product $\prod_{i=1}^{n} G_i$ is a group.*

**Proposition 2.2.** *The non-inverse graph associated with the direct product of $n$ cyclic groups under multiplication of order 2 with generators $a_i$ with $a_i^2 = 1$, $i = 1, 2, \dots, n$ is complete.*

***Proof.*** By Lemma 2.1, the direct product of $n$ cyclic groups of order 2 is a group and since multiplication is commutative, it is an Abelian group with identity element 1. Given that the generators $a_i, i = 1, 2, \dots, n$ satisfy the condition $a_i^2 = 1$, the diagonal elements of the Cayley table are all 1 which means that $a_i$ are inverse of itself. By Definition 1, each $a_i$ is adjacent to all the remaining vertices in the non-inverse graph associated with the direct product of $n$ cyclic groups of order 2 and hence it is complete. ■

**Proposition 2.3.** *The non-inverse graph $\Gamma$ associated with a group $G$ is connected.*

***Proof.*** For any group $G$, there exists an identity element $i_G$ which is inverse of itself. Hence, by Definition 1, $i_G$ is adjacent to all other vertices in $\Gamma$. Therefore, the non-inverse graph $\Gamma$ associated with $G$ is connected. ∎

**Theorem 2.4.** *Given a group $G$ with identity element $i_G$ and its non-inverse graph $\Gamma$, the graph $\Gamma - i_G$ is disconnected if and only if order of $G$ is 3 and $G - i_G$ has no self inverse elements.*

***Proof.*** Assume that $\Gamma - i_G$ is disconnected. We have to prove that $o(G) = 3$ and $i_G$ is the only self-inverse element in $G$.

If $G$ has at least one element, say $a$ which is the inverse of itself, then $a$ is adjacent to all other vertices in $\Gamma$ and hence it is adjacent to all vertices in $\Gamma - i_G$, which is a contradiction to the fact that $\Gamma - i_G$ is disconnected.

If $o(G) > 3$, then there are two cases as follows.

*Case* 1. If $o(G)$ is even, then there exists at least one element, say $x \neq e$, in $G$ which is the inverse of itself. Therefore, as mentioned earlier, $x$ will be adjacent to all other vertices in $\Gamma$ and hence in $\Gamma - i_G$, a contradiction.

*Case* 2. Let $o(G)$ be odd. If $G$ has self-inverse elements other than $i_G$, then, as explained above, $\Gamma - i_G$ is connected. Otherwise, there exist at least two partitions in $\Gamma$ with two elements each. Since every element other than $i_G$ is non-adjacent to its own inverse, any two partitions will form a complete bipartite graph showing that $\Gamma - i_G$ is connected.

Thus, in all cases we arrive at a contradiction to the hypothesis that $\Gamma - i_G$ is disconnected. Therefore, $o(G) = 3$ and $i_G$ is the only self-inverse element in $G$.

Conversely, assume that $o(G) = 3$ and $i_G$ is the only self-inverse element in $G$. Therefore, $G = \{e, a, b\}$ such $a$ and $b$ are inverses of each other. Hence, by Definition 1, $a$ and $b$ are non-adjacent in $\Gamma - i_G$. Therefore, $a$ and $b$ are in two components in $\Gamma - i_G$, completing the proof. ∎

**Theorem 2.5.** *The non-inverse graph associated with any group $(G, *)$ of order $n$ is complete multipartite graph with $\frac{n+l}{2}$ partitions, where $l$ is the number of self-inverse elements in $G$.*

***Proof.*** Let $A \subseteq G$, be the set of all self-inverse elements of $G$. Then, $|A| = l$. By Definition 1, every element $a \in A$ will be adjacent to all other elements of $G$. Thus, the corresponding partition of $G$ containing $a$ will be a singleton. Therefore, there are $l$ partitions consisting of exactly one element. Now, every element $x$ of $\overline{A}$ will be adjacent to all elements of $G$ other than its inverse. Thus, the partition containing $x$ consists of one more element, which is the inverse of

$x$. Therefore, partitions containing the elements of $\overline{A}$ are 2-element sets. Hence, there are $l$ partitions with one element and $\frac{n-l}{2}$ partitions with two elements each. Therefore, the non-inverse graph of $G$ is a complete multipartite graph with $\frac{n-l}{2} + l = \frac{n+l}{2}$ partitions. ■

**Theorem 2.6.** *The size of non-inverse graph associated with a group $G$ of order $n$ with $l$ self-inverse elements is $\frac{n^2-2n+l}{2}$.*

***Proof.*** Let $V(\Gamma) = \{v_1, v_2, \ldots, v_n\}$ and $E(\Gamma)$ be the vertex set and edge set of non-inverse graph $\Gamma$ associated with a group $G$. The vertices $v_i$, $i = 1, 2, \ldots, l$ have degree $n-1$ each and the remaining $n-l$ vertices have degree $n-2$. Therefore,

$$\sum_{i=1}^{n} \deg(v_i) = l(n-1) + (n-l)(n-2) = n^2 - 2n + l.$$

By the first theorem of graph theory, we have

$$\sum_{i=1}^{n} \deg(v_i) = 2|E(\Gamma)|.$$

Therefore, $|E(\Gamma)| = \frac{n^2-2n+l}{2}$. ■

## 3. Some parameters of non-inverse graphs

In this section, we discuss certain parameters of non-inverse graphs.

**Proposition 3.1.** *The chromatic number of the non-inverse graph associated with any group $G$ of order $n$ is $\frac{n+l}{2}$, where $l$ is the number of self-inverse elements in $G$.*

***Proof.*** Since a self-inverse vertex is adjacent to all other vertices in $\Gamma$, all the $l$ self-inverse vertices are assigned $l$ different colors. The remaining $n-l$ vertices are assigned $\frac{n-l}{2}$ colors as each vertex and its inverse can be given the same color. Therefore, $\chi(\Gamma) = l + \frac{n-l}{2} = \frac{n+l}{2}$. ■

**Proposition 3.2.** *The independence number of the non-inverse graph $\Gamma$ associated with a group $G$ with at least one element which is not self-inverse is 2.*

***Proof.*** Let $a$ be an element in $G$ which is not a self-inverse. Then there exists another element $b$ in $G$ which is the inverse of $a$. So the vertices corresponding to $a$ and $b$ in $\Gamma$ will be non-adjacent and hence can be in one independent set. Since all other vertices of $\Gamma$ are adjacent to both $a$ and $b$ in $\Gamma$, the set $\{a, b\}$ is a maximal independence set in $\Gamma$. Therefore, independence number of $\Gamma$ is 2. ■

**Proposition 3.3.** *The domination number of the non-inverse graph associated with any group $G$ is* 1.

***Proof.*** For any group $G$, there exists an identity element $i_G$ which is self-inverse. Thus, by Definition 1, $i_G$ is adjacent to all the remaining vertices in $\Gamma$. Hence, $G(\Gamma) = 1$. ∎

**Theorem 3.4.** *The set of all self inverse elements of a group $G$ form the centre of non-inverse graph of $G$.*

***Proof.*** Consider the partition of vertex set of $\Gamma$ as mentioned in Theorem 2.5. The vertices belonging to the 2-element set have eccentricity 2 and the remaining vertices belonging to singleton set have eccentricity 1. Therefore, the diameter and radius of $\Gamma$ is 2 and 1 respectively. The vertices of singleton sets having eccentricity 1 in $\Gamma$ are the self-inverse elements in $G$. Thus, the self-inverse elements in $G$ are the central points in $\Gamma$. Therefore, the set of all self-inverse elements in $G$ form the centre of $\Gamma$. ∎

The following theorem is used in our next result.

**Theorem 3.5** [14]**.** *If $G$ is a simple graph, then $\kappa(G) \leq \kappa'(G) \leq \delta(G)$, where, $\kappa(G), \kappa'(G), \delta(G)$ are vertex connectivity, edge connectivity and minimum degree of $G$.*

**Theorem 3.6.** *For a non-inverse graph $\Gamma$ associated with a group $G$, the vertex-connectivity and edge-connectivity are always equal.*

***Proof.*** By Theorem 3.5 , $\kappa(\Gamma) \leq \kappa'(\Gamma) \leq \delta(\Gamma)$ where, $\kappa(\Gamma), \kappa'(\Gamma), \delta(\Gamma)$ are vertex connectivity, edge connectivity and minimum degree of the non-inverse graph $\Gamma$ associated with a group $G$ of order $n$ respectively.

*Case* 1. Let all elements of the group $G$ be self-inverses. Then, $\delta(\Gamma) = n-1$. Then every partition of $V(\Gamma)$ will be singleton (as mentioned in Theorem 2.5). Since $\Gamma$ has no cut-edges and cut-vertices and every vertex of $\Gamma$ is adjacent to all other vertices, we have to remove $n-1$ edges or $n-1$ vertices to make the graph disconnected. Therefore, $\kappa(\Gamma) = \kappa'(\Gamma) = \delta(\Gamma) = n-1$.

*Case* 2. Assume that $G$ has some elements which are not self-inverse. Hence, $\delta(\Gamma) = n-2$. Since degree of any self-inverse element in $\Gamma$ is $n-1$, we have to remove all singleton partitions from the vertex set of $\Gamma$. Now, we have only 2-element partitions in $V(\Gamma)$. The two vertices in a partition will be adjacent to all other vertices of all other remaining partitions. Therefore, we have to remove the vertices until only one partition remains to make the graph disconnected. Therefore, $\kappa(\Gamma) = n-2$. Note that the degree of a vertex which belongs to a 2-element partition of $V(\Gamma)$ is $n-2$ which is the minimum degree of $\Gamma$. Therefore, $\kappa'(\Gamma) = n-2 = \delta(\Gamma)$. Therefore, $\kappa(\Gamma) = \kappa'(\Gamma) = \delta(\Gamma) = n-2$. ∎

For the next result, we use the lemma stated below.

**Lemma 3.7** [10]**.** *A complete multipartite graph $G$ of at least three vertices is Hamiltonian if and only if the cardinality of no partite set is larger than sum of the cardinalities of all the other partite sets.*

**Proposition 3.8.** *Let $G$ be a group with $n \geq 4$ elements, then the circumference and girth of the non-inverse graph associated with $G$ are $n$ and 3, respectively.*

***Proof.*** By Theorem 2.5, non-inverse graph $\Gamma$ associated with a group $G$ of order $n$ is a complete multipartite graph. By Lemma 3.7 and since $n \geq 4$, there exists a Hamiltonian cycle in $\Gamma$, which implies that circumference of $\Gamma$ is $n$.

Since $n \geq 4$, there are at least three partitions of $V(\Gamma)$ and vertices from each partition are mutually adjacent which forms a cycle of length 3. Therefore, girth of $\Gamma$ is 3. ■

## 4. Further results on non-inverse graphs

The following results discuss the characterisation for the non-inverse graph of a group to be regular with certain conditions.

**Theorem 4.1.** *Let $G$ be a group of odd order $n$. Then, the non-inverse graph $\Gamma$ associated with $G - i_G$ is $(n-3)$-regular if and only if there are no self-inverse elements in $G - i_G$.*

***Proof.*** Let $G = \{i_G = x_1, x_2, \ldots, x_n\}$ be a group of odd order $n$. Without loss of generality, let the identity element be $x_1$. Let $\Gamma(G) - x_1$ be $(n-3)$-regular. We have to show that there are no self inverse elements in $G - x_1$. If possible, assume the contrary. Without loss of generality, let $x_2$ be a self inverse element in $G - x_1$. Then by Definition 1, $x_2$ is adjacent to all $x_i$'s, $i \neq 2$ in $\Gamma(G) - x_1$. Therefore, degree of $x_1$ in $\Gamma(G) - x_1$ is $n-2$ which is a contradiction to the fact that $\Gamma(G) - x_1$ is $(n-3)$-regular. Therefore, $G - x_1$ cannot have self inverse elements.

Conversely, let there be no self-inverse elements in $G - x_1$. Therefore, by Definition 1, each vertex is adjacent to all vertices in $\Gamma(G) - x_1$ other than its inverse. Hence, every vertex in $\Gamma(G) - x_1$ has degree $n-3$. Therefore, $\Gamma(G) - x_1$ is $(n-3)$-regular. ■

**Theorem 4.2.** *Let $G$ be a group of order $n$. Then, the non-inverse graph associated with $G - i_G$ is $(n-2)$-regular if and only if $G - i_G$ contains only self-inverse elements.*

***Proof.*** Let $G = \{i_G = x_1, x_2, \ldots, x_n\}$ be a group of $n$ elements. Without loss of generality, let the identity element be $x_1$. Let $\Gamma(G) - x_1$ be $(n-2)$-regular. We have to show that there are only self inverse elements in $\Gamma(G) - x_1$. If possible, assume the contrary. Without loss of generality, let there be one element say, $x_2$ in $\Gamma(G) - x_1$ which is not self inverse. Then, there must be another element say $x_3$ which is inverse of $x_2$. Thus, by Definition 1, $x_2$ is adjacent to $x_i$'s, $i \neq 2, 3$ in $\Gamma(G) - x_1$ and degree of $x_2$ is $n-3$. Similarly, degree of $x_3$ is $n-3$ which is a contradiction to the fact that $\Gamma(G) - x_1$ is $(n-2)$-regular.

Conversely, let there be only self-inverse elements in $G - x_1$. Therefore, by Definition 1, each vertex is adjacent to all vertices in $\Gamma(G) - x_1$. Hence, every vertex in $\Gamma(G) - x_1$ has degree $n-2$. Therefore, $\Gamma(G) - x_1$ is $(n-2)$-regular. ∎

Now, recall the following results.

**Lemma 4.3** [13]**.** *For an arbitrary graph $G$ of order $n$,*

$$\omega(G) \geq \sum_{i=1}^{n} \frac{1}{n - d_i},$$

*where $\omega$ is the clique number and $d_i$ is the degree of the vertex $v_i$, $i = 1, 2, \ldots, n$ in $G$.*

**Lemma 4.4** [13]**.** *The chromatic number of a graph $G$ is greater than or equal to its clique number, i.e., $\chi(G) \geq \omega(G)$.*

In view of the above lemmas, we have the next result,

**Theorem 4.5.** *The clique number of non-inverse graph associated with a group $G$ of order $n$ is equal to its chromatic number.*

***Proof.*** Let $G$ be a group of order $n$ and $l$ be the number of self-inverse elements in $G$. By Lemma 4.3 and Lemma 4.4,

$$\sum_{i=1}^{n} \frac{1}{n - d_i} \leq \omega(G) \leq \chi(G). \tag{1}$$

Since there are $l$ vertices with degree $(n-1)$ and $(n-l)$ vertices with degree $(n-2)$, we have,

$$\sum_{i=1}^{n} \frac{1}{n - d_i} = l \frac{1}{n - (n-1)} + (n - l) \frac{1}{n - (n-2)} = \frac{n + l}{2}. \tag{2}$$

In view of Lemma 3.1, Equation (1) and Equation (2) together will give,

$$\frac{n + l}{2} \leq \omega(G) \leq \chi(G) = \frac{n + l}{2}.$$

Therefore, $\omega(G) = \frac{n+l}{2} = \chi(G)$. ∎

Now, recall the following important properties of group isomorphism.

**Lemma 4.6** [9]**.** *If two groups are isomorphic, they must have the same order.*

**Lemma 4.7** [9]**.** *If $\phi$ is a homomorphism of a group $G$ into another group $G'$, then*

(i) $\phi(1) = 1'$ (*where, $1'$ is the unit element of $G'$*),

(ii) $\phi(x^{-1}) = [\phi(x)]^{-1}$, $\forall x \in G$.

In view of the above properties, we have:

**Theorem 4.8.** *The non-inverse graphs of two isomorphic groups are also isomorphic.*

***Proof.*** Let $G$ and $H$ be two groups and let $f$ be an isomorphism of $G$ onto $H$. Then, by Lemma 4.6, $G$ and $H$ must have same order say, $n$. Therefore, each element say $a$ in $G$ is mapped onto an element say, $f(a)$ in $H$. By Lemma 4.7, inverse of $a$ is mapped onto inverse of $f(a)$. Therefore, by Definition 1, the adjacency of the vertices in $\Gamma(G)$ is same as the adjacency of the vertices in $\Gamma(H)$. Therefore, $\Gamma(G) \cong \Gamma(H)$. ∎

**Definition 3.** Let $\Gamma(G)$ be the non-inverse graph associated with a group $G$ of order $n$ and $A(\Gamma(G))$ be its adjacency matrix. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A(\Gamma(G))$ with their multiplicities $m_1, m_2, \ldots, m_r$, $r \leq n$, then, the spectra of $\Gamma(G)$ is given by

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ m_1 & m_2 & \cdots & m_r \end{pmatrix}.$$

**Theorem 4.9.** *Let $G$ be a group of order $n$, then the spectra of $\Gamma(G)$ is given by the following cases:*

(i) *when all the elements in $G$ are self-inverse elements:*

$$\begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix},$$

(ii) *when $G$ contains all non self-inverse elements other than the identity element:*

$$\begin{pmatrix} 0 & \frac{n-3}{2} + \sqrt{\frac{(n-1)^2}{4} + 1} & -2 & \frac{n-3}{2} - \sqrt{\frac{(n-1)^2}{4} + 1} \\ \frac{n-1}{2} & 1 & \frac{n-1}{2} - 1 & 1 \end{pmatrix},$$

(iii) *when $G$ has $l$ self-inverse elements:*

(a) *when $n$ is even:*

$$\begin{pmatrix} 0 & \frac{(n-3)+\sqrt{n^2-2n+4l+1}}{2} & -2 & \frac{(n-3)-\sqrt{n^2-2n+4l+1}}{2} & -1 \\ \frac{n-l}{2} & 1 & \frac{n-l}{2}-1 & 1 & l-1 \end{pmatrix},$$

(b) *when $n$ is odd:*

$$\begin{pmatrix} 0 & \frac{n-3}{2}+\sqrt{\frac{n^2-2n+5}{4}+(l-1)} & -2 & \frac{n-3}{2}-\sqrt{\frac{n^2-2n+5}{4}+(l-1)} & -1 \\ \frac{n-l}{2} & 1 & \frac{n-l}{2}-1 & 1 & l-1 \end{pmatrix}.$$

***Proof.*** Let $G$ be a group of order $n$. Then the adjacency matrix of $\Gamma(G)$ denoted by $A(\Gamma(G))$ is given by

$$A(\Gamma(G)) = \begin{bmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1(n-1)} & a_{1n} \\ a_{12} & 0 & a_{23} & \cdots & a_{2(n-1)} & a_{2n} \\ a_{13} & a_{23} & 0 & \cdots & a_{3(n-1)} & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{1(n-1)} & a_{2(n-1)} & a_{3(n-1)} & \cdots & 0 & a_{(n-1)n} \\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{(n-1)n} & 0 \end{bmatrix}_{n\times n}.$$

Consider the following cases to find the spectra of $\Gamma(G)$.

*Case* 1. Let all the elements in $G$ be self-inverse elements. By Definition 1, $\Gamma(G)$ is a complete graph of order $n$ and the spectra of $K_n$ is given by [4] as

$$\begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}.$$

*Case* 2. Let all the elements in $G$ other than $i_G$ be non self-inverse elements. By Theorem 2.5, it is clear that $o(G)$ must be odd. In $A(\Gamma(G))$,

$$a_{ij}= \begin{cases} 0, & \text{if } i=k,\ j=k+1;\ k=2,4,6,\ldots,n-1, \\ 1, & \text{otherwise.} \end{cases}$$

Consider $\det(\lambda I - A(\Gamma(G)))$.

**Step 1.** If $R_i$ represents the $i$-th row, let $R_i \to R_i + R_{i+1} - R_{i+2} - R_{i+3}$ for $i = 2, 4, \ldots, n-3$, then we get $\det(\lambda I - A(\Gamma(G)))$ of the form $[\lambda+2]^{\frac{n-3}{2}} \det(B)$.

**Step 2.** In $\det(B)$, if $C_i$ refers to $i$-th column, let $C_i \to C_i - C_{i-1}$, where $i = 3, 5, 7, \ldots, n$ to obtain $\det(B)$ of the form $[\lambda]^{\frac{n-1}{2}} \det(C)$.

**Step 3.** Expanding $\det(C)$ by third column and simplifying we get

$$\det(C) = \left[\lambda - \left(\frac{n-3}{2} + \sqrt{\frac{(n-1)^2}{4}+1}\right)\right]\left[\lambda - \left(\frac{n-3}{2} - \sqrt{\frac{(n-1)^2}{4}+1}\right)\right].$$

Therefore, the characteristic polynomial of

$$A(\Gamma(G)) = [\lambda+2]^{\frac{n-3}{2}}[\lambda]^{\frac{n-1}{2}}\left[\lambda-\left(\frac{n-3}{2}+\sqrt{\frac{(n-1)^2}{4}+1}\right)\right]\left[\lambda-\left(\frac{n-3}{2}-\sqrt{\frac{(n-1)^2}{4}+1}\right)\right].$$

*Case* 3. Let there be $l$ self-inverse elements in $G$ and hence the entries of $A(\Gamma(G))$ depend on $l$. Consider $\det(\lambda I - A(\Gamma(G)))$.

**Step 1.** Let $R_i \to R_i + R_{i+1} - R_{i+2} - R_{i+3}$ for $i = l+1, l+2, \ldots, n-3$, then we get $\det(\lambda I - A(\Gamma(G)))$ of the form $[\lambda+2]^{\frac{n-l-2}{2}} \det(B)$.

**Step 2.** In $\det(B)$, let $C_i \to C_i - C_1$, where $i = 2,3,4\ldots,l$ to obtain $\det(B)$ of the form $[\lambda+1]^{l-1} \det(\mathrm{C})$.

**Step 3.** In $\det(C)$, let $C_i \to C_i - C_{i+1}$, where $i = l+1, l+3, \ldots, n-1$ to obtain $\det(C)$ of the form $[\lambda]^{\frac{n-l}{2}} \det(D)$.

Here, we need to consider the following subcases.

*Subcase* 1. Let $n$ be even. Then, on expansion and simplification of $det(D)$, we get

$$\det(D) = \left[\lambda - \frac{(n-3)+\sqrt{n^2-2n+4l+1}}{2}\right]\left[\lambda - \frac{(n-3)-\sqrt{n^2-2n+4l+1}}{2}\right].$$

*Subcase* 2. Let $n$ be odd. Then, on expansion and simplification of $\det(D)$, we get

$$\det(D)=\left[\lambda-\left(\frac{n-3}{2}+\sqrt{\frac{n^2-2n+5}{4}+(l-1)}\right)\right]\left[\lambda-\left(\frac{n-3}{2}-\sqrt{\frac{n^2-2n+5}{4}+(l-1)}\right)\right].$$

Therefore, the characteristic polynomial of $A(\Gamma(G))$ is given by

$$\varphi(A(\Gamma(G))) = \begin{cases} [\lambda+2]^{\frac{n-l-2}{2}}[\lambda+1]^{l-1}[\lambda]^{\frac{n-l}{2}}\left[\lambda-\frac{(n-3)+\eta}{2}\right]\left[\lambda-\frac{(n-3)-\eta}{2}\right]; & \text{when } n \text{ is even;} \\ [\lambda+2]^{\frac{n-l-2}{2}}[\lambda+1]^{l-1}[\lambda]^{\frac{n-l}{2}}\left[\lambda-\left(\frac{n-3}{2}+\zeta\right)\right]\left[\lambda-\left(\frac{n-3}{2}-\zeta\right)\right]; & \text{when } n \text{ is odd,} \end{cases}$$

where $\eta = \sqrt{n^2-2n+4l+1}$ and $\zeta = \sqrt{\frac{n^2-2n+5}{4}+(l-1)}$. Hence the result. ■

## 5. CONCLUSION

This paper discussed certain properties and characteristics of a newly defined graph called non-inverse graph which is a modified version of the inverse graph.

Some of the parameters of non-inverse graph such as size, chromatic number, domination number, centre, vertex-connectivity, edge-connectivity, girth and circumference were also determined. We also investigated graph theoretic properties and characteristics of non-inverse graphs. Being a newly defined graph class, the class of non-inverse graphs offers a wide scope for further studies.

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