Discussiones Mathematicae
General Algebra and Applications 42 (2022) 279–291
https://doi.org/10.7151/dmgaa.1390

EXTENDED ANNIHILATING-IDEAL GRAPH OF A COMMUTATIVE RING

S. Nithya[1,3] and G. Elavarasi[2,3]

[1]*Assistant Professor*
Department of Mathematics
St. Xavier's College (Autonomous)
Palayamkottai 627 002, Tamil Nadu, India

[2]*Reg. No.18111282092014*
PG and Research Department of Mathematics
St. Xavier's College (Autonomous)
Palayamkottai 627 002, Tamil Nadu, India

[3]*Affiliated to Manonmaniam Sundaranar University*
Abishekapatti, Tirunelveli 627 012, Tamil Nadu, India

e-mail: nithyasxc@gmail.com
gelavarasi94@gmail.com

Abstract

Let R be a commutative ring with identity. An ideal I of a ring R is called an annihilating-ideal if there exists a nonzero ideal J of R such that $IJ = (0)$ and we use the notation $\mathbb{A}(R)$ for the set of all annihilating-ideals of R. In this paper, we introduce the extended annihilating-ideal graph of R, denoted by $\mathbb{EAG}(R)$. It is the simple graph with vertices $\mathbb{A}(R)^* = \mathbb{A}(R) \setminus \{(0)\}$, and two distinct vertices I and J are adjacent whenever there exist two positive integers n and m such that $I^n J^m = (0)$ with $I^n \neq (0)$ and $J^m \neq (0)$. Here we discuss in detail the diameter and girth of $\mathbb{EAG}(R)$ and investigate the coincidence of $\mathbb{EAG}(R)$ with the annihilating-ideal graph $\mathbb{AG}(R)$. Moreover we propose open questions in this paper.

Keywords: annihilating-ideal graph, extended annihilating-ideal graph.

2010 Mathematics Subject Classification: Primary: 05C75, 05C25, Secondary: 13A15, 13M05.

1. Introduction

The concept of zero-divisor graph of a commutative ring R was first introduced by Beck in [7]. He let all elements of the ring be vertices of the graph and was interested mainly in coloring. In [5], Anderson *et al.* associated a zero-divisor graph $\Gamma(R)$ to R with vertices $Z(R)^* = Z(R)\backslash\{0\}$, the set of all nonzero zero-divisors and two distinct vertices x and y are adjacent if and only if $xy = 0$. The zero-divisor graphs of commutative rings attracted the attention of several researchers and also this graph was assigned to other algebraic structures (see for instance [4, 13, 14]). In [10], Bennis et al. introduced and studied the extended zero-divisor graph of R which is the extension of the classical zero-divisor graph of R and it is denoted by $\bar{\Gamma}(R)$ whose vertex set consists of all its nonzero zero-divisors and that two distinct vertices x and y are adjacent whenever there exist two non-negative integers n and m such that $x^n y^m = 0$ with $x^n \neq 0$ and $y^m \neq 0$. In [8], Behboodi et al. introduced and investigated the annihilating-ideal graph of R, denoted by $\mathbb{AG}(R)$. An ideal I of a ring R is called an annihilating-ideal if there exists a nonzero ideal J of R such that $IJ = (0)$ and we use the notation $\mathbb{A}(R)$ for the set of all annihilating-ideals of R. It is the simple graph with vertices $\mathbb{A}(R)^* = \mathbb{A}(R)\backslash\{(0)\}$, the set of all nonzero annihilating-ideals of R and two distinct vertices I and J are adjacent if and only if $IJ = (0)$. They obtained some finiteness conditions of $\mathbb{AG}(R)$ and found out the facts of the connectivity of annihilating-ideal graphs. In [9], they discussed the diameter and coloring of annihilating-ideal graphs.

Throughout this paper R denotes a commutative ring with identity $1 \neq 0$. In this paper we introduce an extension of the annihilating-ideal graph of a commutative ring R, denoted by $\mathbb{EAG}(R)$, which we call the extended annihilating-ideal graph of R, such that its vertex set is $\mathbb{A}(R)^*$ which is the set of all nonzero annihilating-ideals of R and that two distinct vertices I and J are adjacent if and only if there exist two positive integers n and m such that $I^n J^m = (0)$ with $I^n \neq (0)$ and $J^m \neq (0)$. Clearly, the annihilating-ideal graph $\mathbb{AG}(R)$ is a spanning subgraph of $\mathbb{EAG}(R)$. Note that $\mathbb{EAG}(R)$ is the empty graph if and only if R is an integral domain.

The main goal of this paper is to establish the relation between $\mathbb{EAG}(R)$ and $\mathbb{AG}(R)$ and the connection between the graph theoretic properties of $\mathbb{EAG}(R)$ and the ring theoretic properties of R. In Section 2, we discuss the basic properties of $\mathbb{EAG}(R)$ and the coincidence of $\mathbb{EAG}(R)$ and $\mathbb{AG}(R)$. Also we determine when $\mathbb{EAG}(R)$ forms a complete graph or a complete bipartite graph. In Sections 3 and 4, we obtain the diameter and girth of $\mathbb{EAG}(R)$ and that compare with $\mathbb{AG}(R)$. In this paper we propose open questions with regard to the diameter of $\mathbb{EAG}(R)$. As usual, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_n$, $\mathbb{F}$ denote the ring of integers, rational numbers, ring of integers modulo n and the field, respectively. For basic definitions on rings, one may refer [6].

For the sake of completeness, we state some definitions and notations used throughout. Let G be a graph and $V(G), E(G)$ be the vertex set and edge set of G respectively. A graph H is called a subgraph of G that is $H \subseteq G$ if $V(H) \subseteq V(G)$

and $E(H) \subseteq E(G)$. A subgraph H of G with $V(H) = V(G)$ is called a spanning subgraph of G. For $S \subseteq V(G)$, the induced subgraph H induced by S is the subgraph of G with vertex set S and two vertices are adjacent in H if and only if they are adjacent in G and it is denoted by $\langle S \rangle$. A closed path is called a cycle. A cycle on $n \geq 3$ vertices is denoted by C_n. We say that G is connected if there is a path between any two distinct vertices of G. For vertices x and y of G, let $d(x, y)$ be the length of the shortest path from x to y ($d(x, x) = 0, d(x, y) = \infty$ if there is no such path). The diameter of G is $diam(G) = \sup\{d(x, y) : x, y \in V(G)\}$. The girth of G denoted by $gr(G)$, is the length of a shortest cycle in G ($gr(G) = \infty$ if G contains no cycles). A graph in which each pair of distinct vertices is joined by an edge is called a complete graph. We denote the complete graph on n vertices by K_n. A bipartite graph is a graph all of whose vertices can be partitioned into two parts V_1 and V_2 such that every edge joins a vertex in V_1 to one in V_2. A complete bipartite graph is a bipartite graph in which every vertex of one part is joined to every vertex of the other part. The complete bipartite graph on m and n vertices is denoted by $K_{m,n}$ and $K_{1,n}$ a star graph. For undefined terms in graph theory we refer [11]. The following results are useful in the subsequent sections.

Theorem 1.1 [8, Theorem 1.1]**.** *Let R be a non-domain ring. Then $\mathbb{AG}(R)$ has ACC (respectively, DCC) on vertices if and only if R is a Noetherian (respectively, an Artinian) ring.*

Corollary 1.2 [8, Corollary 2.3]**.** *Let R be a reduced ring. Then the following statements are equivalent.*

(i) *There is a vertex of $\mathbb{AG}(R)$ which is adjacent to every other vertices.*

(ii) *$\mathbb{AG}(R)$ is a star graph.*

(iii) *$R \cong \mathbb{F} \times D$, where $\mathbb{F}$ is a field and D is an integral domain.*

2. BASIC PROPERTIES OF $\mathbb{EAG}(R)$

In this Section we discuss the basic properties of $\mathbb{EAG}(R)$ and when $\mathbb{EAG}(R)$ and $\mathbb{AG}(R)$ coincide. A ring R is called a reduced ring if it has no nonzero nilpotent elements. The following theorem plays an important role in this paper.

Theorem 2.1. *Let R be a reduced ring. Then $\mathbb{EAG}(R) = \mathbb{AG}(R)$.*

Proof. Since R is reduced, it has no nonzero nilpotent ideals. Then for every nonzero ideal I in R, $I^n \neq (0)$ for all positive integers n. By definitions of $\mathbb{EAG}(R)$ and $\mathbb{AG}(R)$, $\mathbb{AG}(R)$ is a spanning subgraph of $\mathbb{EAG}(R)$. Suppose there exist two nonzero proper ideals I, J in R such that $I^n J^m = (0)$ with $I^n \neq (0)$

and $J^m \neq (0)$ for some positive integers n and m but $IJ \neq (0)$. Now $I^nJ^m = (0)$ implies $I^{m+n}J^{m+n} = (0)$ that is $(IJ)^{m+n} = (0)$. Hence IJ is a nonzero nilpotent ideal of R which is a contradiction. Thus $\mathbb{EAG}(R) = \mathbb{AG}(R)$. ∎

Remark 2.2. In Theorem 2.1, the converse is not true in general. For example, consider the ring $R \cong \mathbb{Z}_p[X]/(X^2)$ where p is a prime number. Here $\mathbb{A}(R)^*$ has only one nonzero proper nilpotent ideal, (X). Thus $\mathbb{EAG}(R) = \mathbb{AG}(R) \cong K_1$ but R is not reduced.

Theorem 2.3. *Let $R \cong \prod_{i=1}^{n} R_i$ where $R_i's$ are rings for every i with $n \geq 2$. Then $\mathbb{EAG}(R) = \mathbb{AG}(R)$ if and only if R_i is reduced for every i.*

Proof. Assume that $\mathbb{EAG}(R) = \mathbb{AG}(R)$. Suppose that R_i is not reduced for some i. Then there exist a nonzero ideal I in R_i such that $I^n = (0)$ for some positive integer n. We have the following non-adjacency in $\mathbb{AG}(R)$, $[(0) \times (0) \times \cdots \times R_i \times (0) \times \cdots \times (0)][R_1 \times R_2 \times \cdots \times I \times R_{i+1} \times \cdots \times R_n] = (0) \times (0) \times \cdots \times I \times (0) \times \cdots \times (0) \neq (0) \times (0) \times \cdots \times (0)$ and the adjacency in $\mathbb{EAG}(R)$, $[(0)\times(0)\times\cdots\times R_i\times(0)\times\cdots\times(0)][R_1\times R_2\times\cdots\times I\times R_{i+1}\times\cdots\times R_n]^n = [(0)\times(0)\times\cdots\times R_i\times(0)\times\cdots\times(0)][R_1\times R_2\times\cdots\times(0)\times R_{i+1}\times\cdots\times R_n] = (0)\times(0)\times\cdots\times(0)$. Therefore a contradiction arises to $\mathbb{EAG}(R) = \mathbb{AG}(R)$. Conversely, assume that $R_i's$ are reduced for every $i = 1$ to n. Since product of reduced ring is reduced and by Theorem 2.1, $\mathbb{EAG}(R) = \mathbb{AG}(R)$. ∎

Recall that an ideal I of R is called a principal ideal if $I = (a) = \{ra : r \in R\}$ for some $a \in R$. If every ideal is a principal ideal in R, then R is called a principal ideal ring (PIR). An integral domain in which every ideal is principal is called a principal ideal domain (PID). A local artinian PIR is called a special principal ring (SPR) and has an extremely simple ideal structure: there are only finitely many ideals, each of which is a power of the maximal ideal.

In the next two theorems, we determine the situations when $\mathbb{EAG}(R)$ forms a complete graph and a complete bipartite graph.

Theorem 2.4. *Let R be a SPR. Then $\mathbb{EAG}(R)$ is a complete graph.*

Proof. Since R is a SPR, the only ideals of R are $R, M, M^2, \ldots$ and $M^n = (0)$. Also all the nonzero proper ideals of R are in $\mathbb{A}(R)^*$. Let $M^i, M^j \in \mathbb{A}(R)^*$. If $i + j \geq n$, then M^i and M^j are adjacent in $\mathbb{EAG}(R)$. If $i + j < n$ and $i < j$, then there exist $k > 1$ such that $ik < n$ and $ik + j \geq n$. Therefore M^i and M^j are adjacent in $\mathbb{EAG}(R)$. Hence $\mathbb{EAG}(R)$ is complete. ∎

Theorem 2.5. *Let $R \cong R_1 \times R_2$ where R_1 is an integral domain and R_2 is a ring with unique nonzero proper ideal. Then $\mathbb{EAG}(R)$ is a complete bipartite graph.*

Proof. Since R_1 is an integral domain, $I_1^n I_2^m \neq (0)$, for all nonzero proper ideals I_1, I_2 in R_1 such that $I_1^n \neq (0), I_2^m \neq (0)$ for all $n, m \in \mathbb{Z}^+$. We know that the ideals in R_2 are $\{(0), J, R_2\}$. Here $(0) \times J$ and $(0) \times R_2$ are adjacent to $R_1 \times (0)$, $R_1 \times J$, $I \times (0)$ and $I \times J$ where I is any nonzero proper ideal of R_1 and there is no other adjacency. Thus it forms a complete bipartite graph. ■

Theorem 2.6. *Let R be a ring. Then the following statements are equivalent.*

(1) $\mathbb{EAG}(R)$ *is a finite graph.*

(2) *R has only finitely many ideals.*

(3) *Every vertex of* $\mathbb{EAG}(R)$ *has finite degree.*

Moreover, $\mathbb{EAG}(R)$ *has $n (n \geq 1)$ vertices if and only if R has only n nonzero proper ideals.*

Proof. Since $\mathbb{AG}(R)$ is a spanning subgraph of $\mathbb{EAG}(R)$, the result follows from [8, Theorem 1.4]. ■

3. Diameter of $\mathbb{EAG}(R)$

In this Section we discuss the diameter of the extended annihilating-ideal graphs of rings. Also we determine some situations when $diam(\mathbb{EAG}(R)) = 0, 1, 2$ or 3.

Theorem 3.1. *Let R be a ring. Then* $\mathbb{EAG}(R)$ *is connected with* $diam(\mathbb{EAG}(R)) \leq 3$ *and if* $\mathbb{EAG}(R)$ *contains a cycle, then* $gr(\mathbb{EAG}(R)) \leq 4$.

Proof. Since $\mathbb{AG}(R)$ is a spanning subgraph of $\mathbb{EAG}(R)$, by [8, Theorem 2.1], the result follows. ■

Theorem 3.2. *Let R be a ring. Then* $diam(\mathbb{EAG}(R)) = 0$ *if and only if it has only one nonzero proper ideal.*

Proof. Assume that $diam(\mathbb{EAG}(R)) = 0$. Since $\mathbb{EAG}(R)$ is always connected, $A(R)^*$ has only one nonzero proper ideal. Hence R has only one nonzero proper ideal. Converse is obviously true. ■

Note that for a nilpotent ideal I of R, the nilpotency index of I is denoted by n_I. The following theorem characterizes artinian rings for which $diam(\mathbb{EAG}(R)) = 1$.

Theorem 3.3. *Let R be an artinian ring. Then* $diam(\mathbb{EAG}(R)) = 1$ *if and only if either $R \cong \mathbb{F}_1 \times \mathbb{F}_2$ or R is a local PIR with at least two nonzero proper ideals or R is local which is not a PIR with at least two nonzero proper ideals for every $I, J \in \mathbb{A}(R)^*$, $I^{n_I - 1} J^{n_J - 1} = (0)$.*

Proof. Since R is an artinian, so, by [6, Theorem 8.7], R is a finite direct product of artinian local rings. Assume that $diam(\mathbb{EAG}(R)) = 1$. Let $R \cong R_1 \times R_2 \times \cdots \times R_n$ where $R_i's$ are artinian local rings with unique maximal ideals M_i. For $n \geq 3$, $R_1 \times R_2 \times (0) \times \cdots \times (0)$ is not adjacent to $R_1 \times (0) \times \cdots \times (0)$ in $\mathbb{EAG}(R)$, a contradiction. Therefore $n \leq 2$ and consider $n = 2$ with $R \ncong \mathbb{F}_1 \times \mathbb{F}_2$. Here $R_1 \times (0)$ is not adjacent to $R_1 \times M_2$, a contradiction. Hence $R \cong \mathbb{F}_1 \times \mathbb{F}_2$. If $n = 1$, then R is a local ring. Suppose that R has less than two nonzero proper ideals, then by Theorem 3.2, a contradiction. Thus R is a local ring with at least two nonzero proper ideals. If R is a PIR, then by Theorem 2.4, the result holds. Suppose that R is not a PIR, for some $I, J \in \mathbb{A}(R)^*$, $I^{n_I - 1}J^{n_J - 1} \neq (0)$, then $diam(\mathbb{EAG}(R) \neq 1$. Thus R is not a PIR for every $I, J \in \mathbb{A}(R)^*, I^{n_I-1}J^{n_J-1} = (0)$. Conversely, assume that $R \cong \mathbb{F}_1 \times \mathbb{F}_2$, then $diam(\mathbb{EAG}(R)) = 1$. If R is a local PIR with at least two nonzero proper ideals, then by Theorem 2.4, $diam(\mathbb{EAG}(R)) = 1$. If R is local which is not a PIR with at least two nonzero proper ideals for every $I, J \in \mathbb{A}(R)^*, I^{n_I-1}J^{n_J-1} = (0)$, then the result is obviously true. ∎

Example 3.4. Let $R \cong \mathbb{Z}[i]/(\pi^n)$ where π is an irreducible gaussian integer and $n \geq 3$. Here R is a SPR with more than one nonzero proper ideals. Thus $diam(\mathbb{EAG}(R)) = 1$.

The following three theorems show that when $\mathbb{EAG}(R)$ has diameter two.

Theorem 3.5. *Let R be a PIR. Then $diam(\mathbb{EAG}(R)) = 2$ if and only if $R \cong R_1 \times R_2$ where R_1 and R_2 are either PID or SPR and any one of R_i is not a field.*

Proof. Since R is a PIR, by [15, Theorem 33], $R \cong \prod_{i=1}^{n} R_i$ where $R_i's$ are either PIDs or SPRs. Assume that $diam(\mathbb{EAG}(R)) = 2$. Consider $n \geq 3$. Then we have the following adjacency in $\mathbb{EAG}(R)$, $(0) \times R_2 \times \cdots \times R_n - R_1 \times (0) \times \cdots \times (0) - (0) \times R_2 \times (0) \times \cdots \times (0) - R_1 \times (0) \times R_3 \times \cdots \times R_n$. This shows that $diam(\mathbb{EAG}(R)) = 3$. Thus $R \cong R_1 \times R_2$. If R_1 and R_2 are fields, then by Theorem 3.3, $diam(\mathbb{EAG}(R)) = 1$. Therefore R_1 and R_2 are not fields. Thus R_1 and R_2 are either PID or SPR and not fields. Suppose that R_1 is a field, then clearly R_2 is not a field, it is either PID or SPR.

Conversely, assume that $R \cong R_1 \times R_2$, where $R_i's$ are either PID or SPR and any one of R_i is not a field, $i = 1, 2$. Consider R_1 and R_2 are SPRs and are not fields. Clearly $(0) \times R_2$ and $M_1 \times R_2$ are not adjacent in $\mathbb{EAG}(R)$ where M_1 is a nonzero proper ideal in R_1, so $diam(\mathbb{EAG}(R)) \geq 2$. By Theorem 3.1, $diam(\mathbb{EAG}(R)) = 2$ or 3. Consider R_1 has a unique nonzero proper ideal, say M_1. Let $V_1 = \{(0) \times {M_2}^j : j = 1 \text{ to } m-1\}$, $V_2 = \{M_1 \times {M_2}^j : j = 1 \text{ to } m-1\}$ and $V_3 = \{R_1 \times {M_2}^j : j = 1 \text{ to } m-1\}$ where M_2 is a nonzero proper ideal in R_2 and ${M_2}^m = (0)$. Then the induced subgraphs $\langle V_1 \rangle$ and $\langle V_2 \rangle$ are complete and $\langle V_3 \rangle$ is totally disconnected. In Figure 2.1, any one edge ends at V_i means that edge adjacent to all the vertices in V_i and also it is the spanning subgraph of $\mathbb{EAG}(R)$ and its diameter is 2. Hence $diam(\mathbb{EAG}(R)) = 2$.

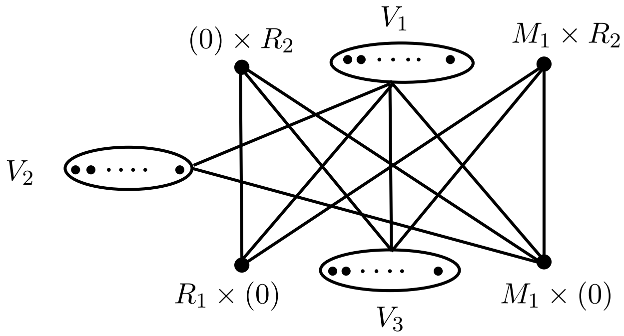

Figure 2.1

Suppose that R_1 and R_2 have more than one nonzero proper ideals. Let $V_1 = \{{M_1}^i \times (0) : i = 1 \text{ to } n-1\}, V_2 = \{{M_1}^i \times {M_2}^j : i = 1 \text{ to } n-1 \text{ and } j = 1 \text{ to } m-1\}, V_3 = \{(0) \times {M_2}^j : j = 1 \text{ to } m-1\}$, $V_4 = \{{M_1}^i \times R_2 : i = 1 \text{ to } n-1\}$ and $V_5 = \{R_1 \times {M_2}^j : j = 1 \text{ to } m-1\}$ where M_1 and M_2 are nonzero proper ideals in R_1 and R_2 respectively, ${M_1}^n = (0)$ and ${M_2}^m = (0)$. Then the induced subgraphs $\langle V_1\rangle, \langle V_2\rangle, \langle V_3\rangle$ are complete graphs and $\langle V_4\rangle$, $\langle V_5\rangle$ are totally disconnected. Figure 2.2 is the spanning subgraph of $\mathbb{EAG}(R)$ and its diameter is 2. Hence $diam(\mathbb{EAG}(R)) = 2$. From the above cases, $diam(\mathbb{EAG}(R)) = 2$.

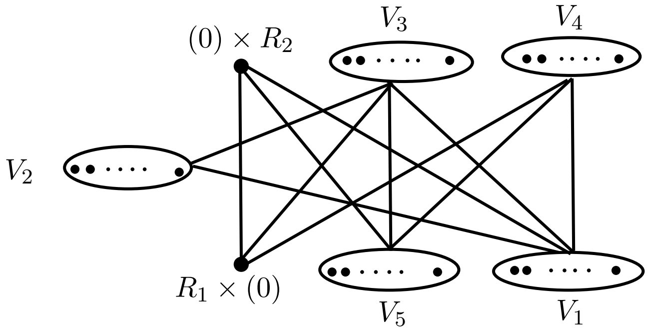

Figure 2.2

Now consider R_1 and R_2 are PIDs which are not fields. Since R_1 and R_2 are reduced and there exist nonzero prime ideals $P = R_1 \times (0)$ and $Q = (0) \times R_2$ of R which are not minimal ideals such that $P \cap Q = (0)$, by Theorem 2.3 and [3, Theorem 2.4], $\mathbb{EAG}(R)$ is a complete bipartite graph. Thus $diam(\mathbb{EAG}(R)) = 2$.

Suppose that R_1 is a PID and R_2 is a SPR and R_1, R_2 are not fields. Since R_1 is a PID, $I_1^{k_1} I_2^{k_2} \neq (0)$ for all nonzero proper ideals I_1, I_2 in R_1 such that $I_1^{k_1} \neq (0), I_2^{k_2} \neq (0)$ for all $k_1, k_2 \in \mathbb{Z}^+$. Let $V_1 = \{R_1 \times {M_2}^j : j = 1 \text{ to } m-1\}$,

$V_2 = \{I \times {M_2}^j : j = 1 \text{ to } m-1\}$, $V_3 = \{I \times (0)\}$ and $V_4 = \{(0) \times {M_2}^j : j = 1 \text{ to } m-1\}$ where I is a nonzero proper ideal in R_1 and M_2 is a nonzero proper ideal in R_2 and ${M_2}^m = (0)$. Then the induced subgraphs $\langle V_1\rangle, \langle V_2\rangle$ and $\langle V_3\rangle$ are totally disconnected and $\langle V_4\rangle$ is complete. Figure 2.3 is the extended annihilating-ideal graph of R and it shows that $diam(\mathbb{EAG}(R)) = 2$.

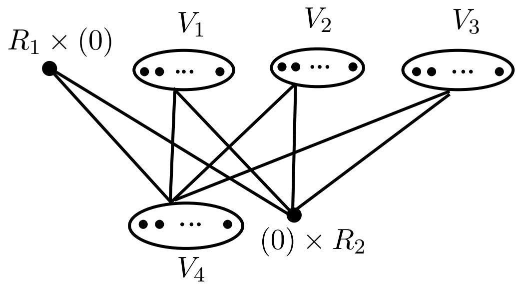

Figure 2.3

Now consider R_1 is a field and R_2 is a PID which is not a field. Since R_1 and R_2 are reduced, by Theorem 2.3 and Corollary 1.2, $\mathbb{EAG}(R)$ is a star graph. Thus $diam(\mathbb{EAG}(R)) = 2$.

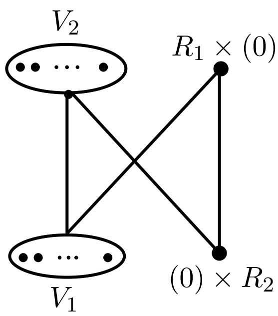

Figure 2.4

Suppose that R_1 is a field and R_2 is a SPR which is not a field. Let $V_1 = \{(0) \times {M_2}^j : j = 1 \text{ to } m-1\}$ and $V_2 = \{R_1 \times {M_2}^j : j = 1 \text{ to } m-1\}$ where M_2 is a nonzero proper ideal in R_2 and ${M_2}^m = (0)$. Then the induced subgraphs $\langle V_1\rangle$ is complete and $\langle V_2\rangle$ is totally disconnected. Figure 2.4 is the extended annihilating-ideal graph of R and it shows that $diam(\mathbb{EAG}(R)) = 2$. Hence in all the cases $diam(\mathbb{EAG}(R)) = 2$. ■

Example 3.6. If $R \cong R_1 \times R_2$ where $R_1 = \mathbb{Z}_2[X]/(X^2 + X + 1)$ and $R_2 = \mathbb{Z}_2[X]/(X^3)$, then $diam(\mathbb{EAG}(R)) = 2$.

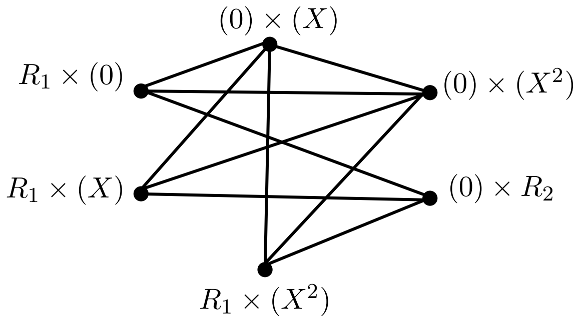

Figure 2.5 $\mathbb{EAG}(R)$

Theorem 3.7. *Let R be a reduced ring such that $Z(R)$ is not an ideal of R. Then $diam(\mathbb{EAG}(R)) = 2$ if and only if R has exactly two minimal prime ideals and at least three nonzero annihilating-ideals.*

Proof. The result is obviously true from the Theorem 2.1 and [9, Theorem 1.2]. ■

Theorem 3.8. *Let $R \cong R_1 \times R_2$ where R_1 is an integral domain and R_2 is a ring with unique nonzero proper ideal. Then $diam(\mathbb{EAG}(R)) = 2$.*

Proof. By Theorem 2.5, $diam(\mathbb{EAG}(R)) = 2$. ■

Open question 3.9. *Determine $diam(\mathbb{EAG}(R))$ for $R \cong R_1 \times R_2$ where R_1 is an integral domain and R_2 is a ring with more than one nonzero proper ideals and not a PIR.*

In the next theorem, we state necessary and sufficient conditions to have $diam(\mathbb{EAG}(R)) = 3$ for artinian PIR.

Theorem 3.10. *Let R be an artinian PIR such that $Z(R)$ is not an ideal of R. Then $diam(\mathbb{EAG}(R)) = 3$ if and only if R is a ring with more than two minimal prime ideals.*

Proof. Since R is an artinian PIR and $Z(R)$ is not an ideal of R, $R \cong \prod_{i=1}^{n} R_i$ where $R_i's$ are SPRs for all $i = 1$ to n and $n \geq 2$. Assume that $diam(\mathbb{EAG}(R)) = 3$. Since $\mathbb{AG}(R)$ is a spanning subgraph of $\mathbb{EAG}(R)$, $diam(\mathbb{AG}(R)) = 3$. Then by [9, Theorem 1.4], R is a reduced ring with more than two minimal prime ideals or R is a non-reduced ring. Consider the case R is non-reduced. For $n = 2$ and $R \cong R_1 \times R_2$, where R_1 is not a field, then as noted in the proof of Theorem 3.5, $diam(\mathbb{EAG}(R)) = 2$. Thus $n \geq 3$. Also note that in a commutative artinian ring, every maximal ideal is a minimal prime ideal. From this, R is a non-reduced ring with more than two minimal prime ideals.

Conversely, assume that R is a ring with more than two minimal prime ideals. Consider R is a reduced ring with more than two minimal prime ideals. Then by Theorem 2.1 and [9, Theorem 1.4], $diam(\mathbb{EAG}(R)) = 3$. Suppose that R is a non-reduced ring with more than two minimal prime ideals and $diam(\mathbb{EAG}(R)) \neq 3$. If $diam(\mathbb{EAG}(R)) = 1$, then by Theorem 3.3, R is a SPR with at least two nonzero proper ideals. Here $Z(R)$ is an ideal of R, a contradiction. If $diam(\mathbb{EAG}(R)) = 2$, then by Theorem 3.5, R does not have more than two minimal prime ideals. Hence $diam(\mathbb{EAG}(R)) = 3$. ∎

Open question 3.11. *Classify the diameter of* $\mathbb{EAG}(R)$ *for all artinian rings which are not PIRs.*

4. Girth of $\mathbb{EAG}(R)$

In this Section we discuss the girth of $\mathbb{EAG}(R)$ and also compare the girth of $\mathbb{AG}(R)$ with $\mathbb{EAG}(R)$. As Theorem 3.1, $gr(\mathbb{EAG}(R)) \leq 4$. Here we characterize the rings for which $gr(\mathbb{EAG}(R)) = 3, 4$ or ∞.

Theorem 4.1. *Let* $R \cong R_1 \times R_2$ *be a ring. If any one of* R_i *is a ring with more than one nonzero proper ideals for* $i = 1, 2$*, then* $gr(\mathbb{EAG}(R)) = 3$.

Proof. Suppose that R_1 is a ring with more than one nonzero proper ideals. Since $\mathbb{EAG}(R_1)$ is connected, there exist two nonzero proper ideals I_1 and I_2 in R_1 such that ${I_1}^n {I_2}^m = (0)$ with ${I_1}^n \neq (0), {I_2}^m \neq (0)$ for some positive integers n, m. Then we have the following cycle of length 3 in $\mathbb{EAG}(R)$, $I_1 \times (0) - I_2 \times (0) - (0) \times R_2 - I_1 \times (0)$. Thus $gr(\mathbb{EAG}(R)) = 3$. ∎

Theorem 4.2. *Let* $R \cong \prod_{i=1}^{n} R_i$ *where* $R_i's$ *are rings for every* i *with* $n \geq 2$*. Then the following hold.*

(i) *For* $n = 2$*,* $gr(\mathbb{EAG}(R)) = \infty$ *if and only if* $R \cong R_1 \times R_2$*, where* R_1 *is a field and* R_2 *is an integral domain.*
(ii) $gr(\mathbb{EAG}(R)) = 3$ *if and only if one of the following statements hold.*
   (a) *When* $n \geq 3$
   (b) *For* $n = 2$*, both* $R_i's$ *are not integral domains.*
   (c) *For* $n = 2$*,* R_1 *is an integral domain and* R_2 *is a ring with more than one nonzero proper ideals.*
(iii) *For* $n = 2$*,* $gr(\mathbb{EAG}(R)) = 4$ *if and only if either* R_1 *and* R_2 *are integral domains which are not fields or* R_1 *is an integral domain and* R_2 *is a ring with unique nonzero proper ideal.*

Proof. (i) For $n = 2$ and assume that $gr(\mathbb{EAG}(R)) = \infty$. Since $\mathbb{AG}(R)$ is a spanning subgraph of $\mathbb{EAG}(R)$, $\mathbb{EAG}(R) \cong \mathbb{AG}(R)$. Then by Theorem 2.3, $R_i's$ are reduced for all $i = 1, 2$. Also by [3, Theorem 3.1], $R \cong R_1 \times R_2$ where R_1 is a field and R_2 is an integral domain. Conversely, assume that $R \cong R_1 \times R_2$ where R_1 is a field and R_2 is an integral domain, then R_1 and R_2 are reduced. By Theorem 2.3 and Corollary 1.2, $\mathbb{EAG}(R)$ is a star graph. Therefore $gr(\mathbb{EAG}(R)) = \infty$.

(ii) (a) Assume that $gr(\mathbb{EAG}(R)) = 3$. Then from the following cycle $R_1 \times (0) \times \cdots \times (0) - (0) \times R_2 \times (0) \times \cdots \times (0) - (0) \times (0) \times R_3 \times (0) \times \cdots \times (0) - R_1 \times (0) \times \cdots \times (0)$, $n \geq 3$ is true. Conversely, when $n \geq 3$, the result is obviously true.

(b) Now consider $n = 2$ and assume that $gr(\mathbb{EAG}(R)) = 3$. Suppose that R_1 and R_2 are integral domains. Since R_1 and R_2 are reduced, by Theorem 2.3, Corollary 1.2 and [3, Corollary 2.5], $gr(\mathbb{EAG}(R)) = 4$ or ∞, a contradiction. From this case $R_i's$ are not integral domains. Conversely, assume that R_1 and R_2 are not integral domains. Consider R_1 and R_2 are rings with unique nonzero proper ideals I and J in R_1 and R_2 respectively such that $I^2 = (0)$ and $J^2 = (0)$. Then in $\mathbb{EAG}(R)$, $I \times (0) - I \times J - (0) \times J - I \times (0)$ is a cycle of length 3 so that $gr(\mathbb{EAG}(R)) = 3$. Also consider R_1 is a ring with more than one nonzero proper ideals. Then by Theorem 4.1, $gr(\mathbb{EAG}(R)) = 3$. From these cases, $gr(\mathbb{EAG}(R)) = 3$.

(c) For $n = 2$, assume that $gr(\mathbb{EAG}(R)) = 3$. If R_1 is an integral domain and R_2 is a ring with unique nonzero proper ideal, then by Theorem 2.5, $gr(\mathbb{EAG}(R)) = 4$, a contradiction. Thus R_1 is an integral domain and R_2 is a ring with more than one nonzero proper ideals. Converse part follows from Theorem 4.1.

(iii) Proof follows from (i) and (ii). ■

We next characterize when $gr(\mathbb{EAG}(\mathbb{Z}_n))$ is $3, 4$ or ∞.

Theorem 4.3. *For $n \in \mathbb{N}$, let $n = \prod_{i=1}^{k} p_i^{\alpha_i}$ be the distinct prime factorization of n. Then the following assertions are true.*

(i) $gr(\mathbb{EAG}(\mathbb{Z}_n)) = 3$ *if and only if one of the following assertions must occur.*
  (a) *When $k \geq 3$.*
  (b) *When $k = 1$ and $\alpha_1 \geq 4$.*
  (c) *When $k = 2$, either $\alpha_1 = 1$, $\alpha_2 > 2$ or $\alpha_1, \alpha_2 \geq 2$.*

(ii) $gr(\mathbb{EAG}(\mathbb{Z}_n)) = 4$ *if and only if $k = 2$ with $\alpha_1 = 1, \alpha_2 = 2$.*

(iii) $gr(\mathbb{EAG}(\mathbb{Z}_n)) = \infty$ *if and only if either $k = 1$ with $\alpha_1 = 2$ or 3 or $k = 2$ with $\alpha_1 = 1, \alpha_2 = 1$.*

Proof. When $k \geq 2$, the result holds by Theorem 4.2. It remains to consider the case $k = 1$. Assume that $gr(\mathbb{EAG}(\mathbb{Z}_n)) = 3$ and $\alpha_1 < 4$. Then $|\mathbb{A}(R)^*| = 0, 1$ or 2

and hence $gr(\mathbb{EAG}(\mathbb{Z}_n)) = \infty$. This shows that $\alpha_1 \geq 4$. Conversely, assume that $\alpha_1 \geq 4$, then $\mathbb{Z}_n$ is a local ring with maximal ideal (p_1) and $|\mathbb{A}(R)^*| = \alpha_1 - 1$. Since $\mathbb{Z}_n$ is a SPR, by Theorem 2.4, $gr(\mathbb{EAG}(\mathbb{Z}_n)) = 3$. By above, (iii) holds for the case $k = 1$. ∎

We conclude this paper with the following theorem to have a better comparison of the girth between $\mathbb{EAG}(R)$ and $\mathbb{AG}(R)$.

Theorem 4.4. *Let R be a ring. Then the following hold.*

(i) *If $gr(\mathbb{AG}(R)) = 3$, then $gr(\mathbb{EAG}(R)) = 3$.*

(ii) *If $gr(\mathbb{AG}(R)) = 4$, then $gr(\mathbb{EAG}(R)) = 4$.*

(iii) *If $gr(\mathbb{AG}(R)) = \infty$, then $gr(\mathbb{EAG}(R)) = 3, 4$ or ∞.*

(iv) *If $gr(\mathbb{EAG}(R)) = 3$, then $gr(\mathbb{AG}(R)) = 3$ or ∞.*

(v) *If $gr(\mathbb{EAG}(R)) = 4$, then $gr(\mathbb{AG}(R)) = 4$ or ∞.*

(vi) *If $gr(\mathbb{EAG}(R)) = \infty$, then $gr(\mathbb{AG}(R)) = \infty$.*

Proof. Since $\mathbb{AG}(R)$ is a spanning subgraph of $\mathbb{EAG}(R)$, (i), (iii), (v), (vi) are obviously true.

(ii) Assume that $gr(\mathbb{AG}(R)) = 4$. This shows that $\mathbb{AG}(R)$ is a triangle-free graph. Then by [2, Lemma 1], $R \cong R_1 \times R_2$ where either R_1 and R_2 are integral domains which are not fields or R_1 is an integral domain which is not a field and R_2 is a ring with unique nonzero proper ideal. By Theorem 4.2(iii), $gr(\mathbb{EAG}(R)) = 4$.

(iv) Assume that $gr(\mathbb{EAG}(R)) = 3$. Since $\mathbb{AG}(R)$ is a spanning subgraph of $\mathbb{EAG}(R)$, $gr(\mathbb{AG}(R)) = 3, 4$ or ∞. By part (ii), $gr(\mathbb{AG}(R)) \neq 4$ and so $gr(\mathbb{AG}(R)) = 3$ or ∞. ∎

References

[1] G. Aalipour, S. Akbari, M. Behboodi, R. Nikandish, M.J. Nikmehr and F. Shaveisi, *The classification of the annihilating-ideal graph of commutative rings*, Algebra Colloq. **21** (2014) 249–256.
https://doi.org/10.1142/S1005386714000200

[2] G. Aalipour, S. Akbari, R. Nikandish, M.J. Nikmehr and F. shaveisi, *Minimal prime ideals and cycles in annihilating-ideal graphs*, Rocky Mountain J. Math. **43** (2013) 1415–1425.
https://doi.org/10.1216/RMJ-2013-43-5-1415

[3] M. Ahrari, Sh. A. Safari Sabet and B. Amini, *On the girth of the annihilating-ideal graph of a commutative ring*, J. Linear and Topological Algebra **4** (2015) 209–216.

[4] G. Alan Cannon, K. M. Neuerburg and S.P. Redmond, *Zero-divisor graphs of near-rings and semigroups*, Near-rings and Near-fields: Proc. Conf. Near-rings and Near-fields II, (2005) 189–200.
https://doi.org/10.1007/1-4020-3391-5_8

[5] D.F. Anderson and P.S. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra **217** (1999) 434–447.
https://doi.org/10.1006/jabr.1998.7840

[6] M.F. Atiyah and I.G. MacDonald, Introduction to Commutative Algebra (Addison-Wesley Publishing Company, London, 1969).

[7] I. Beck, *Coloring of commutative rings*, J. Algebra **116** (1988) 208–226.
https://doi.org/10.1016/0021-8693(88)90202-5

[8] M. Behboodi and Z. Rakeei, *The annihilating-ideal graph of commutative rings* I, J. Algebra Appl. **10** (2011) 727–739.
https://doi.org/10.1142/S0219498811004896

[9] M. Behboodi and Z. Rakeei, *The annihilating-ideal graph of commutative rings* II, J. Algebra Appl. **10** (2011) 741–753.
https://doi.org/10.1142/S0219498811004902

[10] D. Bennis, J. Mikram and F. Taraza, *On the extended zero-divisor graph of a commutative rings*, Turk. J. Math. **40** (2016) 376–388.
https://doi.org/10.3906/mat-1504-61

[11] G. Chartrand and Ping Zhang, Introduction to Graph Theory (Tata McGraw Hill, India, 2006).

[12] K.R. McLean, *Commutative Artinian principal ideal rings*, Proc. London Math. Soc. **26** (1973) 249–272.
https://doi.org/10.1112/plms/s3-26.2.249

[13] T. Tamizh Chelvam and S. Nithya, *Zero-divisor graph of an ideal of a near-ring*, Discrete Math. Alg. Appl. **5** (2013) 1350007 (1–11).
https://doi.org/10.1142/S1793830913500079

[14] T. Tamizh Chelvam and S. Nithya, *Domination in the zero-divisor graph of an ideal of a near-ring*, Taiwanese J. Math. **17** (2013) 1613–1625.
https://doi.org/10.11650/tjm.17.2013.2739

[15] O. Zariski and P. Samuel, Commutative Algebra, I (Van Nostrand, Princeton, 1958).

Received 27 June 2020
Revised 4 February 2021
Accepted 22 March 2022