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ON SHEFFER STROKE BE-ALGEBRAS

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Abstract

In this paper we introduce Sheffer stroke BE-algebras (briefly, SBE-algebras) and investigate a relationship between SBE-algebras and BE- algebras. By presenting a SBE-filter, an upper set and a SBE-subalgebra on a SBE-algebra, it is shown that any SBE-filter of a SBE-algebra is a SBE-subalgebra but the converse of this statement is not true. Besides we construct quotient SBE-algebras via a congruence relation defined by a special SBE-filter. We discuss SBE-homomorphisms and their properties between SBE-algebras. Finally, a relation between Sheffer stroke Hilbert algebras and SBE-algebras is established.

Keywords: Sheffer stroke, SBE-algebra, congruence, SBE-homomorphism.

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1. Introduction

The notion of BE-algebra is originally introduced by H.S. Kim and Y.H. Kim as a generalization of a dual BCK-algebra. They gave the definitions of a filter and an upper set on this algebraic structure [6]. Ahn and So defined ideals on BE-algebras and provided various characterizations of these ideals [2, 3]. Rezaei and Saeid presented a definition of regular congruence relation to construct quotient BE-algebras from self-distributive BE-algebras [14], and introduced commutative ideals in BE-algebras with several properties [17]. Recently, Rezaei *et al.* presented some relations between generalized Hilbert (in short, g-Hilbert) algebras, CI/BE-algebras, implication algebras and other algebraic structures [13, 15, 16].

Sheffer stroke is introduced by Sheffer [19]. Sheffer stroke, which is a negation of a conjunction and is said to be NAND operator as well, is one of the two operators that can be used by itself, without any other logical operators, to constitute a logical formal system. There is an interest in finding simple axiom systems for various algebras and logics, where simplicity is characterized by the number of axioms in a system. As a well-known example, all widely accepted Boolean algebra axioms can be written in a single axiom using the Sheffer stroke [7]. It provides new and easily applicable axiom systems for many algebraic structures, and leads to various similarities and discrepancies among algebraic structures due to its commutative property. This operation has many applications in algebraic structures such as ortholattices [4], and orthoimplication algebras [1]. Especially, Chajda *et al.* introduced Sheffer stroke NMV-algebras which are the Sheffer stroke operation reducts of non-associative MV-algebras [5]. Recently, Oner *et al.* presented Sheffer stroke Hilbert algebras [8] and fuzzy filters [11], Sheffer stroke BL-algebras and their (fuzzy) filters [9], filters of strong Sheffer stroke NMV-algebras [10] and Sheffer stroke UP-algebras [12]. These new structures are convenient in logic and related areas since a system containing only the Sheffer stroke is functionally complete (completeness of a logical system) and consistent.

In second section, we present basic definitions and notions about Sheffer operation, BE-algebras, Sheffer stroke Hilbert algebras and lattices. In third section, a definition of Sheffer Stroke BE-algebras is given. It is shown that a relation $\preceq$ is only a partial order on commutative and self-distributive SBE-algebras. Also, it is shown that the class of all SBE-algebras forms a variety. Then we demonstrate that a Sheffer Stroke BE-algebra is a BE-algebra with $x * y := x \circ (y \circ y)$, and also a BE-algebra with the least element 0 is a SBE-algebra, where $x \circ y := x * y'$ and $y' := y * 0$. In addition, it is proved that every Sheffer stroke Hilbert algebra is a SBE-algebra, and every commutative and self-distributive SBE-algebra is a Sheffer stroke Hilbert algebra. In fourth section, a SBE-filter and an upper set on a SBE-algebra are defined and then we state that any upper set of a SBE-algebra is not in general a SBE-filter but any upper set of a self-distributive SBE-algebra is a SBE-filter. It is proved that the family of all SBE-filters of a SBE-algebra forms a complete lattice, and hence for a subset of a SBE-algebra there exists a minimal SBE-filter containing this subset. By defining a SBE-subalgebra of SBE-algebra, it is shown that any SBE-filter of a SBE-algebra is a SBE-subalgebra, but the inverse is not true. In fact, a congruence relation on a SBE-algebra determined by its special SBE-filter and related notions are presented. Besides, it is constructed quotient SBE-algebras by a congruence relation defined by a special SBE-filter. In fifth section, we give a SBE-homomorphism and its properties on SBE-algebras. Moreover SBE-filters of the quotient SBE-algebras are constructed. By determining the kernel of a SBE-homomorphism, the isomorphism theorems on SBE-algebras are stated.

2. Preliminaries

In this section, basic definitions and notions about Sheffer stroke and BE-algebras are given.

Definition 1 [4]**.** Let $\mathcal{S} = \langle S, \circ \rangle$ be a groupoid. The operation $\circ$ on S is said to be a *Sheffer operation (or Sheffer stroke)* if it satisfies the following conditions for all $x, y, z \in S$:
$(S1)$ $x \circ y = y \circ x$,
$(S2)$ $(x \circ x) \circ (x \circ y) = x$,
$(S3)$ $x \circ ((y \circ z) \circ (y \circ z)) = ((x \circ y) \circ (x \circ y)) \circ z$,
$(S4)$ $(x \circ ((x \circ x) \circ (y \circ y))) \circ (x \circ ((x \circ x) \circ (y \circ y))) = x$.

Lemma 1 [4]**.** *Let $\mathcal{S} = \langle S, \circ \rangle$ be a groupoid with a Sheffer operation. The binary relation $\leq$ defined on S as follows*

$$x \leq y \text{ if and only if } x \circ y = x \circ x$$

is an order on S.

Definition 2 [6]**.** An algebra $\mathcal{S} = \langle S; *, 1 \rangle$ of type $(2, 0)$ is called a BE-algebra if it satisfies the following conditions for all $x, y, z \in S$:
$(BE-1)$ $x * x = 1$,
$(BE-2)$ $x * 1 = 1$,
$(BE-3)$ $1 * x = x$, and
$(BE-4)$ $x * (y * z) = y * (x * z)$.

Lemma 2 [17]**.** *In a BE-algebra $\langle S; *, 1 \rangle$, the following identities are satisfied for all $x, y \in S$:*
$(p1)$ $x * (y * x) = 1$,
$(p2)$ $x * ((x * y) * y) = 1$.

Definition 3 [20]**.** A BE-algebra $\langle S; *, 1\rangle$ is said to be commutative if $(x*y)*y = (y*x)*x$, for all $x, y \in S$.

Definition 4 [6]**.** A BE-algebra $\langle S; *, 1\rangle$ is said to be self-distributive if $x*(y*z) = (x*y)*(x*z)$, for all $x, y, z \in S$.

Lemma 3 [17]**.** *Let $\langle S; *, 1\rangle$ be a BE-algebra. Define a relation $\leq$ on S by*

$$x \preceq y \text{ if and only if } x*y = 1,$$

for all $x, y \in S$.

However, the relation $\preceq$ is not a partial order on S since it is only reflexive by $(BE-1)$. If S is a commutative and self-distributive, then this relation is a partial order on S [21].

Definition 5 [8]**.** A Sheffer stroke Hilbert algebra is a structure $\langle S, \circ\rangle$ of type (2), in which S is a non-empty set and $\circ$ is Sheffer stroke on S such that the following identities are satisfied for all $x, y, z \in S$:

(SHa$_1$) $(x\circ((y\circ(z\circ z))\circ(y\circ(z\circ z))))\circ(((x\circ(y\circ y))\circ((x\circ(z\circ z))\circ(x\circ(z\circ z))))\circ((x\circ(y\circ y))\circ((x\circ(z\circ z))\circ(x\circ(z\circ z))))) = x\circ(x\circ x)$,

(SHa$_2$) If $x\circ(y\circ y) = y\circ(x\circ x) = x\circ(x\circ x)$, then $x = y$.

Lemma 4 [8]**.** *Let $\langle S, \circ\rangle$ be a Sheffer stroke Hilbert algebra. Then $x\circ(x\circ x) = y\circ(y\circ y)$, for all $x, y \in S$.*

Remark 5 [8]**.** The Sheffer stroke Hilbert algebra $\langle S, \circ\rangle$ satisfies the identity $x\circ(x\circ x) = y\circ(y\circ y)$, for all $x, y \in S$, by Lemma 4. It means that $\langle S, \circ\rangle$ has an algebraic constant which will be denoted by 1.

Lemma 6 [8]**.** *Let $\langle S, \circ\rangle$ be a Sheffer stroke Hilbert algebra. Then*

$$x\circ((y\circ(z\circ z))\circ(y\circ(z\circ z))) = y\circ((x\circ(z\circ z))\circ(x\circ(z\circ z))),$$

for all $x, y, z \in S$.

Definition 6 [18]**.** A poset S is a lattice if and only if for every $a, b \in S$ both $\sup\{a, b\}$ and $\inf\{a, b\}$ exist in S.

Definition 7 [18]**.** A poset S is complete if for every subset A of S both $supA$ and $infA$ exist in S. The elements $supA$ and $infA$ are denoted by $\bigvee A$ and $\bigwedge A$, respectively. All complete posets are lattices, and a lattice S which is complete as a poset is a complete lattice.

3. Sheffer stroke BE-Algebras

In this section, we define a Sheffer stroke BE-algebra and give some of its properties. Besides, we state relationships between Sheffer stroke BE-algebras and Sheffer stroke Hilbert algebras.

Definition 8. A Sheffer stroke BE-algebra (shortly, SBE-algebra) is a structure $\langle S; \circ, 1\rangle$ of type $(2, 0)$ such that 1 is the constant in S, $\circ$ is a Sheffer operation on S and the following axioms are satisfied for all $x, y, z \in S$:
$(SBE-1)$ $x \circ (x \circ x) = 1$,
$(SBE-2)$ $x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z))) = y \circ ((x \circ (z \circ z)) \circ (x \circ (z \circ z)))$.

Remark 7. The class of all SBE-algebras forms a variety.

Remark 8. The axioms $(SBE-1)$ and $(SBE-2)$ are independent.

Proof. (a) Given a set $S = \{u, v, 1\}$ with Cayley table as below:

$\circ$	1	u	v
1	1	1	1
u	1	u	1
v	1	1	v

Then $(SBE-2)$ holds while $(SBE-1)$ does not hold since $u\circ(u\circ u) = u \neq 1$.
(b) Consider a set $S = \{u, v, 1\}$ with Cayley table as below:

$\circ$	1	u	v
1	1	u	v
u	u	v	1
v	v	1	u

Then $(SBE-1)$ holds but $(SBE-2)$ does not hold because $1 \circ ((u \circ (1 \circ 1)) \circ (u \circ (1 \circ 1))) = v \neq u = u \circ ((1 \circ (1 \circ 1)) \circ (1 \circ (1 \circ 1)))$. ∎

Now, we give examples of SBE-algebras.

Example 9. Given a structure $\langle S; \circ, 1\rangle$ where $S = \{0, u, v, 1\}$ and a binary operation $\circ$ with Cayley table as below:

$\circ$	0	u	v	1
0	1	1	1	1
u	1	v	1	v
v	1	1	u	u
1	1	v	u	0

It is easy to show that this structure is a SBE-algebra.

Example 10. Consider a structure $\langle S;\circ,1\rangle$ where $S=\{0,u,v,w,t,1\}$ and a binary operation $\circ$ with the following Cayley table. Then this structure is a SBE-algebra.

$\circ$	0	u	v	w	t	1
0	1	1	1	1	1	1
u	1	v	1	1	1	v
v	1	1	u	1	1	u
w	1	1	1	t	1	t
t	1	1	1	1	w	w
1	1	v	u	t	w	0

Example 11. Consider a structure $\langle S;\circ,1\rangle$ where $S=\{0,u,v,w,t,1\}$ and a binary operation $\circ$ with the following Cayley table

$\circ$	0	u	v	w	t	1
0	1	1	1	1	1	1
u	1	t	w	1	1	t
v	1	w	w	1	1	w
w	1	1	1	v	u	v
t	1	1	1	u	u	u
1	1	t	w	v	u	0

Then it is a SBE-algebra.

Lemma 12. *Let $\langle S;\circ,1\rangle$ be a SBE-algebra. Then the following hold for all x, $y\in S$:*

(i) $x\circ(1\circ 1)=1$,

(ii) $1\circ(x\circ x)=x$,

(iii) $x\circ((y\circ(x\circ x))\circ(y\circ(x\circ x)))=1$.

(iv) $x\circ(((x\circ(y\circ y))\circ(y\circ y))\circ((x\circ(y\circ y))\circ(y\circ y)))=1$,

(v) $(x\circ 1)\circ(x\circ 1)=x$,

(vi) $((x\circ y)\circ(x\circ y))\circ(x\circ x)=1$ *and* $((x\circ y)\circ(x\circ y))\circ(y\circ y)=1$,

(vii) $x\circ((x\circ y)\circ(x\circ y))=x\circ y=((x\circ y)\circ(x\circ y))\circ y$.

Proof. (i) We have $x\circ(1\circ 1)=x\circ((x\circ(x\circ x))\circ(x\circ(x\circ x)))=(x\circ x)\circ((x\circ x)\circ(x\circ x))=1$ from $(SBE-1)$, $(S1)$ and $(S3)$.

(ii) We obtain $1\circ(x\circ x)=(x\circ(x\circ x))(x\circ x)=(x\circ x)\circ(x\circ(x\circ x))=x$ from $(SBE-1)$, $(S1)$ and $(S2)$.

(iii) We get $x\circ((y\circ(x\circ x))\circ(y\circ(x\circ x)))=y\circ((x\circ(x\circ x))\circ(x\circ(x\circ x)))=y\circ(1\circ 1)=1$ from $(SBE-2)$, $(SBE-1)$ and (i).

(iv) It follows from $(SBE-2)$ and $(SBE-1)$, respectively.

(v) It is obtained from (S1), (S2) and (ii) that

$$\begin{aligned}(x \circ 1) \circ (x \circ 1) &= (1 \circ ((x \circ x) \circ (x \circ x))) \circ (1 \circ ((x \circ x) \circ (x \circ x))) \\ &= (x \circ x) \circ (x \circ x) \\ &= x.\end{aligned}$$

(vi)
$$\begin{aligned}((x \circ y) \circ (x \circ y)) \circ (x \circ x) &= y \circ ((x \circ (x \circ x)) \circ (x \circ (x \circ x))) \\ &= y \circ (1 \circ 1) \\ &= 1\end{aligned}$$

and

$$\begin{aligned}((x \circ y) \circ (x \circ y)) \circ (y \circ y) &= x \circ ((y \circ (y \circ y)) \circ (y \circ (y \circ y))) \\ &= x \circ (1 \circ 1) \\ &= 1\end{aligned}$$

from (S1), (S3), (SBE-1) and (i).

(vii)
$$\begin{aligned}x \circ ((x \circ y) \circ (x \circ y)) &= ((x \circ x) \circ (x \circ x)) \circ y \\ &= x \circ y \\ &= x \circ ((y \circ y) \circ (y \circ y)) \\ &= ((x \circ y) \circ (x \circ y)) \circ y\end{aligned}$$

from (S2) and (S3). ■

Definition 9. A SBE-algebra $\langle S; \circ, 1\rangle$ is called commutative if

(1) $$(x \circ (y \circ y)) \circ (y \circ y) = (y \circ (x \circ x)) \circ (x \circ x),$$

for any $x, y, z \in S$.

Example 13. The SBE-algebra $\langle S; \circ, 1\rangle$ in Example 9 is commutative while the SBE-algebra in Example 11 is not since $(u \circ (v \circ v)) \circ (v \circ v) = v \neq u = (v \circ (u \circ u)) \circ (u \circ u)$.

Definition 10. A SBE-algebra $\langle S; \circ, 1\rangle$ is called self-distributive if

(2) $$x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z))) = (x \circ (y \circ y)) \circ ((x \circ (z \circ z)) \circ (x \circ (z \circ z))),$$

for any $x, y, z \in S$.

Example 14. The SBE-algebra $\langle S; \circ, 1\rangle$ in Example 9 is self-distributive while the SBE-algebra in Example 10 is not since $u \circ ((w \circ (v \circ v)) \circ (w \circ (v \circ v))) = 1 \neq v = (u \circ (w \circ w)) \circ ((u \circ (v \circ v)) \circ (u \circ (v \circ v)))$.

Definition 11. Let $\langle S; \circ, 1\rangle$ be a SBE-algebra. Define a relation $\preceq$ on S by

$$x \preceq y \text{ if and only if } x \circ (y \circ y) = 1,$$

for all $x, y \in S$.

The relation is not a partial order on S, since it is only reflexive by $(SBE-1)$.

Lemma 15. *Let $\langle S;\circ,1\rangle$ be a SBE-algebra. Then*

1. *If $x \preceq y$, then $y\circ y \preceq x\circ x$,*
2. *$x \preceq y\circ(x\circ x)$,*
3. *$y \preceq (y\circ(x\circ x))\circ(x\circ x)$,*
4. *If S is self-distributive, then $x \preceq y$ implies $y\circ z \preceq x\circ z$,*
5. *If S is self-distributive, then $y\circ(z\circ z) \preceq (z\circ(x\circ x))\circ((y\circ(x\circ x))\circ(y\circ(x\circ x)))$.*

Proof. 1. Let $x \preceq y$, i.e., $x\circ(y\circ y) = 1$. Since $(y\circ y)((x\circ x)\circ(x\circ x)) = x\circ(y\circ y) = 1$ from $(S1)$–$(S2)$, it follows $y\circ y \preceq x\circ x$.

2. It is obtained from $(SBE-1)$–$(SBE-2)$ and Lemma 12(i) that

$$\begin{aligned} x\circ((y\circ(x\circ x))\circ(y\circ(x\circ x))) &= y\circ((x\circ(x\circ x))\circ(y\circ(x\circ x))) \\ &= y\circ(1\circ 1) \\ &= 1. \end{aligned}$$

Hence, $x \preceq y\circ(x\circ x)$.

3. Since

$$\begin{aligned} &y\circ(((y\circ(x\circ x))\circ(x\circ x))\circ((y\circ(x\circ x))\circ(x\circ x))) \\ &= (y\circ(x\circ x))\circ((y\circ(x\circ x))\circ(y\circ(x\circ x))) \\ &= 1 \end{aligned}$$

from $(SBE-1)$–$(SBE-2)$, we have $y \preceq (y\circ(x\circ x))\circ(x\circ x)$.

4. Let S be self-distributive and $x \preceq y$, i.e., $x\circ(y\circ y) = 1$. Then

$$\begin{aligned} (y\circ z)\circ((x\circ z)\circ(x\circ z)) &= (z\circ((y\circ y)\circ(y\circ y)))\circ((z\circ((x\circ x) \\ &\quad \circ(x\circ x)))\circ(z\circ((x\circ x)\circ(x\circ x)))) \\ &= z\circ(((y\circ y)\circ((x\circ x)\circ(x\circ x))) \\ &\quad \circ((y\circ y)\circ((x\circ x)\circ(x\circ x)))) \\ &= z\circ((x\circ(y\circ y))\circ(x\circ(y\circ y))) \\ &= z\circ(1\circ 1) \\ &= 1 \end{aligned}$$

from $(S1)$–$(S2)$, Equation (2), and Lemma 12(i). So, $y\circ z \preceq x\circ z$.

5. Let S be self-distributive. Then

$$\begin{aligned} y \circ (z \circ z) &\preceq (x \circ x) \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z))) \\ &= (x \circ x) \circ (((z \circ z) \circ ((y \circ y) \circ (y \circ y))) \\ &\quad \circ ((z \circ z) \circ ((y \circ y) \circ (y \circ y)))) \\ &= ((x \circ x) \circ ((z \circ z) \circ (z \circ z))) \circ (((x \circ x) \circ ((y \circ y) \\ &\quad \circ (y \circ y))) \circ ((x \circ x) \circ ((y \circ y) \circ (y \circ y)))) \\ &= (z \circ (x \circ x)) \circ ((y \circ (x \circ x)) \circ (y \circ (x \circ x))) \end{aligned}$$

from the condition (2), Equation (2) and ($S1$)–($S2$). ■

Theorem 16. *Let $\langle S; \circ, 1\rangle$ be a commutative and self-distributive SBE-algebra. Then the relation $\preceq$ is a partial order on S.*

Proof. Let $\langle S; \circ, 1\rangle$ be a commutative and self-distributive SBE-algebra.

• *Reflexive:* It follows from $(SBE-1)$ that $x \preceq x$, for all $x \in S$.

• *Antisymmetric:* Let $x \preceq y$ and $y \preceq x$, i.e., $x \circ (y \circ y) = 1$ and $y \circ (x \circ x) = 1$. Then

$$\begin{aligned} x &= 1 \circ (x \circ x) && \text{(Lemma 12(ii))} \\ &= y \circ (x \circ x) \circ (x \circ x) \\ &= (x \circ (y \circ y)) \circ (y \circ y) && \text{(Equation (1))} \\ &= 1 \circ (y \circ y) \\ &= y. && \text{(Lemma 12(ii))} \end{aligned}$$

• *Transitive:* Let $x \preceq y$ and $y \preceq z$, i.e., $x \circ (y \circ y) = 1$ and $y \circ (z \circ z) = 1$. Then

$$\begin{aligned} x \circ (z \circ z) &= 1 \circ ((x \circ (z \circ z)) \circ (x \circ (z \circ z))) && \text{(Lemma 12(ii))} \\ &= (x \circ (y \circ y))((x \circ (z \circ z))(x \circ (z \circ z))) \\ &= x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z))) && \text{(Equation (2))} \\ &= x \circ (1 \circ 1) \\ &= 1, && \text{(Lemma 12(i))} \end{aligned}$$

and so, $x \preceq z$. ■

Theorem 17. *Let $\langle S; \circ, 1\rangle$ be a SBE-algebra. If we define $x * y := x \circ (y \circ y)$, then $\langle S; *, 1\rangle$ is a BE-algebra.*

Proof. Let $\langle S; \circ, 1\rangle$ be a SBE-algebra and x, y, z be any elements of S.
$(BE-1)$: We have $x * x = x \circ (x \circ x) = 1$ from $(SBE-1)$.
$(BE-2)$: It follows from Lemma 12(i) that $x * 1 = x \circ (1 \circ 1) = 1$.
$(BE-3)$: By Lemma 12(ii), $1 * x = 1 \circ (x \circ x) = x$.
$(BE-4)$: We get $x * (y * z) = x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z))) = y \circ ((x \circ (z \circ z)) \circ (x \circ (z \circ z))) = y * (x * z)$ from $(SBE-2)$. ■

Example 18. Consider the SBE-algebra $\langle S; \circ, 1\rangle$ in Example 10. Then BE-algebra $\langle S; *, 1\rangle$ constructed from this SBE-algebra has the following Cayley table:

$*$	0	u	v	w	t	1
0	1	1	1	1	1	1
u	v	1	v	1	1	1
v	u	u	1	1	1	1
w	t	1	1	1	t	1
t	w	1	1	w	1	1
1	0	u	v	w	t	1

In the next theorem, we construct a SBE-algebra from BE-algebra.

Theorem 19. *Let $\langle S; *, 1\rangle$ be a BE-algebra and 0 be a constant of S such that $0 \neq 1$. Define a unary operation $'$ on S by $x' = x * 0$ and $x \circ y := x * y'$, for all $x, y \in S$. Then $\langle S; \circ, 1\rangle$ is a SBE-algebra.*

Proof. Let $\langle S; *, 1\rangle$ be a BE-algebra and 0 be a constant of S such that $0 \neq 1$. Then
$(SBE-1)$: It is obtained from $(BE-1)$ that $x \circ (x \circ x) = x * x = 1$.
$(SBE-2)$: It follows from $(BE-4)$ that $x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z))) = x * (y * z) = y * (x * z) = y \circ ((x \circ (z \circ z)) \circ (x \circ (z \circ z)))$,
for all $x, y, z \in S$.

Also, $0' = 0 * 0 = 1$ and $1' = 1 * 0 = 0$ from $(BE-1)$ and $(BE-3)$. ∎

Example 20. Consider a BE-algebra $\langle S; *, 1\rangle$ where $S = \{0, u, v, w, t, 1\}$ and a binary operation $*$ on S has the following Cayley table:

$*$	0	u	v	w	t	1
0	1	1	1	1	1	1
u	t	1	1	w	t	1
v	w	1	1	w	w	1
w	v	u	v	1	1	1
t	u	u	u	1	1	1
1	0	u	v	w	t	1

Then a SBE-algebra $\langle S; \circ, 1\rangle$ defined by this BE-algebra is the SBE-algebra in Example 11.

Theorem 21. *Every Sheffer stroke Hilbert algebra is a SBE-algebra.*

Proof. It is proved Lemma 4, Remark 5 and Lemma 6. ∎

The converse of Theorem 21 is not true.

Example 22. The SBE-algebra $\langle S; \circ, 1\rangle$ in Example 10 is not a Sheffer stroke Hilbert algebra since $u \neq t$ when $u \circ (t \circ t) = t \circ (u \circ u) = 1 = u \circ (u \circ u)$.

Theorem 23. *Every commutative and self-distributive SBE-algebra is a Sheffer stroke Hilbert algebra.*

Proof. Let $\langle S; \circ, 1\rangle$ be a commutative and self-distributive SBE-algebra. So, (SHa_1):

$$\begin{aligned}
&(x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z)))) \circ (((x \circ (y \circ y)) \circ ((x \circ (z \circ z)) \\
&(x \circ (z \circ z)))) \circ ((x \circ (y \circ y)) \circ ((x \circ (z \circ z)) \circ (x \circ (z \circ z))))) \\
&= (x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z)))) \circ ((x \circ ((y \circ (z \circ z)) \\
&\quad (y \circ (z \circ z)))) \circ (x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z))))) \\
&= 1 \\
&= x \circ (x \circ x)
\end{aligned}$$

from Equation (2) and $(SBE-1)$.

(SHa_2) Let $x \circ (y \circ y) = y \circ (x \circ x) = x \circ (x \circ x)$, i.e., $x \circ (y \circ y) = y \circ (x \circ x) = 1$. Then

$$\begin{aligned}
x &= 1 \circ (x \circ x) \\
&= (y \circ (x \circ x)) \circ (x \circ x) \\
&= (x \circ (y \circ y)) \circ (y \circ y) \\
&= 1 \circ (y \circ y) \\
&= y
\end{aligned}$$

from Equation (1) and Lemma 12(i).

Thus, $\langle S, \circ\rangle$ is a Sheffer stroke Hilbert algebra. ∎

Example 24. Since the SBE-algebra $\langle S; \circ, 1\rangle$ in Example 9 is commutative and self-distributive, it is a Sheffer stroke Hilbert algebra.

4. OnfFilters of SBE-algebras

In this section, we give definitions of upper sets and filters of a SBE-algebra and analyse relationships between them. Here, S presents a SBE-algebra.

Definition 12. A nonempty subset $F \subseteq S$ is called a SBE-filter of S if it satisfies the following properties:
$(SBEf-1)$ $1 \in F$,
$(SBEf-2)$ For all $x, y \in S$, $x \circ (y \circ y) \in F$ and $x \in F$ imply $y \in F$.

In other structures related with algebraic logics (BCK-algebras, Heyting algebras, Boolean algebras) these kind of structures are called deductive filters.

Example 25. Consider the SBE-algebra S in Example 9. Then it is trivial that S itself and $\{1\}$ are SBE-filters of S. Also $\{1,u\}$ and $\{1,v\}$ are SBE-filters of S.

Lemma 26. *Let S be a SBE-algebra. Then a nonempty subset $F \subseteq S$ is a SBE-filter of S if and only if for all $x,y \in S$*

(i) *$x \in F$ and $y \in F$ imply $(x \circ y) \circ (x \circ y) \in F$,*

(ii) *$x \in F$ and $x \preceq y$ imply $y \in F$.*

Proof. $(\Rightarrow)$ Let F be a SBE-filter of S.

(i) Let $x \in F$ and $y \in F$. Then $x\circ(((x\circ y)\circ y)\circ((x\circ y)\circ y)) = (x\circ y)\circ((x\circ y)\circ(x\circ y)) = 1 \in F$ from $(SBE-2)$, $(SBE-1)$ and $(SBEf-1)$, respectively. It follows from $(SBEf-2)$ that $(x\circ y)\circ y \in F$. Since $y\circ(((x\circ y)\circ(x\circ y))\circ((x\circ y)\circ(x\circ y))) = (x\circ y)\circ y \in F$ from $(S1)-(S2)$, we have from $(SBEf-2)$ that $(x\circ y)\circ(x\circ y) \in F$.

(ii) Let $x \in F$ and $x \preceq y$, i.e., $x \circ (y \circ y) = 1$. Then $x \circ (y \circ y) = 1 \in F$ from $(SBEf-1)$. Thus, $y \in F$ from $(SBEf-2)$.

$(\Leftarrow)$ Let F be a nonempty subset of S satisfying (i) and (ii).

• Then $x \in F$. Since $x \circ (1 \circ 1) = 1$ from Lemma 12(i), we have that $x \preceq 1$, for all $x \in S$. Hence, $1 \in F$ from (ii).

• Let $x \in F$ and $x \circ (y \circ y) \in F$. Then $(x \circ (x \circ (y \circ y))) \circ (x \circ (x \circ (y \circ y))) \in F$ from (i). Since $((x \circ (x \circ (y \circ y))) \circ (x \circ (x \circ (y \circ y)))) \circ (y \circ y) = (x \circ (y \circ y)) \circ ((x \circ (y \circ y)) \circ (x \circ (y \circ y))) = 1$ from $(S1)$, $(S3)$ and $(SBE-1)$, it is obtained that $(x \circ (x \circ (y \circ y))) \circ (x \circ (x \circ (y \circ y))) \preceq y$. Hence, $y \in F$ from (ii). ■

Definition 13. Let $x,y \in S$ and define $U(x,y) = \{z \in S : x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z))) = 1\}$. Then $U(x,y)$ is called an upper set of x and y.

For $x,y \in S$, $U(x,y)$ is not a SBE-filter of S in general.

Example 27. Consider the SBE-algebra S in Example 10. Then $U(1,u) = \{u,w,t,1\}$ is not a SBE-filter of this SBE-algebra since its SBE-filters are only S itself and $\{1\}$.

Lemma 28. *Let $\langle S;\circ,1\rangle$ be a SBE-algebra. Then $U(x,y) = U(y,x)$, for all $x,y \in S$.*

Proof. We have $U(x,y) = \{z \in S : x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z))) = 1\} = \{z \in S : y \circ ((x \circ (z \circ z)) \circ (x \circ (z \circ z))) = 1\} = U(y,x)$ from $(SBE-2)$. ■

Theorem 29. *Let $\langle S;,1\rangle$ be a self-distributive SBE-algebra. Then $U(x,y)$ is a SBE-filter of S.*

Proof. We know $1, x, y \in U(x,y)$ from Lemma 12. Let $u \circ (v \circ v) \in U(x,y)$ and $u \in U(x,y)$, i.e., $x\circ((y\circ((u\circ(v\circ v))\circ(u\circ(v\circ v))))\circ(y\circ((u\circ(v\circ v))\circ(u\circ(v\circ v))))) = 1$ and $x \circ ((y \circ (u \circ u)) \circ (y \circ (u \circ u))) = 1$. Then we obtain

$$\begin{aligned}
1 &= x \circ ((y \circ ((u \circ (v \circ v)) \circ (u \circ (v \circ v)))) \circ (y \circ ((u \circ (v \circ v)) \circ (u \circ (v \circ v))))) \\
&= x \circ (((y \circ (u \circ u)) \circ ((y \circ (v \circ v)) \circ (y \circ (v \circ v)))) \\
&\quad \circ((y \circ (u \circ u)) \circ (y \circ (v \circ v)) \circ (y \circ (v \circ v))))) \\
&= (x \circ ((y \circ (u \circ u)) \circ (y \circ (u \circ u)))) \circ ((x \circ ((y \circ (v \circ v)) \\
&\quad (y \circ (v \circ v)))) \circ (x \circ ((y \circ (v \circ v)) \circ (y \circ (v \circ v))))) \\
&= 1 \circ ((x \circ ((y \circ (v \circ v)) \circ (y \circ (v \circ v)))) \circ (x \circ ((y \circ (v \circ v)) \circ (y \circ (v \circ v))))) \\
&= x \circ ((y \circ (v \circ v)) \circ (y \circ (v \circ v)))
\end{aligned}$$

from the self-distributivity of S and Lemma 12(ii). Thus, it follows $v \in U(x,y)$. ■

The converse of Theorem 29 is not true.

Example 30. Consider the SBE-algebra S in Example 11. $U(u,v) = \{u, v, 1\}$ is a SBE-filter of S but S is not a self-distributive.

Theorem 31. *Let F be a nonempty subset of a SBE-algebra $\langle S; \circ, 1\rangle$. Then F is a SBE-filter of S if and only if $U(x,y) \subseteq F$, for all $x, y \in F$.*

Proof. ($\Rightarrow$) Suppose that F is a SBE-filter of S, and let x and y be arbitrary elements of F. If $z \in U(x,y)$, then $x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z))) = 1 \in F$ from $(SBEf-1)$. Since $x, y \in F$, we get $z \in F$ by applying twice $(SBEf-2)$. So, $U(x,y) \subseteq F$.

($\Leftarrow$) Assume that $U(x,y) \subseteq F$, for all $x, y \in F$. It follows Lemma 12(i) that $1 \in U(x,y) \subseteq F$. Let $u \circ (v \circ v), u \in F$. Because $(u \circ (v \circ v)) \circ ((u \circ (v \circ v)) \circ (u \circ (v \circ v))) = 1$, we have $v \in U(u \circ (v \circ v), u) \subseteq F$. Thus, F is a SBE-filter of S. ■

Theorem 32. *If F is a SBE-filter of a SBE-algebra $\langle S; \circ, 1\rangle$, then*

$$F = \bigcup_{x\in F} U(x,1).$$

Proof. Let v be any element of $\bigcup_{x\in F} U(x,1)$. Then there exists $u \in F$ such that $v \in U(u,1)$ which yields $u \circ (v \circ v) = u \circ ((1 \circ (v \circ v)) \circ (1 \circ (v \circ v))) = 1 \in F$ from Lemma 12(ii) and $(SBEf-1)$. Since F is a SBE-filter of S, $u \circ (v \circ v), u \in F$ implies $v \in F$. Thus, it follows $\bigcup_{x\in F} U(x,1) \subseteq F$.

Let $x \in F$. Since we know $x \circ ((1 \circ (x \circ x)) \circ (1 \circ (x \circ x))) = 1 \in F$ from Lemma 12(ii) and $(SBE-1)$, we get $x \in U(x,1)$, which gives $F \subseteq \bigcup_{x\in F} U(x,1)$. ■

Corollary 33. *If F is a SBE-filter of a SBE-algebra $\langle S; \circ, 1\rangle$, then*

$$F = \bigcup_{x,y \in F} U(x,y).$$

Proof. Let F be a SBE-filter of S and let z be any element of F. Then $z \in F = \bigcup_{z \in F} U(z,1) \subseteq \bigcup_{x,y \in F} U(x,y)$ by Theorem 32.

Let $z \in \bigcup_{x,y \in F} U(x,y)$. Then there exist $u, v \in F$ such that $z \in U(u,v)$. It follows from Theorem 31 that $z \in U(u,v) \subseteq F$, which gives $\bigcup_{x,y \in F} U(x,y) \subseteq F$. ■

Lemma 34. *Let Σ_S be a family of all SBE-filters of a SBE-algebra $\langle S; \circ, 1\rangle$. Then $(\Sigma_S, \subseteq)$ is a partially ordered set (poset) where a relation $\subseteq$ is the set inclusion.*

Theorem 35. *The poset Σ_S forms a complete lattice.*

Proof. Let $\{F_i\}_{i \in I}$ be a family of SBE-filters of S. Since we know $1 \in F_i$, for all $i \in I$, it follows that $1 \in \bigcup_{i \in I} F_i$ and $1 \in \bigcap_{i \in I} F_i$.

(a) Assume that $x \circ (y \circ y) \in \bigcap_{i \in I} F_i$ and $x \in \bigcap_{i \in I} F_i$ hold for any $x, y \in S$, i.e., $x \circ (y \circ y) \in F_i$ and $x \in F_i$ hold for all $i \in I$. Since every F_i is a SBE-filter of S for all $i \in I$, we have $y \in F_i$, for all $i \in I$. Thus, $y \in \bigcap_{i \in I} F_i$.

(b) Let X be the family of all SBE-filters of S containing the union $\bigcup_{i \in I} F_i$, i.e., $X = \{F : \bigcup_{i \in I} F_i \subset F \; and \; F \; is \; a \; SBE-filter \; of \; S\}$. Then $\bigcap X$ is a SBE-filter of S from (a). If $\bigwedge_{i \in I} F_i = \bigcap_{i \in I} F_i$ and $\bigvee_{i \in I} F_i = \bigcap X$, then $(\Sigma_S, \bigwedge, \bigvee)$ is a complete lattice. ■

Corollary 36. *Let T be a subset of a SBE-algebra S. Then there is the minimal SBE-filter (T) containing the subset T.*

Proof. Let $Y = \{F \in \Sigma_S : T \subseteq S\}$. Then $\langle T\rangle = \{s \in S : s \in \bigcap_{F \in Y} F\}$ is the minimal SBE-filter of S containing $T \subseteq S$. ■

Definition 14. A subset T of a SBE-algebra S is called a SBE-subalgebra of S if $x \circ (y \circ y) \in T$, for $x, y \in T$. Clearly, S itself and $\{1\}$ are SBE-subalgebras of S.

Lemma 37. *Any SBE-filter of a SBE-algebra S is a SBE-subalgebra of S.*

Proof. Let F be a SBE-filter of S and $x, y \in F$. Since $y \circ ((x \circ (y \circ y)) \circ (x \circ (y \circ y))) = x \circ ((y \circ (y \circ y)) \circ (y \circ (y \circ y))) = x \circ (1 \circ 1) = 1 \in F$ from $(SBE-2)$, $(SBE-1)$ Lemma 12(i) and $(SBEf-1)$, respectively, it is obtained from $(SBEf-2)$ that $x \circ (y \circ y) \in F$. Thus, F is a SBE-subalgebra of S. ■

However, the converse of Lemma 37 is not true.

Example 38. Consider the SBE-algebra S in Example 10. Then a subset $F = \{0, u, v, 1\} \subseteq S$ is a SBE-subalgebra of S but it is not SBE-filter of S because $w \notin F$ when $u \circ (w \circ w) = 1 \in F$ and $u \in F$.

Definition 15. Let F be a SBE-filter of S. Define the binary relation σ_F on S as follows: for all $x, y \in S$

$$(x, y) \in \sigma_F \text{ if and only if } x \circ (y \circ y) \in F \text{ and } y \circ (x \circ x) \in F. \tag{3}$$

Example 39. Consider the SBE-algebra S in Example 11. For the SBE-filter $F_1 = \{u, v, 1\}$, $\sigma_{F_1} = \{(0,0), (u,u), (v,v), (w,w), (t,t), (1,1), (u,v), (v,u), (w,t), (t,w), (0,w), (w,0), (0,t), (t,0), (u,1), (1,u), (v,1), (1,v)\}$ is a binary relation on S.

Definition 16. An equivalence relation β is called a congruence relation on S if $(x, y) \in \beta$ and $(u, v) \in \beta$ imply $(x \circ u, y \circ v) \in \beta$, for $x, y, u, v \in S$.

Example 40. Consider the SBE-algebra S in Example 9. The equivalence relation $\beta = \{(0,0), (u,u), (v,v), (1,1), (u,1), (1,u), (0,v), (v,0)\}$ is a congruence on S.

Lemma 41. *An equivalence relation β is a congruence on S if and only if $(x, y) \in \beta$ implies $(x \circ z, y \circ z) \in \beta$, for all $x, y, z \in S$.*

Proof. Let β be a congruence on S, and let x, y, z be any elements of S such that $(x, y) \in \beta$. Since $(z, z) \in \beta$, we have that $(x \circ z, y \circ z) \in \beta$.

Conversely, let $(x, y) \in \beta$ imply $(x \circ z, y \circ z) \in \beta$, for all $x, y, z \in S$. Then it follows from $(S1)$ that $(x \circ u, y \circ u)\beta$ and $(y \circ u, y \circ v) \in \beta$. Thus, we get $(x \circ u, y \circ v) \in \beta$ from transitivity of β. Thus, β is a congruence on S. ∎

Definition 17. Let F be a SBE-filter of a SBE-algebra S. If $x \circ (y \circ y) \in F$ and $y \circ (x \circ x) \in F$ imply $(z \circ (x \circ x)) \circ ((z \circ (y \circ y)) \circ (z \circ (y \circ y))) \in F$, for all $z \in S$, then F is called a special SBE-filter of S. It is obvious that S itself is a special SBE-filter of S.

Example 42. Consider the SBE-algebra S in Example 11. Then $F_1 = \{u, v, 1\}$ and $F_2 = \{1\}$ are special SBE-filters of S.

Remark 43. Every SBE-filter of a SBE-algebra S is usually not a special SBE-filter of S.

Example 44. Consider the SBE-algebra S in Example 10. Then $F = \{1\}$ is a SBE-filter of S but it is not a special SBE-filter of S since $(t \circ (u \circ u)) \circ ((t \circ (w \circ w)) \circ (t \circ (w \circ w))) = (t \circ v) \circ ((t \circ t) \circ (t \circ t)) = 1 \circ (w \circ w) = 1 \circ t = w \notin F$ when $u \circ (w \circ w) = u \circ t = 1 \in F$ and $w \circ (u \circ u) = w \circ v = 1 \in F$.

Theorem 45. *Let F be a special SBE-filter of S and the binary relation σ_F is defined as the statement* (3)*. Then σ_F is a congruence on S.*

Proof. We first demonstrate that the binary relation σ_F is an equivalence relation on S.

• Reflexive: It follows from $(SBE-1)$ and $(SBEf-1)$ that $x \circ (x \circ x) = 1 \in F$, i.e., $(x, x) \in \sigma_F$, for all $x \in S$.

• Symmetric: it is obvious from the definition of σ_F.

• Transitive: Let x, y, z be any elements of S such that $(x, y) \in \sigma_F$ and $(y, z) \in \sigma_F$, i.e., $x \circ (y \circ y), y \circ (x \circ x) \in F$ and $y \circ (z \circ z), z \circ (y \circ y) \in F$. Since F is a special SBE-filter of S, $y \circ (x \circ x) \in F$ and $x \circ (y \circ y) \in F$ imply $(z \circ (y \circ y)) \circ ((z \circ (x \circ x)) \circ (z \circ (x \circ x))) \in F$. By $z \circ (y \circ y) \in F$ and $(SBEf-2)$, we have $z \circ (x \circ x) \in F$. Similarly, $y \circ (z \circ z) \in F$ and $z \circ (y \circ y) \in F$ imply $(x \circ (y \circ y)) \circ ((x \circ (z \circ z)) \circ (x \circ (z \circ z))) \in F$ because it is a special SBE-filter of S. Then $x \circ (z \circ z) \in F$ by $x \circ (y \circ y) \in F$ and $(SBEf-2)$. Thus, it follows $(x, z) \in \sigma_F$.

Now, we show that σ_F is a congruence on S. Let x, y, u, v be arbitrary elements of S such that $(x, u) \in \sigma_F$ and $(y, v) \in \sigma_F$, i.e., $x \circ (u \circ u), u \circ (x \circ x) \in F$ and $y \circ (v \circ v)$, $v \circ (y \circ y) \in F$. From $(S2)$, we have $(v \circ v) \circ ((y \circ y) \circ (y \circ y)), (y \circ y) \circ ((v \circ v) \circ (v \circ v)) \in F$. Since F is a special SBE-filter of S, we have $(x \circ v) \circ ((x \circ y) \circ (x \circ y)) = (x \circ ((v \circ v) \circ (v \circ v))) \circ ((x \circ ((y \circ y) \circ (y \circ y))) \circ (x \circ ((y \circ y) \circ (y \circ y)))) \in F$ from $(S2)$, and similarly, $(x \circ y) \circ ((x \circ v) \circ (x \circ v)) \in F$. So, we obtain $(x \circ y, x \circ v) \in \sigma_F$. Besides, we get from $(S2)$ that $(u \circ u) \circ ((x \circ x) \circ (x \circ x)), (x \circ x) \circ ((u \circ u) \circ (u \circ u)) \in F$. Since F is a special SBE-filter of S, we obtain $(u \circ v) \circ ((x \circ v) \circ (x \circ v)) = (v \circ ((u \circ u) \circ (u \circ u))) \circ ((v \circ ((x \circ x) \circ (x \circ x))) \circ (v \circ ((x \circ x) \circ (x \circ x)))) \in F$ from $(S1)-(S2)$, and similarly, we have $(x \circ v) \circ ((u \circ v) \circ (u \circ v)) \in F$. Hence, $(x \circ v, u \circ v) \in \sigma_F$.

Thus, it follows $(x \circ y, u \circ v) \in \sigma_F$ from transitivity of σ_F. ■

Corollary 46. *If F is a special SBE-filter of S and σ is the congruence on S defined by F, then $S/F \equiv S/\sigma = \{[x]_\sigma : x \in S\}$ is also SBE-algebra with the Sheffer stroke operation $_\sigma$ defined by $[x]_\sigma \circ_\sigma [y]_\sigma = [x \circ y]_\sigma$, for all $x, y \in S$ and the constant $F = [1]_\sigma$.*

Example 47. Consider a SBE-algebra $\langle S; \circ, 1\rangle$ where $S = \{0, u, v, w, t, p, q, 1\}$ and Sheffer stroke $\circ$ with the following Cayley table:

$\circ$	0	u	v	w	t	p	q	1
0	1	1	1	1	1	1	1	1
u	1	q	1	1	q	q	1	q
v	1	1	p	1	p	1	p	p
w	1	1	1	t	1	t	t	t
t	1	q	p	1	w	q	p	w
p	1	q	1	t	q	v	t	v
q	1	1	p	t	p	t	u	u
1	1	q	p	t	w	v	u	0

For a special SBE-filter $F=\{1,t\}$, $\sigma_F=\{(0,0),(u,u),(v,v),(w,w),(t,t),(p,p),$ $(q,q),(1,1),(0,w),(w,0),(t,1),(1,t),(v,q),(q,v),(u,p),(p,u)\}$ is a congruence on S defined by F. Then $S/F\equiv S/\sigma_F=\{[0]_{\sigma_F},[u]_{\sigma_F},[v]_{\sigma_F},[1]_{\sigma_F}\}$ and the constant is $[1]_{\sigma_F}=F$. Thus, $\langle S/F;\circ_{\sigma_F},F\rangle$ is a SBE-algebra with Cayley table as below:

$\circ_{\sigma_F}$	$[0]_{\sigma_F}$	$[u]_{\sigma_F}$	$[v]_{\sigma_F}$	$[1]_{\sigma_F}$
$[0]_{\sigma_F}$	$[1]_{\sigma_F}$	$[1]_{\sigma_F}$	$[1]_{\sigma_F}$	$[1]_{\sigma_F}$
$[u]_{\sigma_F}$	$[1]_{\sigma_F}$	$[v]_{\sigma_F}$	$[1]_{\sigma_F}$	$[v]_{\sigma_F}$
$[v]_{\sigma_F}$	$[1]_{\sigma_F}$	$[1]_{\sigma_F}$	$[u]_{\sigma_F}$	$[u]_{\sigma_F}$
$[1]_{\sigma_F}$	$[1]_{\sigma_F}$	$[v]_{\sigma_F}$	$[u]_{\sigma_F}$	$[0]_{\sigma_F}$

5. SBE-homomorphisms

In this section, we introduce some definitions and notions about SBE- homomorphisms. Also, three basic isomorphism theorems are presented.

Definition 18. Let $\langle S;\circ_S,1_S\rangle$ and $\langle T;\circ_T,1_T\rangle$ be SBE-algebras. A mapping $f:S\longrightarrow T$ is called a SBE-homomorphism if

$$f(x\circ_S y)=f(x)\circ_T f(y),$$

for all $x,y\in S$ and $f(1_S)=1_T$.

Theorem 48. *Let $\langle S;\circ_S,1_S\rangle$ and $\langle T;\circ_T,1_T\rangle$ be SBE-algebras, and let the mapping $f:S\longrightarrow T$ be a SBE-homomorphism. Then, the following conditions are satisfied:*

(a) *If F is a (special) SBE-filter of S, then $f(F)$ is a (special) SBE-filter of T.*

(b) *If G is a (special) SBE-filter of T and f is bijective, then $f^{-1}(G)$ is a (special) SBE-filter of S.*

Proof. Let $\langle S;\circ_S,1_S\rangle$ and $\langle T;\circ_T,1_T\rangle$ be SBE-algebras, and let the mapping $f:S\longrightarrow T$ be a SBE-homomorphism.

(a) Assume that F is a (special) SBE-filter of S. Then we get $1_T=f(1_S)\in f(F)$. Let $f(x\circ_S(y\circ_S y))=f(x)\circ_T(f(y)\circ_T f(y))\in f(F)$ and $f(x)\in f(F)$, i.e., $x\circ_S(y\circ_S y)\in F$ and $x\in F$. Since F is a SBE-filter of S, we have $y\in F$, i.e., $f(y)\in f(F)$.

Now, we show that $f(F)$ is a special SBE-filter of T. Let $f(x\circ_S(y\circ_S y))=f(x)\circ_T(f(y)\circ_T f(y))\in f(F)$ and $f(y\circ_S(x\circ_S x))=f(y)\circ_T(f(x)\circ_T f(x))\in f(F)$, i.e., $x\circ_S(y\circ_S y),y_S(x\circ_S x)\in F$. Since F is a special SBE-filter of S, it follows $(z\circ_S(x\circ_S x))\circ_S((z\circ_S(y\circ_S y))\circ_S(z\circ_S(y\circ_S y)))\in F$, i.e., $(f(z)\circ_T(f(x)\circ_T f(x)))\circ_T((f(z)\circ_T(f(y)\circ_T f(y)))\circ_T(f(z)\circ_T(f(y)\circ_T f(y))))=f((z\circ_S(x\circ_S x))\circ_S((z\circ_S(y\circ_S y))\circ_S(z\circ_S(y\circ_S y))))\in f(F)$.

(b) Suppose that G is a (special) SBE-filter of T and f is bijective. Since $f(1_S) = 1_T \in G$, we obtain $1_S = f^{-1}(1_T) \in f^{-1}(G)$. Let $x \circ_S (y \circ_S y) \in f^{-1}(G)$ and $x \in f^{-1}(G)$, i.e., $f(x) \circ_T (f(y) \circ_T f(y)) = f(x \circ_S (y \circ_S y)) \in G$ and $f(x) \in G$. Because G is a SBE-filter of T, we have $f(y) \in G$, i.e., $y \in f^{-1}(G)$.

Now, we show that $f^{-1}(G)$ is a special SBE-filter of S. Let $x \circ_S (y \circ_S y), y \circ_S (x \circ_S x) \in f^{-1}(G)$, i.e., $f(x) \circ_T (f(y) \circ_T f(y)) = f(x \circ_S (y \circ_S y)), f(y) \circ_T (f(x) \circ_T f(x)) = f(y \circ_S (x \circ_S x)) \in G$. Since G is a special SBE-filter of T, it follows $f((z \circ_S (x \circ_S x)) \circ_S ((z \circ_S (y \circ_S y)) \circ_S (z \circ_S (y \circ_S y)))) = (f(z) \circ_T (f(x) \circ_T f(x))) \circ_T ((f(z) \circ_T (f(y) \circ_T f(y))) \circ_T (f(z) \circ_T (f(y) \circ_T f(y)))) \in G$, i.e., $(z \circ_S (x \circ_S x))_S((z \circ_S (y \circ_S y)) \circ_S (z \circ_S (y \circ_S y))) \in f^{-1}(G)$. ■

Corollary 49. *Let $\langle S; \circ_S, 1_S\rangle$ and $\langle T; \circ_T, 1_T\rangle$ be SBE-algebras, and the mapping $f : S \longrightarrow T$ be a SBE-homomorphism. Then, $f(S)$ is a (special) SBE-filter of T. Moreover, $f(S)$ is a SBE-subalgebra of T.*

Theorem 50. *Let F be a special SBE-filter of S and G be such that $F \subseteq G \subseteq S$. Then G is a special SBE-filter of S if and only if the set G/F is a special SBE-filter of S/F.*

Proof. ($\Rightarrow$) We know that $\langle S/F,_F, F\rangle$ is a SBE-algebra and $F = [1]_F \in S/F$ is the constant in S/F from Corollary 46. Then we obtain $F = [1]_F \in G/F$ by the definition of G/F. Let $[x]_F \circ_F ([y]_F \circ_F [y]_F) \in G/F$ and $[x]_F \in G/F$, i.e., $[x \circ (y \circ y)]_F \in G/F$ and $[x]_F \in G/F$. So, we have $x \circ (y \circ y) \in G$ and $x \in G$. Because G is a SBE-filter of S, it follows $y \in G$, i.e., $[y]_F \in G/F$.

Now we show that G/F is a special SBE-filter of S/F. Let $[x]_F \circ_F ([y]_F \circ_F [y]_F)$, $[y]_F \circ_F ([x]_F \circ_F [x]_F) \in G/F$, i.e., $[x \circ (y \circ y)]_F, [y \circ (x \circ x)]_F \in G/F$. Then it follows $x \circ (y \circ y), y \circ (x \circ x) \in G$. Since G is a special SBE-filter of S, we obtain $(z \circ (x \circ x)) \circ ((z \circ (y \circ y)) \circ (z \circ (y \circ y))) \in G$, i.e., $([z]_F \circ_F ([x]_F \circ_F [x]_F)) \circ_F (([z]_F \circ_F ([y]_F \circ_F [y]_F)) \circ_F ([z]_F \circ_F ([y]_F \circ_F [y]_F))) = [(z \circ (x \circ x)) \circ ((z \circ (y \circ y)) \circ (z \circ (y \circ y)))]_F \in G/F$.

($\Leftarrow$) It follows from the definition of G/F. ■

Theorem 51. *Let $\langle S; \circ_S, 1_S\rangle$ and $\langle T; \circ_T, 1_T\rangle$ be SBE-algebras, and let the mapping $f : S \longrightarrow T$ be a SBE-homomorphism. Then, the following conditions are satisfied:*

(a) *$Kerf = \{s \in S : f(x) = 1_T\}$ is a SBE-filter of S.*

(b) *If S is commutative, then $Kerf$ is a special SBE-filter of S.*

Proof. (a) Since $f(1_S) = 1_T$, $1_S \in Kerf$. Let $x \in Kerf$ and $x \circ_S (y \circ_S y) \in Kerf$. Then $f(x) = 1_T$ and $f(x \circ_S (y \circ_S y)) = 1_T$. Since

$$\begin{aligned} f(y) &= 1_T \circ_T (f(y) \circ_T f(y)) \\ &= f(x) \circ_T (f(y) \circ_T f(y)) \end{aligned}$$

$$\begin{aligned}&= f(x \circ_S (y \circ_S y))\\ &= 1_T,\end{aligned}$$

it follows $y \in Kerf$. Thus, $Kerf$ is a SBE-filter of S. Moreover, it is a SBE-subalgebra of S by Lemma 37.

(b) Assume that S is a commutative. Let $x \circ_S (y \circ_S y) \in Kerf$ and $y \circ_S (x \circ_S x) \in Kerf$. Then $f(x) \circ_T (f(y) \circ_T f(y)) = f(x \circ_S (y \circ_S y)) = 1_T$ and $f(y) \circ_T (f(x) \circ_T f(x)) = f(y \circ_S (x \circ_S x)) = 1_T$. Thus,

$$\begin{aligned} f(x) &= 1_T \circ_T (f(x) \circ_T f(x))\\ &= (f(y) \circ_T (f(x) \circ_T f(x))) \circ_T (f(x) \circ_T f(x))\\ &= f((y \circ_S (x \circ_S x)) \circ_S (x \circ_S x))\\ &= f((x \circ_S (y \circ_S y)) \circ_S (y \circ_S y))\\ &= (f(x) \circ_T (f(y) \circ_T f(y))) \circ_T (f(y) \circ_T f(y))\\ &= 1_T \circ_T (f(y) \circ_T f(y))\\ &= f(y)\end{aligned}$$

from Lemma 12(ii).
$f((z \circ_S (x \circ_S x)) \circ_S ((z \circ_S (y \circ_S y)) \circ_S (z \circ_S (y \circ_S y))))$
$= (f(z) \circ_T (f(x) \circ_T f(x))) \circ_T ((f(z) \circ_T (f(y) \circ_T f(y))) \circ_T (f(z) \circ_T (f(y) \circ_T f(y))))$
$= (f(z) \circ_T (f(x) \circ_T f(x))) \circ_T ((f(z) \circ_T (f(x) \circ_T f(x))) \circ_T (f(z) \circ_T (f(x) \circ_T f(x))))$
$= 1_T$
from $(SBE-1)$, and so $(z \circ_S (x \circ_S x)) \circ_S ((z \circ_S (y \circ_S y)) \circ_S (z \circ_S (y \circ_S y))) \in Kerf$. Hence, $Kerf$ is a special SBE-filter of S. ■

Remark 52. Let $\langle S; \circ, 1\rangle$ be a SBE-algebra and let F be a special SBE-filter of S. Then there exists the natural SBE-homomorphism $g : S \longrightarrow S/F$ defined by $s \mapsto [s]_F$.

Remark 53. Let $\langle S; \circ, 1\rangle$ be a SBE-algebra and let F be a special SBE-filter of S. Then there exists a canonical surjective SBE-homomorphism $h : S \longrightarrow S/F$ defined by $h(s) = [s]_F$ and $Kerh = F$.

Corollary 54. *Let $\langle S; \circ_S, 1_S\rangle$ be a commutative SBE-algebraisch, $\langle T; \circ_T, 1_T\rangle$ be a SBE-algebra, and let the mapping $f : S \longrightarrow T$ be a SBE-homomorphism. Then there exists a unique SBE-homomorphism $g : S/Kerf \longrightarrow T$ such that $f = g \circ h$ where h is the natural SBE-homomorphism $S \longrightarrow S/Kerf$. Moreover, h is surjective and g is injective.*

Corollary 55 (1st Isomorphism Theorem)**.** *Let $\langle S; \circ_S, 1_S\rangle$ be a commutative SBE-algebra, $\langle T; \circ_T, 1_T\rangle$ be a SBE-algebra, and let the mapping $f : S \longrightarrow T$ be a SBE-homomorphism. Then $S/Kerf \cong f(S)$. If f is onto, then $S/Kerf \cong T$.*

Corollary 56 (2nd Isomorphism Theorem). *Let G be a SBE-subalgebra of a commutative SBE-algebra $\langle S; \circ_S, 1_S\rangle$, and let F be a special SBE-filter of S. Then $GF/F \cong G/G \cap F$ where $GF = \bigcup [s]_F$ for $s \in G$ and $GF/F = \{[s]_F \in S/F : s \in GF\}$.*

Corollary 57 (3rd Isomorphism Theorem). *Let F and G be special SBE-filters of S such that $F \subseteq G$. Then $(S/F)/(G/F) \cong S/G$.*

6. Conclusion

By studying SBE-filters of SBE-algebras, a congruence relation induced by a special SBE-filter of a SBE-algebra is defined and a quotient set is constructed by means of the congruence relation on the algebraic structure. As a consequence, we reveal an open problem that "Do a congruence relation on SBE-algebras induce their special filters?" In the future works, we wish to study on different types of filters, fuzzy and neutrosophic structures on SBE-algebras to solve this problem.

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