

## EXTENDED ANNIHILATING-IDEAL GRAPH OF A COMMUTATIVE RING

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### Abstract

Let  $R$  be a commutative ring with identity. An ideal  $I$  of a ring  $R$  is called an annihilating-ideal if there exists a nonzero ideal  $J$  of  $R$  such that  $IJ = (0)$  and we use the notation  $\mathbb{A}(R)$  for the set of all annihilating-ideals of  $R$ . In this paper, we introduce the extended annihilating-ideal graph of  $R$ , denoted by  $\mathbb{EAG}(R)$ . It is the simple graph with vertices  $\mathbb{A}(R)^* = \mathbb{A}(R) \setminus \{(0)\}$ , and two distinct vertices  $I$  and  $J$  are adjacent whenever there exist two positive integers  $n$  and  $m$  such that  $I^n J^m = (0)$  with  $I^n \neq (0)$  and  $J^m \neq (0)$ . Here we discuss in detail the diameter and girth of  $\mathbb{EAG}(R)$  and investigate the coincidence of  $\mathbb{EAG}(R)$  with the annihilating-ideal graph  $\mathbb{AG}(R)$ . Moreover we propose open questions in this paper.

**Keywords:** annihilating-ideal graph, extended annihilating-ideal graph.

**2010 Mathematics Subject Classification:** Primary: 05C75, 05C25, Secondary: 13A15, 13M05.

### 1. INTRODUCTION

The concept of zero-divisor graph of a commutative ring  $R$  was first introduced by Beck in [7]. He let all elements of the ring be vertices of the graph and was

interested mainly in coloring. In [5], Anderson *et al.* associated a zero-divisor graph  $\Gamma(R)$  to  $R$  with vertices  $Z(R)^* = Z(R) \setminus \{0\}$ , the set of all nonzero zero-divisors and two distinct vertices  $x$  and  $y$  are adjacent if and only if  $xy = 0$ . The zero-divisor graphs of commutative rings attracted the attention of several researchers and also this graph was assigned to other algebraic structures (see for instance [4, 13, 14]). In [10], Bennis *et al.* introduced and studied the extended zero-divisor graph of  $R$  which is the extension of the classical zero-divisor graph of  $R$  and it is denoted by  $\bar{\Gamma}(R)$  whose vertex set consists of all its nonzero zero-divisors and that two distinct vertices  $x$  and  $y$  are adjacent whenever there exist two non-negative integers  $n$  and  $m$  such that  $x^n y^m = 0$  with  $x^n \neq 0$  and  $y^m \neq 0$ . In [8], Behboodi *et al.* introduced and investigated the annihilating-ideal graph of  $R$ , denoted by  $\mathbb{AG}(R)$ . An ideal  $I$  of a ring  $R$  is called an annihilating-ideal if there exists a nonzero ideal  $J$  of  $R$  such that  $IJ = (0)$  and we use the notation  $\mathbb{A}(R)$  for the set of all annihilating-ideals of  $R$ . It is the simple graph with vertices  $\mathbb{A}(R)^* = \mathbb{A}(R) \setminus \{(0)\}$ , the set of all nonzero annihilating-ideals of  $R$  and two distinct vertices  $I$  and  $J$  are adjacent if and only if  $IJ = (0)$ . They obtained some finiteness conditions of  $\mathbb{AG}(R)$  and found out the facts of the connectivity of annihilating-ideal graphs. In [9], they discussed the diameter and coloring of annihilating-ideal graphs.

Throughout this paper  $R$  denotes a commutative ring with identity  $1 \neq 0$ . In this paper we introduce an extension of the annihilating-ideal graph of a commutative ring  $R$ , denoted by  $\mathbb{EAG}(R)$ , which we call the extended annihilating-ideal graph of  $R$ , such that its vertex set is  $\mathbb{A}(R)^*$  which is the set of all nonzero annihilating-ideals of  $R$  and that two distinct vertices  $I$  and  $J$  are adjacent if and only if there exist two positive integers  $n$  and  $m$  such that  $I^n J^m = (0)$  with  $I^n \neq (0)$  and  $J^m \neq (0)$ . Clearly, the annihilating-ideal graph  $\mathbb{AG}(R)$  is a spanning subgraph of  $\mathbb{EAG}(R)$ . Note that  $\mathbb{EAG}(R)$  is the empty graph if and only if  $R$  is an integral domain.

The main goal of this paper is to establish the relation between  $\mathbb{EAG}(R)$  and  $\mathbb{AG}(R)$  and the connection between the graph theoretic properties of  $\mathbb{EAG}(R)$  and the ring theoretic properties of  $R$ . In Section 2, we discuss the basic properties of  $\mathbb{EAG}(R)$  and the coincidence of  $\mathbb{EAG}(R)$  and  $\mathbb{AG}(R)$ . Also we determine when  $\mathbb{EAG}(R)$  forms a complete graph or a complete bipartite graph. In Sections 3 and 4, we obtain the diameter and girth of  $\mathbb{EAG}(R)$  and that compare with  $\mathbb{AG}(R)$ . In this paper we propose open questions with regard to the diameter of  $\mathbb{EAG}(R)$ . As usual,  $\mathbb{Z}$ ,  $\mathbb{Q}$ ,  $\mathbb{Z}_n$ ,  $\mathbb{F}$  denote the ring of integers, rational numbers, ring of integers modulo  $n$  and the field, respectively. For basic definitions on rings, one may refer [6].

For the sake of completeness, we state some definitions and notations used throughout. Let  $G$  be a graph and  $V(G)$ ,  $E(G)$  be the vertex set and edge set of  $G$  respectively. A graph  $H$  is called a subgraph of  $G$  that is  $H \subseteq G$  if  $V(H) \subseteq V(G)$

and  $E(H) \subseteq E(G)$ . A subgraph  $H$  of  $G$  with  $V(H) = V(G)$  is called a spanning subgraph of  $G$ . For  $S \subseteq V(G)$ , the induced subgraph  $H$  induced by  $S$  is the subgraph of  $G$  with vertex set  $S$  and two vertices are adjacent in  $H$  if and only if they are adjacent in  $G$  and it is denoted by  $\langle S \rangle$ . A closed path is called a cycle. A cycle on  $n \geq 3$  vertices is denoted by  $C_n$ . We say that  $G$  is connected if there is a path between any two distinct vertices of  $G$ . For vertices  $x$  and  $y$  of  $G$ , let  $d(x, y)$  be the length of the shortest path from  $x$  to  $y$  ( $d(x, x) = 0, d(x, y) = \infty$  if there is no such path). The diameter of  $G$  is  $\text{diam}(G) = \sup \{d(x, y) : x, y \in V(G)\}$ . The girth of  $G$  denoted by  $\text{gr}(G)$ , is the length of a shortest cycle in  $G$  ( $\text{gr}(G) = \infty$  if  $G$  contains no cycles). A graph in which each pair of distinct vertices is joined by an edge is called a complete graph. We denote the complete graph on  $n$  vertices by  $K_n$ . A bipartite graph is a graph all of whose vertices can be partitioned into two parts  $V_1$  and  $V_2$  such that every edge joins a vertex in  $V_1$  to one in  $V_2$ . A complete bipartite graph is a bipartite graph in which every vertex of one part is joined to every vertex of the other part. The complete bipartite graph on  $m$  and  $n$  vertices is denoted by  $K_{m,n}$  and  $K_{1,n}$  a star graph. For undefined terms in graph theory we refer [11]. The following results are useful in the subsequent sections.

**Theorem 1.1** [8, Theorem 1.1]. *Let  $R$  be a non-domain ring. Then  $\mathbb{AG}(R)$  has ACC (respectively, DCC) on vertices if and only if  $R$  is a Noetherian (respectively, an Artinian) ring.*

**Corollary 1.2** [8, Corollary 2.3]. *Let  $R$  be a reduced ring. Then the following statements are equivalent.*

- (i) *There is a vertex of  $\mathbb{AG}(R)$  which is adjacent to every other vertices.*
- (ii)  *$\mathbb{AG}(R)$  is a star graph.*
- (iii)  *$R \cong \mathbb{F} \times D$ , where  $\mathbb{F}$  is a field and  $D$  is an integral domain.*

## 2. BASIC PROPERTIES OF $\mathbb{EAG}(R)$

In this Section we discuss the basic properties of  $\mathbb{EAG}(R)$  and when  $\mathbb{EAG}(R)$  and  $\mathbb{AG}(R)$  coincide. A ring  $R$  is called a reduced ring if it has no nonzero nilpotent elements. The following theorem plays an important role in this paper.

**Theorem 2.1.** *Let  $R$  be a reduced ring. Then  $\mathbb{EAG}(R) = \mathbb{AG}(R)$ .*

**Proof.** Since  $R$  is reduced, it has no nonzero nilpotent ideals. Then for every nonzero ideal  $I$  in  $R$ ,  $I^n \neq (0)$  for all positive integers  $n$ . By definitions of  $\mathbb{EAG}(R)$  and  $\mathbb{AG}(R)$ ,  $\mathbb{AG}(R)$  is a spanning subgraph of  $\mathbb{EAG}(R)$ . Suppose there exist two nonzero proper ideals  $I, J$  in  $R$  such that  $I^n J^m = (0)$  with  $I^n \neq (0)$

and  $J^m \neq (0)$  for some positive integers  $n$  and  $m$  but  $IJ \neq (0)$ . Now  $I^n J^m = (0)$  implies  $I^{m+n} J^{m+n} = (0)$  that is  $(IJ)^{m+n} = (0)$ . Hence  $IJ$  is a nonzero nilpotent ideal of  $R$  which is a contradiction. Thus  $\mathbb{EAG}(R) = \mathbb{AG}(R)$ . ■

**Remark 2.2.** In Theorem 2.1, the converse is not true in general. For example, consider the ring  $R \cong \mathbb{Z}_p[X]/(X^2)$  where  $p$  is a prime number. Here  $\mathbb{A}(R)^*$  has only one nonzero proper nilpotent ideal,  $(X)$ . Thus  $\mathbb{EAG}(R) = \mathbb{AG}(R) \cong K_1$  but  $R$  is not reduced.

**Theorem 2.3.** Let  $R \cong \prod_{i=1}^n R_i$  where  $R_i$ 's are rings for every  $i$  with  $n \geq 2$ . Then  $\mathbb{EAG}(R) = \mathbb{AG}(R)$  if and only if  $R_i$  is reduced for every  $i$ .

**Proof.** Assume that  $\mathbb{EAG}(R) = \mathbb{AG}(R)$ . Suppose that  $R_i$  is not reduced for some  $i$ . Then there exist a nonzero ideal  $I$  in  $R_i$  such that  $I^n = (0)$  for some positive integer  $n$ . We have the following non-adjacency in  $\mathbb{AG}(R)$ ,  $[(0) \times (0) \times \cdots \times R_i \times (0) \times \cdots \times (0)][R_1 \times R_2 \times \cdots \times I \times R_{i+1} \times \cdots \times R_n] = (0) \times (0) \times \cdots \times I \times (0) \times \cdots \times (0) \neq (0) \times (0) \times \cdots \times (0)$  and the adjacency in  $\mathbb{EAG}(R)$ ,  $[(0) \times (0) \times \cdots \times R_i \times (0) \times \cdots \times (0)][R_1 \times R_2 \times \cdots \times I \times R_{i+1} \times \cdots \times R_n]^n = [(0) \times (0) \times \cdots \times R_i \times (0) \times \cdots \times (0)][R_1 \times R_2 \times \cdots \times (0) \times R_{i+1} \times \cdots \times R_n] = (0) \times (0) \times \cdots \times (0)$ . Therefore a contradiction arises to  $\mathbb{EAG}(R) = \mathbb{AG}(R)$ . Conversely, assume that  $R_i$ 's are reduced for every  $i = 1$  to  $n$ . Since product of reduced ring is reduced and by Theorem 2.1,  $\mathbb{EAG}(R) = \mathbb{AG}(R)$ . ■

Recall that an ideal  $I$  of  $R$  is called a principal ideal if  $I = (a) = \{ra : r \in R\}$  for some  $a \in R$ . If every ideal is a principal ideal in  $R$ , then  $R$  is called a principal ideal ring (PIR). An integral domain in which every ideal is principal is called a principal ideal domain (PID). A local artinian PIR is called a special principal ring (SPR) and has an extremely simple ideal structure: there are only finitely many ideals, each of which is a power of the maximal ideal.

In the next two theorems, we determine the situations when  $\mathbb{EAG}(R)$  forms a complete graph and a complete bipartite graph.

**Theorem 2.4.** Let  $R$  be a SPR. Then  $\mathbb{EAG}(R)$  is a complete graph.

**Proof.** Since  $R$  is a SPR, the only ideals of  $R$  are  $R, M, M^2, \dots$  and  $M^n = (0)$ . Also all the nonzero proper ideals of  $R$  are in  $\mathbb{A}(R)^*$ . Let  $M^i, M^j \in \mathbb{A}(R)^*$ . If  $i + j \geq n$ , then  $M^i$  and  $M^j$  are adjacent in  $\mathbb{EAG}(R)$ . If  $i + j < n$  and  $i < j$ , then there exist  $k > 1$  such that  $ik < n$  and  $ik + j \geq n$ . Therefore  $M^i$  and  $M^j$  are adjacent in  $\mathbb{EAG}(R)$ . Hence  $\mathbb{EAG}(R)$  is complete. ■

**Theorem 2.5.** Let  $R \cong R_1 \times R_2$  where  $R_1$  is an integral domain and  $R_2$  is a ring with unique nonzero proper ideal. Then  $\mathbb{EAG}(R)$  is a complete bipartite graph.

**Proof.** Since  $R_1$  is an integral domain,  $I_1^n I_2^m \neq (0)$ , for all nonzero proper ideals  $I_1, I_2$  in  $R_1$  such that  $I_1^n \neq (0), I_2^m \neq (0)$  for all  $n, m \in \mathbb{Z}^+$ . We know that the ideals in  $R_2$  are  $\{(0), J, R_2\}$ . Here  $(0) \times J$  and  $(0) \times R_2$  are adjacent to  $R_1 \times (0)$ ,  $R_1 \times J$ ,  $I \times (0)$  and  $I \times J$  where  $I$  is any nonzero proper ideal of  $R_1$  and there is no other adjacency. Thus it forms a complete bipartite graph. ■

**Theorem 2.6.** *Let  $R$  be a ring. Then the following statements are equivalent.*

- (1)  $\mathbb{EAG}(R)$  is a finite graph.
- (2)  $R$  has only finitely many ideals.
- (3) Every vertex of  $\mathbb{EAG}(R)$  has finite degree.

Moreover,  $\mathbb{EAG}(R)$  has  $n(n \geq 1)$  vertices if and only if  $R$  has only  $n$  nonzero proper ideals.

**Proof.** Since  $\mathbb{AG}(R)$  is a spanning subgraph of  $\mathbb{EAG}(R)$ , the result follows from [8, Theorem 1.4]. ■

### 3. DIAMETER OF $\mathbb{EAG}(R)$

In this Section we discuss the diameter of the extended annihilating-ideal graphs of rings. Also we determine some situations when  $\text{diam}(\mathbb{EAG}(R)) = 0, 1, 2$  or 3.

**Theorem 3.1.** *Let  $R$  be a ring. Then  $\mathbb{EAG}(R)$  is connected with  $\text{diam}(\mathbb{EAG}(R)) \leq 3$  and if  $\mathbb{EAG}(R)$  contains a cycle, then  $\text{gr}(\mathbb{EAG}(R)) \leq 4$ .*

**Proof.** Since  $\mathbb{AG}(R)$  is a spanning subgraph of  $\mathbb{EAG}(R)$ , by [8, Theorem 2.1], the result follows. ■

**Theorem 3.2.** *Let  $R$  be a ring. Then  $\text{diam}(\mathbb{EAG}(R)) = 0$  if and only if it has only one nonzero proper ideal.*

**Proof.** Assume that  $\text{diam}(\mathbb{EAG}(R)) = 0$ . Since  $\mathbb{EAG}(R)$  is always connected,  $A(R)^*$  has only one nonzero proper ideal. Hence  $R$  has only one nonzero proper ideal. Converse is obviously true. ■

Note that for a nilpotent ideal  $I$  of  $R$ , the nilpotency index of  $I$  is denoted by  $n_I$ . The following theorem characterizes artinian rings for which  $\text{diam}(\mathbb{EAG}(R)) = 1$ .

**Theorem 3.3.** *Let  $R$  be an artinian ring. Then  $\text{diam}(\mathbb{EAG}(R)) = 1$  if and only if either  $R \cong \mathbb{F}_1 \times \mathbb{F}_2$  or  $R$  is a local PIR with at least two nonzero proper ideals or  $R$  is local which is not a PIR with at least two nonzero proper ideals for every  $I, J \in A(R)^*$ ,  $I^{n_I-1} J^{n_J-1} = (0)$ .*

**Proof.** Since  $R$  is an artinian, so, by [6, Theorem 8.7],  $R$  is a finite direct product of artinian local rings. Assume that  $\text{diam}(\mathbb{EAG}(R)) = 1$ . Let  $R \cong R_1 \times R_2 \times \cdots \times R_n$  where  $R'_i$ s are artinian local rings with unique maximal ideals  $M_i$ . For  $n \geq 3$ ,  $R_1 \times R_2 \times (0) \times \cdots \times (0)$  is not adjacent to  $R_1 \times (0) \times \cdots \times (0)$  in  $\mathbb{EAG}(R)$ , a contradiction. Therefore  $n \leq 2$  and consider  $n = 2$  with  $R \not\cong \mathbb{F}_1 \times \mathbb{F}_2$ . Here  $R_1 \times (0)$  is not adjacent to  $R_1 \times M_2$ , a contradiction. Hence  $R \cong \mathbb{F}_1 \times \mathbb{F}_2$ . If  $n = 1$ , then  $R$  is a local ring. Suppose that  $R$  has less than two nonzero proper ideals, then by Theorem 3.2, a contradiction. Thus  $R$  is a local ring with at least two nonzero proper ideals. If  $R$  is a PIR, then by Theorem 2.4, the result holds. Suppose that  $R$  is not a PIR, for some  $I, J \in \mathbb{A}(R)^*$ ,  $I^{n_I-1}J^{n_J-1} \neq (0)$ , then  $\text{diam}(\mathbb{EAG}(R)) \neq 1$ . Thus  $R$  is not a PIR for every  $I, J \in \mathbb{A}(R)^*$ ,  $I^{n_I-1}J^{n_J-1} = (0)$ . Conversely, assume that  $R \cong \mathbb{F}_1 \times \mathbb{F}_2$ , then  $\text{diam}(\mathbb{EAG}(R)) = 1$ . If  $R$  is a local PIR with at least two nonzero proper ideals, then by Theorem 2.4,  $\text{diam}(\mathbb{EAG}(R)) = 1$ . If  $R$  is local which is not a PIR with at least two nonzero proper ideals for every  $I, J \in \mathbb{A}(R)^*$ ,  $I^{n_I-1}J^{n_J-1} = (0)$ , then the result is obviously true. ■

**Example 3.4.** Let  $R \cong \mathbb{Z}[i]/(\pi^n)$  where  $\pi$  is an irreducible gaussian integer and  $n \geq 3$ . Here  $R$  is a SPR with more than one nonzero proper ideals. Thus  $\text{diam}(\mathbb{EAG}(R)) = 1$ .

The following three theorems show that when  $\mathbb{EAG}(R)$  has diameter two.

**Theorem 3.5.** Let  $R$  be a PIR. Then  $\text{diam}(\mathbb{EAG}(R)) = 2$  if and only if  $R \cong R_1 \times R_2$  where  $R_1$  and  $R_2$  are either PID or SPR and any one of  $R_i$  is not a field.

**Proof.** Since  $R$  is a PIR, by [15, Theorem 33],  $R \cong \prod_{i=1}^n R_i$  where  $R'_i$ s are either PIDs or SPRs. Assume that  $\text{diam}(\mathbb{EAG}(R)) = 2$ . Consider  $n \geq 3$ . Then we have the following adjacency in  $\mathbb{EAG}(R)$ ,  $(0) \times R_2 \times \cdots \times R_n - R_1 \times (0) \times \cdots \times (0) - (0) \times R_2 \times (0) \times \cdots \times (0) - R_1 \times (0) \times R_3 \times \cdots \times R_n$ . This shows that  $\text{diam}(\mathbb{EAG}(R)) = 3$ . Thus  $R \cong R_1 \times R_2$ . If  $R_1$  and  $R_2$  are fields, then by Theorem 3.3,  $\text{diam}(\mathbb{EAG}(R)) = 1$ . Therefore  $R_1$  and  $R_2$  are not fields. Thus  $R_1$  and  $R_2$  are either PID or SPR and not fields. Suppose that  $R_1$  is a field, then clearly  $R_2$  is not a field, it is either PID or SPR.

Conversely, assume that  $R \cong R_1 \times R_2$ , where  $R'_i$ s are either PID or SPR and any one of  $R_i$  is not a field,  $i = 1, 2$ . Consider  $R_1$  and  $R_2$  are SPRs and are not fields. Clearly  $(0) \times R_2$  and  $M_1 \times R_2$  are not adjacent in  $\mathbb{EAG}(R)$  where  $M_1$  is a nonzero proper ideal in  $R_1$ , so  $\text{diam}(\mathbb{EAG}(R)) \geq 2$ . By Theorem 3.1,  $\text{diam}(\mathbb{EAG}(R)) = 2$  or 3. Consider  $R_1$  has a unique nonzero proper ideal, say  $M_1$ . Let  $V_1 = \{(0) \times M_2^j : j = 1 \text{ to } m-1\}$ ,  $V_2 = \{M_1 \times M_2^j : j = 1 \text{ to } m-1\}$  and  $V_3 = \{R_1 \times M_2^j : j = 1 \text{ to } m-1\}$  where  $M_2$  is a nonzero proper ideal in  $R_2$  and  $M_2^m = (0)$ . Then the induced subgraphs  $\langle V_1 \rangle$  and  $\langle V_2 \rangle$  are complete and  $\langle V_3 \rangle$  is totally disconnected. In Figure 2.1, any one edge ends at  $V_i$  means that edge

adjacent to all the vertices in  $V_i$  and also it is the spanning subgraph of  $\mathbb{EAG}(R)$  and its diameter is 2. Hence  $\text{diam}(\mathbb{EAG}(R)) = 2$ .

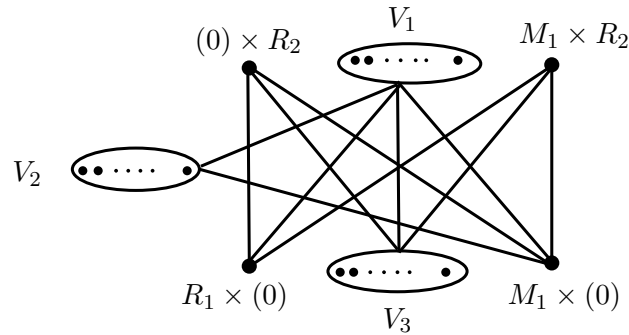


Figure 2.1

Suppose that  $R_1$  and  $R_2$  have more than one nonzero proper ideals. Let  $V_1 = \{M_1^i \times (0) : i = 1 \text{ to } n-1\}$ ,  $V_2 = \{M_1^i \times M_2^j : i = 1 \text{ to } n-1 \text{ and } j = 1 \text{ to } m-1\}$ ,  $V_3 = \{(0) \times M_2^j : j = 1 \text{ to } m-1\}$ ,  $V_4 = \{M_1^i \times R_2 : i = 1 \text{ to } n-1\}$  and  $V_5 = \{R_1 \times M_2^j : j = 1 \text{ to } m-1\}$  where  $M_1$  and  $M_2$  are nonzero proper ideals in  $R_1$  and  $R_2$  respectively,  $M_1^n = (0)$  and  $M_2^m = (0)$ . Then the induced subgraphs  $\langle V_1 \rangle, \langle V_2 \rangle, \langle V_3 \rangle$  are complete graphs and  $\langle V_4 \rangle, \langle V_5 \rangle$  are totally disconnected. Figure 2.2 is the spanning subgraph of  $\mathbb{EAG}(R)$  and its diameter is 2. Hence  $\text{diam}(\mathbb{EAG}(R)) = 2$ . From the above cases,  $\text{diam}(\mathbb{EAG}(R)) = 2$ .

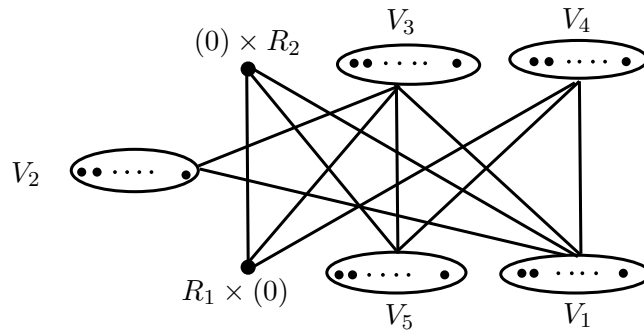


Figure 2.2

Now consider  $R_1$  and  $R_2$  are PIDs which are not fields. Since  $R_1$  and  $R_2$  are reduced and there exist nonzero prime ideals  $P = R_1 \times (0)$  and  $Q = (0) \times R_2$  of  $R$  which are not minimal ideals such that  $P \cap Q = (0)$ , by Theorem 2.3 and [3, Theorem 2.4],  $\mathbb{EAG}(R)$  is a complete bipartite graph. Thus  $\text{diam}(\mathbb{EAG}(R)) = 2$ .

Suppose that  $R_1$  is a PID and  $R_2$  is a SPR and  $R_1, R_2$  are not fields. Since  $R_1$  is a PID,  $I_1^{k_1} I_2^{k_2} \neq (0)$  for all nonzero proper ideals  $I_1, I_2$  in  $R_1$  such that  $I_1^{k_1} \neq (0), I_2^{k_2} \neq (0)$  for all  $k_1, k_2 \in \mathbb{Z}^+$ . Let  $V_1 = \{R_1 \times M_2^j : j = 1 \text{ to } m-1\}$ ,

$V_2 = \{I \times M_2^j : j = 1 \text{ to } m - 1\}$ ,  $V_3 = \{I \times (0)\}$  and  $V_4 = \{(0) \times M_2^j : j = 1 \text{ to } m - 1\}$  where  $I$  is a nonzero proper ideal in  $R_1$  and  $M_2$  is a nonzero proper ideal in  $R_2$  and  $M_2^m = (0)$ . Then the induced subgraphs  $\langle V_1 \rangle$ ,  $\langle V_2 \rangle$  and  $\langle V_3 \rangle$  are totally disconnected and  $\langle V_4 \rangle$  is complete. Figure 2.3 is the extended annihilating-ideal graph of  $R$  and it shows that  $\text{diam}(\mathbb{EAG}(R)) = 2$ .

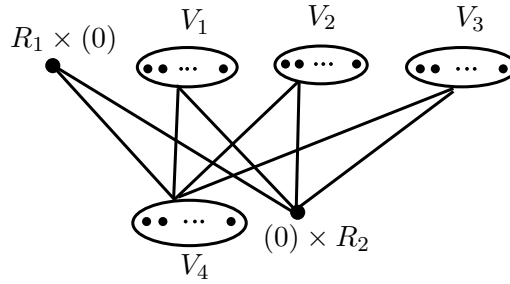


Figure 2.3

Now consider  $R_1$  is a field and  $R_2$  is a PID which is not a field. Since  $R_1$  and  $R_2$  are reduced, by Theorem 2.3 and Corollary 1.2,  $\mathbb{EAG}(R)$  is a star graph. Thus  $\text{diam}(\mathbb{EAG}(R)) = 2$ .

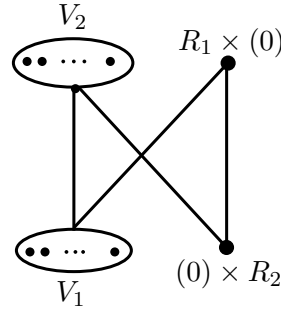
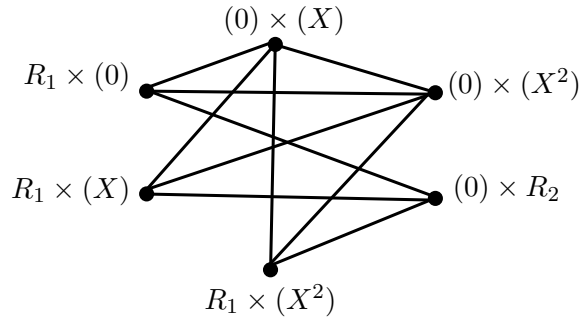


Figure 2.4

Suppose that  $R_1$  is a field and  $R_2$  is a SPR which is not a field. Let  $V_1 = \{(0) \times M_2^j : j = 1 \text{ to } m - 1\}$  and  $V_2 = \{R_1 \times M_2^j : j = 1 \text{ to } m - 1\}$  where  $M_2$  is a nonzero proper ideal in  $R_2$  and  $M_2^m = (0)$ . Then the induced subgraphs  $\langle V_1 \rangle$  is complete and  $\langle V_2 \rangle$  is totally disconnected. Figure 2.4 is the extended annihilating-ideal graph of  $R$  and it shows that  $\text{diam}(\mathbb{EAG}(R)) = 2$ . Hence in all the cases  $\text{diam}(\mathbb{EAG}(R)) = 2$ . ■

**Example 3.6.** If  $R \cong R_1 \times R_2$  where  $R_1 = \mathbb{Z}_2[X]/(X^2 + X + 1)$  and  $R_2 = \mathbb{Z}_2[X]/(X^3)$ , then  $\text{diam}(\mathbb{EAG}(R)) = 2$ .



Figure 2.5  $\mathbb{EAG}(R)$ 

**Theorem 3.7.** *Let  $R$  be a reduced ring such that  $Z(R)$  is not an ideal of  $R$ . Then  $\text{diam}(\mathbb{EAG}(R)) = 2$  if and only if  $R$  has exactly two minimal prime ideals and at least three nonzero annihilating-ideals.*

**Proof.** The result is obviously true from the Theorem 2.1 and [9, Theorem 1.2]. ■

**Theorem 3.8.** *Let  $R \cong R_1 \times R_2$  where  $R_1$  is an integral domain and  $R_2$  is a ring with unique nonzero proper ideal. Then  $\text{diam}(\mathbb{EAG}(R)) = 2$ .*

**Proof.** By Theorem 2.5,  $\text{diam}(\mathbb{EAG}(R)) = 2$ . ■

**Open question 3.9.** *Determine  $\text{diam}(\mathbb{EAG}(R))$  for  $R \cong R_1 \times R_2$  where  $R_1$  is an integral domain and  $R_2$  is a ring with more than one nonzero proper ideals and not a PIR.*

In the next theorem, we state necessary and sufficient conditions to have  $\text{diam}(\mathbb{EAG}(R)) = 3$  for artinian PIR.

**Theorem 3.10.** *Let  $R$  be an artinian PIR such that  $Z(R)$  is not an ideal of  $R$ . Then  $\text{diam}(\mathbb{EAG}(R)) = 3$  if and only if  $R$  is a ring with more than two minimal prime ideals.*

**Proof.** Since  $R$  is an artinian PIR and  $Z(R)$  is not an ideal of  $R$ ,  $R \cong \prod_{i=1}^n R_i$  where  $R_i$ 's are SPRs for all  $i = 1$  to  $n$  and  $n \geq 2$ . Assume that  $\text{diam}(\mathbb{EAG}(R)) = 3$ . Since  $\mathbb{AG}(R)$  is a spanning subgraph of  $\mathbb{EAG}(R)$ ,  $\text{diam}(\mathbb{AG}(R)) = 3$ . Then by [9, Theorem 1.4],  $R$  is a reduced ring with more than two minimal prime ideals or  $R$  is a non-reduced ring. Consider the case  $R$  is non-reduced. For  $n = 2$  and  $R \cong R_1 \times R_2$ , where  $R_1$  is not a field, then as noted in the proof of Theorem 3.5,  $\text{diam}(\mathbb{EAG}(R)) = 2$ . Thus  $n \geq 3$ . Also note that in a commutative artinian ring, every maximal ideal is a minimal prime ideal. From this,  $R$  is a non-reduced ring with more than two minimal prime ideals.

Conversely, assume that  $R$  is a ring with more than two minimal prime ideals. Consider  $R$  is a reduced ring with more than two minimal prime ideals. Then by Theorem 2.1 and [9, Theorem 1.4],  $\text{diam}(\mathbb{EAG}(R)) = 3$ . Suppose that  $R$  is a non-reduced ring with more than two minimal prime ideals and  $\text{diam}(\mathbb{EAG}(R)) \neq 3$ . If  $\text{diam}(\mathbb{EAG}(R)) = 1$ , then by Theorem 3.3,  $R$  is a SPR with at least two nonzero proper ideals. Here  $Z(R)$  is an ideal of  $R$ , a contradiction. If  $\text{diam}(\mathbb{EAG}(R)) = 2$ , then by Theorem 3.5,  $R$  does not have more than two minimal prime ideals. Hence  $\text{diam}(\mathbb{EAG}(R)) = 3$ . ■

**Open question 3.11.** *Classify the diameter of  $\mathbb{EAG}(R)$  for all artinian rings which are not PIRs.*

#### 4. GIRTH OF $\mathbb{EAG}(R)$

In this Section we discuss the girth of  $\mathbb{EAG}(R)$  and also compare the girth of  $\mathbb{AG}(R)$  with  $\mathbb{EAG}(R)$ . As Theorem 3.1,  $\text{gr}(\mathbb{EAG}(R)) \leq 4$ . Here we characterize the rings for which  $\text{gr}(\mathbb{EAG}(R)) = 3, 4$  or  $\infty$ .

**Theorem 4.1.** *Let  $R \cong R_1 \times R_2$  be a ring. If any one of  $R_i$  is a ring with more than one nonzero proper ideals for  $i = 1, 2$ , then  $\text{gr}(\mathbb{EAG}(R)) = 3$ .*

**Proof.** Suppose that  $R_1$  is a ring with more than one nonzero proper ideals. Since  $\mathbb{EAG}(R_1)$  is connected, there exist two nonzero proper ideals  $I_1$  and  $I_2$  in  $R_1$  such that  $I_1^n I_2^m = (0)$  with  $I_1^n \neq (0), I_2^m \neq (0)$  for some positive integers  $n, m$ . Then we have the following cycle of length 3 in  $\mathbb{EAG}(R)$ ,  $I_1 \times (0) - I_2 \times (0) - (0) \times R_2 - I_1 \times (0)$ . Thus  $\text{gr}(\mathbb{EAG}(R)) = 3$ . ■

**Theorem 4.2.** *Let  $R \cong \prod_{i=1}^n R_i$  where  $R_i$ 's are rings for every  $i$  with  $n \geq 2$ . Then the following hold.*

- (i) *For  $n = 2$ ,  $\text{gr}(\mathbb{EAG}(R)) = \infty$  if and only if  $R \cong R_1 \times R_2$ , where  $R_1$  is a field and  $R_2$  is an integral domain.*
- (ii)  *$\text{gr}(\mathbb{EAG}(R)) = 3$  if and only if one of the following statements hold.*
  - (a) *When  $n \geq 3$*
  - (b) *For  $n = 2$ , both  $R_i$ 's are not integral domains.*
  - (c) *For  $n = 2$ ,  $R_1$  is an integral domain and  $R_2$  is a ring with more than one nonzero proper ideals.*
- (iii) *For  $n = 2$ ,  $\text{gr}(\mathbb{EAG}(R)) = 4$  if and only if either  $R_1$  and  $R_2$  are integral domains which are not fields or  $R_1$  is an integral domain and  $R_2$  is a ring with unique nonzero proper ideal.*

**Proof.** (i) For  $n = 2$  and assume that  $gr(\mathbb{EAG}(R)) = \infty$ . Since  $\mathbb{AG}(R)$  is a spanning subgraph of  $\mathbb{EAG}(R)$ ,  $\mathbb{EAG}(R) \cong \mathbb{AG}(R)$ . Then by Theorem 2.3,  $R'_i$ s are reduced for all  $i = 1, 2$ . Also by [3, Theorem 3.1],  $R \cong R_1 \times R_2$  where  $R_1$  is a field and  $R_2$  is an integral domain. Conversely, assume that  $R \cong R_1 \times R_2$  where  $R_1$  is a field and  $R_2$  is an integral domain, then  $R_1$  and  $R_2$  are reduced. By Theorem 2.3 and Corollary 1.2,  $\mathbb{EAG}(R)$  is a star graph. Therefore  $gr(\mathbb{EAG}(R)) = \infty$ .

(ii) (a) Assume that  $gr(\mathbb{EAG}(R)) = 3$ . Then from the following cycle  $R_1 \times (0) \times \cdots \times (0) - (0) \times R_2 \times (0) \times \cdots \times (0) - (0) \times (0) \times R_3 \times (0) \times \cdots \times (0) - R_1 \times (0) \times \cdots \times (0)$ ,  $n \geq 3$  is true. Conversely, when  $n \geq 3$ , the result is obviously true.

(b) Now consider  $n = 2$  and assume that  $gr(\mathbb{EAG}(R)) = 3$ . Suppose that  $R_1$  and  $R_2$  are integral domains. Since  $R_1$  and  $R_2$  are reduced, by Theorem 2.3, Corollary 1.2 and [3, Corollary 2.5],  $gr(\mathbb{EAG}(R)) = 4$  or  $\infty$ , a contradiction. From this case  $R'_i$ s are not integral domains. Conversely, assume that  $R_1$  and  $R_2$  are not integral domains. Consider  $R_1$  and  $R_2$  are rings with unique nonzero proper ideals  $I$  and  $J$  in  $R_1$  and  $R_2$  respectively such that  $I^2 = (0)$  and  $J^2 = (0)$ . Then in  $\mathbb{EAG}(R)$ ,  $I \times (0) - I \times J - (0) \times J - I \times (0)$  is a cycle of length 3 so that  $gr(\mathbb{EAG}(R)) = 3$ . Also consider  $R_1$  is a ring with more than one nonzero proper ideals. Then by Theorem 4.1,  $gr(\mathbb{EAG}(R)) = 3$ . From these cases,  $gr(\mathbb{EAG}(R)) = 3$ .

(c) For  $n = 2$ , assume that  $gr(\mathbb{EAG}(R)) = 3$ . If  $R_1$  is an integral domain and  $R_2$  is a ring with unique nonzero proper ideal, then by Theorem 2.5,  $gr(\mathbb{EAG}(R)) = 4$ , a contradiction. Thus  $R_1$  is an integral domain and  $R_2$  is a ring with more than one nonzero proper ideals. Converse part follows from Theorem 4.1.

(iii) Proof follows from (i) and (ii). ■

We next characterize when  $gr(\mathbb{EAG}(\mathbb{Z}_n))$  is 3, 4 or  $\infty$ .

**Theorem 4.3.** For  $n \in \mathbb{N}$ , let  $n = \prod_{i=1}^k p_i^{\alpha_i}$  be the distinct prime factorization of  $n$ . Then the following assertions are true.

- (i)  $gr(\mathbb{EAG}(\mathbb{Z}_n)) = 3$  if and only if one of the following assertions must occur.
  - (a) When  $k \geq 3$ .
  - (b) When  $k = 1$  and  $\alpha_1 \geq 4$ .
  - (c) When  $k = 2$ , either  $\alpha_1 = 1, \alpha_2 > 2$  or  $\alpha_1, \alpha_2 \geq 2$ .
- (ii)  $gr(\mathbb{EAG}(\mathbb{Z}_n)) = 4$  if and only if  $k = 2$  with  $\alpha_1 = 1, \alpha_2 = 2$ .
- (iii)  $gr(\mathbb{EAG}(\mathbb{Z}_n)) = \infty$  if and only if either  $k = 1$  with  $\alpha_1 = 2$  or 3 or  $k = 2$  with  $\alpha_1 = 1, \alpha_2 = 1$ .

**Proof.** When  $k \geq 2$ , the result holds by Theorem 4.2. It remains to consider the case  $k = 1$ . Assume that  $gr(\mathbb{EAG}(\mathbb{Z}_n)) = 3$  and  $\alpha_1 < 4$ . Then  $|\mathbb{A}(R)^*| = 0, 1$  or 2

and hence  $gr(\mathbb{E}\mathbb{A}\mathbb{G}(\mathbb{Z}_n)) = \infty$ . This shows that  $\alpha_1 \geq 4$ . Conversely, assume that  $\alpha_1 \geq 4$ , then  $\mathbb{Z}_n$  is a local ring with maximal ideal  $(p_1)$  and  $|\mathbb{A}(R)^*| = \alpha_1 - 1$ . Since  $\mathbb{Z}_n$  is a SPR, by Theorem 2.4,  $gr(\mathbb{E}\mathbb{A}\mathbb{G}(\mathbb{Z}_n)) = 3$ . By above, (iii) holds for the case  $k = 1$ . ■

We conclude this paper with the following theorem to have a better comparison of the girth between  $\mathbb{E}\mathbb{A}\mathbb{G}(R)$  and  $\mathbb{A}\mathbb{G}(R)$ .

**Theorem 4.4.** *Let  $R$  be a ring. Then the following hold.*

- (i) *If  $gr(\mathbb{A}\mathbb{G}(R)) = 3$ , then  $gr(\mathbb{E}\mathbb{A}\mathbb{G}(R)) = 3$ .*
- (ii) *If  $gr(\mathbb{A}\mathbb{G}(R)) = 4$ , then  $gr(\mathbb{E}\mathbb{A}\mathbb{G}(R)) = 4$ .*
- (iii) *If  $gr(\mathbb{A}\mathbb{G}(R)) = \infty$ , then  $gr(\mathbb{E}\mathbb{A}\mathbb{G}(R)) = 3, 4$  or  $\infty$ .*
- (iv) *If  $gr(\mathbb{E}\mathbb{A}\mathbb{G}(R)) = 3$ , then  $gr(\mathbb{A}\mathbb{G}(R)) = 3$  or  $\infty$ .*
- (v) *If  $gr(\mathbb{E}\mathbb{A}\mathbb{G}(R)) = 4$ , then  $gr(\mathbb{A}\mathbb{G}(R)) = 4$  or  $\infty$ .*
- (vi) *If  $gr(\mathbb{E}\mathbb{A}\mathbb{G}(R)) = \infty$ , then  $gr(\mathbb{A}\mathbb{G}(R)) = \infty$ .*

**Proof.** Since  $\mathbb{A}\mathbb{G}(R)$  is a spanning subgraph of  $\mathbb{E}\mathbb{A}\mathbb{G}(R)$ , (i), (iii), (v), (vi) are obviously true.

(ii) Assume that  $gr(\mathbb{A}\mathbb{G}(R)) = 4$ . This shows that  $\mathbb{A}\mathbb{G}(R)$  is a triangle-free graph. Then by [2, Lemma 1],  $R \cong R_1 \times R_2$  where either  $R_1$  and  $R_2$  are integral domains which are not fields or  $R_1$  is an integral domain which is not a field and  $R_2$  is a ring with unique nonzero proper ideal. By Theorem 4.2(iii),  $gr(\mathbb{E}\mathbb{A}\mathbb{G}(R)) = 4$ .

(iv) Assume that  $gr(\mathbb{E}\mathbb{A}\mathbb{G}(R)) = 3$ . Since  $\mathbb{A}\mathbb{G}(R)$  is a spanning subgraph of  $\mathbb{E}\mathbb{A}\mathbb{G}(R)$ ,  $gr(\mathbb{A}\mathbb{G}(R)) = 3, 4$  or  $\infty$ . By part (ii),  $gr(\mathbb{A}\mathbb{G}(R)) \neq 4$  and so  $gr(\mathbb{A}\mathbb{G}(R)) = 3$  or  $\infty$ . ■

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Received 27 June 2020

Revised 4 February 2021

Accepted 22 March 2022