# EXTENDED ANNIHILATING-IDEAL GRAPH OF A COMMUTATIVE RING 

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#### Abstract

Let $R$ be a commutative ring with identity. An ideal $I$ of a ring $R$ is called an annihilating-ideal if there exists a nonzero ideal $J$ of $R$ such that $I J=(0)$ and we use the notation $\mathbb{A}(R)$ for the set of all annihilating-ideals of $R$. In this paper, we introduce the extended annihilating-ideal graph of $R$, denoted by $\mathbb{E A} \mathbb{G}(R)$. It is the simple graph with vertices $\mathbb{A}(R)^{*}=\mathbb{A}(R) \backslash\{(0)\}$, and two distinct vertices $I$ and $J$ are adjacent whenever there exist two positive integers $n$ and $m$ such that $I^{n} J^{m}=(0)$ with $I^{n} \neq(0)$ and $J^{m} \neq(0)$. Here we discuss in detail the diameter and girth of $\mathbb{E} \mathbb{G}(R)$ and investigate the coincidence of $\mathbb{E} \mathbb{A}(R)$ with the annihilating-ideal graph $\mathbb{A} \mathbb{G}(R)$. Moreover we propose open questions in this paper.


Keywords: annihilating-ideal graph, extended annihilating-ideal graph.
2010 Mathematics Subject Classification: Primary: 05C75, 05C25, Secondary: 13A15, 13M05.

## 1. InTroduction

The concept of zero-divisor graph of a commutative ring $R$ was first introduced by Beck in [7]. He let all elements of the ring be vertices of the graph and was
interested mainly in coloring. In [5], Anderson et al. associated a zero-divisor graph $\Gamma(R)$ to $R$ with vertices $Z(R)^{*}=Z(R) \backslash\{0\}$, the set of all nonzero zerodivisors and two distinct vertices $x$ and $y$ are adjacent if and only if $x y=0$. The zero-divisor graphs of commutative rings attracted the attention of several researchers and also this graph was assigned to other algebraic structures (see for instance $[4,13,14])$. In [10], Bennis et al. introduced and studied the extended zero-divisor graph of $R$ which is the extension of the classical zero-divisor graph of $R$ and it is denoted by $\bar{\Gamma}(R)$ whose vertex set consists of all its nonzero zerodivisors and that two distinct vertices $x$ and $y$ are adjacent whenever there exist two non-negative integers $n$ and $m$ such that $x^{n} y^{m}=0$ with $x^{n} \neq 0$ and $y^{m} \neq 0$. In [8], Behboodi et al. introduced and investigated the annihilating-ideal graph of $R$, denoted by $\mathbb{A} \mathbb{G}(R)$. An ideal $I$ of a ring $R$ is called an annihilating-ideal if there exists a nonzero ideal $J$ of $R$ such that $I J=(0)$ and we use the notation $\mathbb{A}(R)$ for the set of all annihilating-ideals of $R$. It is the simple graph with vertices $\mathbb{A}(R)^{*}=\mathbb{A}(R) \backslash\{(0)\}$, the set of all nonzero annihilating-ideals of $R$ and two distinct vertices $I$ and $J$ are adjacent if and only if $I J=(0)$. They obtained some finiteness conditions of $\mathbb{A} \mathbb{G}(R)$ and found out the facts of the connectivity of annihilating-ideal graphs. In [9], they discussed the diameter and coloring of annihilating-ideal graphs.

Throughout this paper $R$ denotes a commutative ring with identity $1 \neq 0$. In this paper we introduce an extension of the annihilating-ideal graph of a commutative ring $R$, denoted by $\mathbb{E} \mathbb{A}(R)$, which we call the extended annihilating-ideal graph of $R$, such that its vertex set is $\mathbb{A}(R)^{*}$ which is the set of all nonzero annihilating-ideals of $R$ and that two distinct vertices $I$ and $J$ are adjacent if and only if there exist two positive integers $n$ and $m$ such that $I^{n} J^{m}=(0)$ with $I^{n} \neq(0)$ and $J^{m} \neq(0)$. Clearly, the annihilating-ideal graph $\mathbb{A} \mathbb{G}(R)$ is a spanning subgraph of $\mathbb{E} \mathbb{A} \mathbb{G}(R)$. Note that $\mathbb{E} \mathbb{A} \mathbb{G}(R)$ is the empty graph if and only if $R$ is an integral domain.

The main goal of this paper is to establish the relation between $\mathbb{E} \mathbb{A} \mathbb{G}(R)$ and $\mathbb{A} \mathbb{G}(R)$ and the connection between the graph theoretic properties of $\mathbb{E} \mathbb{A} \mathbb{G}(R)$ and the ring theoretic properties of $R$. In Section 2, we discuss the basic properties of $\mathbb{E} \mathbb{A} \mathbb{G}(R)$ and the coincidence of $\mathbb{E} \mathbb{A} \mathbb{G}(R)$ and $\mathbb{A} \mathbb{G}(R)$. Also we determine when $\mathbb{E} \mathbb{A}(R)$ forms a complete graph or a complete bipartite graph. In Sections 3 and 4 , we obtain the diameter and girth of $\mathbb{E} \mathbb{A} \mathbb{G}(R)$ and that compare with $\mathbb{A} \mathbb{G}(R)$. In this paper we propose open questions with regard to the diameter of $\mathbb{E} \mathbb{A} \mathbb{G}(R)$. As usual, $\mathbb{Z}, \mathbb{Q}, \mathbb{Z}_{n}, \mathbb{F}$ denote the ring of integers, rational numbers, ring of integers modulo $n$ and the field, respectively. For basic definitions on rings, one may refer [6].

For the sake of completeness, we state some definitions and notations used throughout. Let $G$ be a graph and $V(G), E(G)$ be the vertex set and edge set of $G$ respectively. A graph $H$ is called a subgraph of $G$ that is $H \subseteq G$ if $V(H) \subseteq V(G)$
and $E(H) \subseteq E(G)$. A subgraph $H$ of $G$ with $V(H)=V(G)$ is called a spanning subgraph of $G$. For $S \subseteq V(G)$, the induced subgraph $H$ induced by $S$ is the subgraph of $G$ with vertex set $S$ and two vertices are adjacent in $H$ if and only if they are adjacent in $G$ and it is denoted by $\langle S\rangle$. A closed path is called a cycle. A cycle on $n \geq 3$ vertices is denoted by $C_{n}$. We say that $G$ is connected if there is a path between any two distinct vertices of $G$. For vertices $x$ and $y$ of $G$, let $d(x, y)$ be the length of the shortest path from $x$ to $y(d(x, x)=0, d(x, y)=\infty$ if there is no such path). The diameter of $G$ is $\operatorname{diam}(G)=\sup \{d(x, y): x, y \in V(G)\}$. The girth of $G$ denoted by $\operatorname{gr}(G)$, is the length of a shortest cycle in $G(\operatorname{gr}(G)=\infty$ if $G$ contains no cycles). A graph in which each pair of distinct vertices is joined by an edge is called a complete graph. We denote the complete graph on n vertices by $K_{n}$. A bipartite graph is a graph all of whose vertices can be partitioned into two parts $V_{1}$ and $V_{2}$ such that every edge joins a vertex in $V_{1}$ to one in $V_{2}$. A complete bipartite graph is a bipartite graph in which every vertex of one part is joined to every vertex of the other part. The complete bipartite graph on $m$ and $n$ vertices is denoted by $K_{m, n}$ and $K_{1, n}$ a star graph. For undefined terms in graph theory we refer [11]. The following results are useful in the subsequent sections.

Theorem 1.1 [8, Theorem 1.1]. Let $R$ be a non-domain ring. Then $\mathbb{A} \mathbb{G}(R)$ has $A C C$ (respectively, $D C C$ ) on vertices if and only if $R$ is a Noetherian (respectively, an Artinian) ring.

Corollary 1.2 [8, Corollary 2.3]. Let $R$ be a reduced ring. Then the following statements are equivalent.
(i) There is a vertex of $\mathbb{A} \mathbb{G}(R)$ which is adjacent to every other vertices.
(ii) $\mathbb{A} \mathbb{G}(R)$ is a star graph.
(iii) $R \cong \mathbb{F} \times D$, where $\mathbb{F}$ is a field and $D$ is an integral domain.

## 2. BASIC PROPERTIES OF $\mathbb{E} \mathbb{A} \mathbb{G}(R)$

In this Section we discuss the basic properties of $\mathbb{E} \mathbb{A}(R)$ and when $\mathbb{E} \mathbb{A} \mathbb{G}(R)$ and $\mathbb{A} \mathbb{G}(R)$ coincide. A ring $R$ is called a reduced ring if it has no nonzero nilpotent elements. The following theorem plays an important role in this paper.

Theorem 2.1. Let $R$ be a reduced ring. Then $\mathbb{E} \mathbb{A} \mathbb{G}(R)=\mathbb{A} \mathbb{G}(R)$.
Proof. Since $R$ is reduced, it has no nonzero nilpotent ideals. Then for every nonzero ideal $I$ in $R, I^{n} \neq(0)$ for all positive integers $n$. By definitions of $\mathbb{E} \mathbb{A} \mathbb{G}(R)$ and $\mathbb{A} \mathbb{G}(R), \mathbb{A} \mathbb{G}(R)$ is a spanning subgraph of $\mathbb{E} \mathbb{A} \mathbb{G}(R)$. Suppose there exist two nonzero proper ideals $I, J$ in $R$ such that $I^{n} J^{m}=(0)$ with $I^{n} \neq(0)$
and $J^{m} \neq(0)$ for some positive integers $n$ and $m$ but $I J \neq(0)$. Now $I^{n} J^{m}=(0)$ implies $I^{m+n} J^{m+n}=(0)$ that is $(I J)^{m+n}=(0)$. Hence $I J$ is a nonzero nilpotent ideal of $R$ which is a contradiction. Thus $\mathbb{E} \mathbb{A} \mathbb{G}(R)=\mathbb{A} \mathbb{G}(R)$.

Remark 2.2. In Theorem 2.1, the converse is not true in general. For example, consider the ring $R \cong \mathbb{Z}_{p}[X] /\left(X^{2}\right)$ where $p$ is a prime number. Here $\mathbb{A}(R)^{*}$ has only one nonzero proper nilpotent ideal, $(X)$. Thus $\mathbb{E} \mathbb{A} \mathbb{G}(R)=\mathbb{A} \mathbb{G}(R) \cong K_{1}$ but $R$ is not reduced.

Theorem 2.3. Let $R \cong \prod_{i=1}^{n} R_{i}$ where $R_{i}^{\prime}$ s are rings for every $i$ with $n \geq 2$. Then $\mathbb{E} \mathbb{A}(R)=\mathbb{A} \mathbb{G}(R)$ if and only if $R_{i}$ is reduced for every $i$.

Proof. Assume that $\mathbb{E} \mathbb{A}(R)=\mathbb{A} \mathbb{G}(R)$. Suppose that $R_{i}$ is not reduced for some $i$. Then there exist a nonzero ideal $I$ in $R_{i}$ such that $I^{n}=(0)$ for some positive integer $n$. We have the following non-adjacency in $\mathbb{A} \mathbb{G}(R),[(0) \times(0) \times$ $\left.\cdots \times R_{i} \times(0) \times \cdots \times(0)\right]\left[R_{1} \times R_{2} \times \cdots \times I \times R_{i+1} \times \cdots \times R_{n}\right]=(0) \times(0) \times$ $\cdots \times I \times(0) \times \cdots \times(0) \neq(0) \times(0) \times \cdots \times(0)$ and the adjacency in $\mathbb{E} \mathbb{A} \mathbb{G}(R)$, $\left[(0) \times(0) \times \cdots \times R_{i} \times(0) \times \cdots \times(0)\right]\left[R_{1} \times R_{2} \times \cdots \times I \times R_{i+1} \times \cdots \times R_{n}\right]^{n}=[(0) \times(0) \times$ $\left.\cdots \times R_{i} \times(0) \times \cdots \times(0)\right]\left[R_{1} \times R_{2} \times \cdots \times(0) \times R_{i+1} \times \cdots \times R_{n}\right]=(0) \times(0) \times \cdots \times(0)$. Therefore a contradiction arises to $\mathbb{E} \mathbb{A}(R)=\mathbb{A} \mathbb{G}(R)$. Conversely, assume that $R_{i}^{\prime} s$ are reduced for every $i=1$ to $n$. Since product of reduced ring is reduced and by Theorem 2.1, $\mathbb{E} \mathbb{A} \mathbb{G}(R)=\mathbb{A} \mathbb{G}(R)$.

Recall that an ideal $I$ of $R$ is called a principal ideal if $I=(a)=\{r a: r \in R\}$ for some $a \in R$. If every ideal is a principal ideal in $R$, then $R$ is called a principal ideal ring (PIR). An integral domain in which every ideal is principal is called a principal ideal domain (PID). A local artinian PIR is called a special principal ring (SPR) and has an extremely simple ideal structure: there are only finitely many ideals, each of which is a power of the maximal ideal.

In the next two theorems, we determine the situations when $\mathbb{E} \mathbb{A}(R)$ forms a complete graph and a complete bipartite graph.

Theorem 2.4. Let $R$ be a $S P R$. Then $\mathbb{E} \mathbb{A}(R)$ is a complete graph.
Proof. Since $R$ is a SPR, the only ideals of $R$ are $R, M, M^{2}, \ldots$ and $M^{n}=(0)$. Also all the nonzero proper ideals of $R$ are in $\mathbb{A}(R)^{*}$. Let $M^{i}, M^{j} \in \mathbb{A}(R)^{*}$. If $i+j \geq n$, then $M^{i}$ and $M^{j}$ are adjacent in $\mathbb{E} \mathbb{G}(R)$. If $i+j<n$ and $i<j$, then there exist $k>1$ such that $i k<n$ and $i k+j \geq n$. Therefore $M^{i}$ and $M^{j}$ are adjacent in $\mathbb{E} \mathbb{A} \mathbb{G}(R)$. Hence $\mathbb{E} \mathbb{A} \mathbb{G}(R)$ is complete.

Theorem 2.5. Let $R \cong R_{1} \times R_{2}$ where $R_{1}$ is an integral domain and $R_{2}$ is a ring with unique nonzero proper ideal. Then $\mathbb{E} \mathbb{A}(R)$ is a complete bipartite graph.

Proof. Since $R_{1}$ is an integral domain, $I_{1}^{n} I_{2}^{m} \neq(0)$, for all nonzero proper ideals $I_{1}, I_{2}$ in $R_{1}$ such that $I_{1}^{n} \neq(0), I_{2}^{m} \neq(0)$ for all $n, m \in \mathbb{Z}^{+}$. We know that the ideals in $R_{2}$ are $\left\{(0), J, R_{2}\right\}$. Here $(0) \times J$ and $(0) \times R_{2}$ are adjacent to $R_{1} \times(0)$, $R_{1} \times J, I \times(0)$ and $I \times J$ where $I$ is any nonzero proper ideal of $R_{1}$ and there is no other adjacency. Thus it forms a complete bipartite graph.

Theorem 2.6. Let $R$ be a ring. Then the following statements are equivalent.
(1) $\mathbb{E} \mathbb{A}(R)$ is a finite graph.
(2) $R$ has only finitely many ideals.
(3) Every vertex of $\mathbb{E A} \mathbb{G}(R)$ has finite degree.

Moreover, $\mathbb{E} \mathbb{A}(R)$ has $n(n \geq 1)$ vertices if and only if $R$ has only $n$ nonzero proper ideals.

Proof. Since $\mathbb{A} \mathbb{G}(R)$ is a spanning subgraph of $\mathbb{E A} \mathbb{G}(R)$, the result follows from [8, Theorem 1.4].

## 3. Diameter of $\mathbb{E} \mathbb{A}(R)$

In this Section we discuss the diameter of the extended annihilating-ideal graphs of rings. Also we determine some situations when $\operatorname{diam}(\mathbb{E} \mathbb{A}(R))=0,1,2$ or 3 .

Theorem 3.1. Let $R$ be a ring. Then $\mathbb{E A}(R)$ is connected with diam $(\mathbb{E A} \mathbb{G}(R))$ $\leq 3$ and if $\mathbb{E} \mathbb{G}(R)$ contains a cycle, then $\operatorname{gr}(\mathbb{E} \mathbb{A}(R)) \leq 4$.

Proof. Since $\mathbb{A} \mathbb{G}(R)$ is a spanning subgraph of $\mathbb{E} \mathbb{A}(R)$, by [8, Theorem 2.1], the result follows.

Theorem 3.2. Let $R$ be a ring. Then $\operatorname{diam}(\mathbb{E} \mathbb{A}(R))=0$ if and only if it has only one nonzero proper ideal.

Proof. Assume that $\operatorname{diam}(\mathbb{E} \mathbb{A}(R))=0$. Since $\mathbb{E} \mathbb{A}(R)$ is always connected, $A(R)^{*}$ has only one nonzero proper ideal. Hence $R$ has only one nonzero proper ideal. Converse is obviously true.

Note that for a nilpotent ideal I of $R$, the nilpotency index of $I$ is denoted by $n_{I}$. The following theorem characterizes artinian rings for which $\operatorname{diam}(\mathbb{E A} \mathbb{G}(R))$ $=1$.

Theorem 3.3. Let $R$ be an artinian ring. Then $\operatorname{diam}(\mathbb{E} \mathbb{A}(R))=1$ if and only if either $R \cong \mathbb{F}_{1} \times \mathbb{F}_{2}$ or $R$ is a local PIR with at least two nonzero proper ideals or $R$ is local which is not a PIR with at least two nonzero proper ideals for every $I, J \in \mathbb{A}(R)^{*}, I^{n_{I}-1} J^{n_{J}-1}=(0)$.

Proof. Since $R$ is an artinian, so, by [6, Theorem 8.7], $R$ is a finite direct product of artinian local rings. Assume that $\operatorname{diam}(\mathbb{E A G}(R))=1$. Let $R \cong R_{1} \times R_{2} \times$ $\cdots \times R_{n}$ where $R_{i}^{\prime} s$ are artinian local rings with unique maximal ideals $M_{i}$. For $n \geq 3, R_{1} \times R_{2} \times(0) \times \cdots \times(0)$ is not adjacent to $R_{1} \times(0) \times \cdots \times(0)$ in $\mathbb{E} \mathbb{A} \mathbb{G}(R)$, a contradiction. Therefore $n \leq 2$ and consider $n=2$ with $R \not \not \mathbb{F}_{1} \times \mathbb{F}_{2}$. Here $R_{1} \times$ ( 0 ) is not adjacent to $R_{1} \times M_{2}$, a contradiction. Hence $R \cong \mathbb{F}_{1} \times \mathbb{F}_{2}$. If $n=1$, then $R$ is a local ring. Suppose that $R$ has less than two nonzero proper ideals, then by Theorem 3.2, a contradiction. Thus $R$ is a local ring with at least two nonzero proper ideals. If $R$ is a PIR, then by Theorem 2.4, the result holds. Suppose that $R$ is not a PIR, for some $I, J \in \mathbb{A}(R)^{*}, I^{n_{I}-1} J^{n_{J}-1} \neq(0)$, then $\operatorname{diam}(\mathbb{E} \mathbb{A}(R) \neq$ 1. Thus $R$ is not a PIR for every $I, J \in \mathbb{A}(R)^{*}, I^{n_{I}-1} J^{n_{J}-1}=(0)$. Conversely, assume that $R \cong \mathbb{F}_{1} \times \mathbb{F}_{2}$, then $\operatorname{diam}(\mathbb{E} \mathbb{A}(R))=1$. If $R$ is a local PIR with at least two nonzero proper ideals, then by Theorem 2.4, $\operatorname{diam}(\mathbb{E} \mathbb{A} \mathbb{G}(R))=1$. If $R$ is local which is not a PIR with at least two nonzero proper ideals for every $I, J \in \mathbb{A}(R)^{*}, I^{n_{I}-1} J^{n_{J}-1}=(0)$, then the result is obviously true.

Example 3.4. Let $R \cong \mathbb{Z}[i] /\left(\pi^{n}\right)$ where $\pi$ is an irreducible gaussian integer and $n \geq 3$. Here $R$ is a SPR with more than one nonzero proper ideals. Thus $\operatorname{diam}(\mathbb{E A G}(R))=1$.

The following three theorems show that when $\mathbb{E} \mathbb{A} \mathbb{G}(R)$ has diameter two.
Theorem 3.5. Let $R$ be a PIR. Then $\operatorname{diam}(\mathbb{E} \mathbb{G}(R))=2$ if and only if $R \cong$ $R_{1} \times R_{2}$ where $R_{1}$ and $R_{2}$ are either PID or $S P R$ and any one of $R_{i}$ is not a field.

Proof. Since $R$ is a PIR, by [15, Theorem 33], $R \cong \prod_{i=1}^{n} R_{i}$ where $R_{i}^{\prime} s$ are either PIDs or SPRs. Assume that $\operatorname{diam}(\mathbb{E} \mathbb{G}(R))=2$. Consider $n \geq 3$. Then we have the following adjacency in $\mathbb{E} \mathbb{A}(R),(0) \times R_{2} \times \cdots \times R_{n}-R_{1} \times(0) \times$ $\cdots \times(0)-(0) \times R_{2} \times(0) \times \cdots \times(0)-R_{1} \times(0) \times R_{3} \times \cdots \times R_{n}$. This shows that $\operatorname{diam}(\mathbb{E A G}(R))=3$. Thus $R \cong R_{1} \times R_{2}$. If $R_{1}$ and $R_{2}$ are fields, then by Theorem 3.3, $\operatorname{diam}(\mathbb{E A G}(R))=1$. Therefore $R_{1}$ and $R_{2}$ are not fields. Thus $R_{1}$ and $R_{2}$ are either PID or SPR and not fields. Suppose that $R_{1}$ is a field, then clearly $R_{2}$ is not a field, it is either PID or SPR.

Conversely, assume that $R \cong R_{1} \times R_{2}$, where $R_{i}^{\prime} s$ are either PID or SPR and any one of $R_{i}$ is not a field, $i=1,2$. Consider $R_{1}$ and $R_{2}$ are SPRs and are not fields. Clearly $(0) \times R_{2}$ and $M_{1} \times R_{2}$ are not adjacent in $\mathbb{E} \mathbb{A} \mathbb{G}(R)$ where $M_{1}$ is a nonzero proper ideal in $R_{1}$, so $\operatorname{diam}(\mathbb{E} \mathbb{A}(R)) \geq 2$. By Theorem 3.1, $\operatorname{diam}(\mathbb{E} \mathbb{A}(R))=2$ or 3 . Consider $R_{1}$ has a unique nonzero proper ideal, say $M_{1}$. Let $V_{1}=\left\{(0) \times M_{2}{ }^{j}: j=1\right.$ to $\left.m-1\right\}, V_{2}=\left\{M_{1} \times M_{2}{ }^{j}: j=1\right.$ to $\left.m-1\right\}$ and $V_{3}=\left\{R_{1} \times M_{2}{ }^{j}: j=1\right.$ to $\left.m-1\right\}$ where $M_{2}$ is a nonzero proper ideal in $R_{2}$ and $M_{2}{ }^{m}=(0)$. Then the induced subgraphs $\left\langle V_{1}\right\rangle$ and $\left\langle V_{2}\right\rangle$ are complete and $\left\langle V_{3}\right\rangle$ is totally disconnected. In Figure 2.1, any one edge ends at $V_{i}$ means that edge
adjacent to all the vertices in $V_{i}$ and also it is the spanning subgraph of $\mathbb{E} \mathbb{A} \mathbb{G}(R)$ and its diameter is 2 . Hence $\operatorname{diam}(\mathbb{E} \mathbb{A} \mathbb{G}(R))=2$.


Figure 2.1
Suppose that $R_{1}$ and $R_{2}$ have more than one nonzero proper ideals. Let $V_{1}=\left\{M_{1}{ }^{i} \times(0): i=1\right.$ to $\left.n-1\right\}, V_{2}=\left\{M_{1}{ }^{i} \times M_{2}{ }^{j}: i=1\right.$ to $n-1$ and $j=1$ to $m-1\}, V_{3}=\left\{(0) \times M_{2}{ }^{j}: j=1\right.$ to $\left.m-1\right\}, V_{4}=\left\{M_{1}{ }^{i} \times R_{2}: i=1\right.$ to $n-1\}$ and $V_{5}=\left\{R_{1} \times M_{2}{ }^{j}: j=1\right.$ to $\left.m-1\right\}$ where $M_{1}$ and $M_{2}$ are nonzero proper ideals in $R_{1}$ and $R_{2}$ respectively, $M_{1}{ }^{n}=(0)$ and $M_{2}{ }^{m}=(0)$. Then the induced subgraphs $\left\langle V_{1}\right\rangle,\left\langle V_{2}\right\rangle,\left\langle V_{3}\right\rangle$ are complete graphs and $\left\langle V_{4}\right\rangle,\left\langle V_{5}\right\rangle$ are totally disconnected. Figure 2.2 is the spanning subgraph of $\mathbb{E A} \mathbb{G}(R)$ and its diameter is 2. Hence $\operatorname{diam}(\mathbb{E} \mathbb{G}(R))=2$. From the above cases, $\operatorname{diam}(\mathbb{E} \mathbb{A}(R))=2$.


Figure 2.2
Now consider $R_{1}$ and $R_{2}$ are PIDs which are not fields. Since $R_{1}$ and $R_{2}$ are reduced and there exist nonzero prime ideals $P=R_{1} \times(0)$ and $Q=(0) \times R_{2}$ of $R$ which are not minimal ideals such that $P \cap Q=(0)$, by Theorem 2.3 and [3, Theorem 2.4], $\mathbb{E} \mathbb{A}(R)$ is a complete bipartite graph. Thus $\operatorname{diam}(\mathbb{E} \mathbb{A}(R))=2$.

Suppose that $R_{1}$ is a PID and $R_{2}$ is a SPR and $R_{1}, R_{2}$ are not fields. Since $R_{1}$ is a PID, $I_{1}^{k_{1}} I_{2}^{k_{2}} \neq(0)$ for all nonzero proper ideals $I_{1}, I_{2}$ in $R_{1}$ such that $I_{1}^{k_{1}} \neq(0), I_{2}^{k_{2}} \neq(0)$ for all $k_{1}, k_{2} \in \mathbb{Z}^{+}$. Let $V_{1}=\left\{R_{1} \times M_{2}{ }^{j}: j=1\right.$ to $\left.m-1\right\}$,
$V_{2}=\left\{I \times M_{2}{ }^{j}: j=1\right.$ to $\left.m-1\right\}, V_{3}=\{I \times(0)\}$ and $V_{4}=\left\{(0) \times M_{2}{ }^{j}: j=1\right.$ to $m-1\}$ where $I$ is a nonzero proper ideal in $R_{1}$ and $M_{2}$ is a nonzero proper ideal in $R_{2}$ and $M_{2}{ }^{m}=(0)$. Then the induced subgraphs $\left\langle V_{1}\right\rangle,\left\langle V_{2}\right\rangle$ and $\left\langle V_{3}\right\rangle$ are totally disconnected and $\left\langle V_{4}\right\rangle$ is complete. Figure 2.3 is the extended annihilating-ideal graph of $R$ and it shows that $\operatorname{diam}(\mathbb{E} \mathbb{A}(R))=2$.


Figure 2.3
Now consider $R_{1}$ is a field and $R_{2}$ is a PID which is not a field. Since $R_{1}$ and $R_{2}$ are reduced, by Theorem 2.3 and Corollary 1.2, $\mathbb{E} \mathbb{G}(R)$ is a star graph. Thus $\operatorname{diam}(\mathbb{E} \mathbb{A}(R))=2$.


Figure 2.4

Suppose that $R_{1}$ is a field and $R_{2}$ is a SPR which is not a field. Let $V_{1}=$ $\left\{(0) \times M_{2}{ }^{j}: j=1\right.$ to $\left.m-1\right\}$ and $V_{2}=\left\{R_{1} \times M_{2}{ }^{j}: j=1\right.$ to $\left.m-1\right\}$ where $M_{2}$ is a nonzero proper ideal in $R_{2}$ and $M_{2}{ }^{m}=(0)$. Then the induced subgraphs $\left\langle V_{1}\right\rangle$ is complete and $\left\langle V_{2}\right\rangle$ is totally disconnected. Figure 2.4 is the extended annihilating-ideal graph of $R$ and it shows that $\operatorname{diam}(\mathbb{E} \mathbb{G}(R))=2$. Hence in all the cases $\operatorname{diam}(\mathbb{E A} \mathbb{G}(R))=2$.

Example 3.6. If $R \cong R_{1} \times R_{2}$ where $R_{1}=\mathbb{Z}_{2}[X] /\left(X^{2}+X+1\right)$ and $R_{2}=$ $\mathbb{Z}_{2}[X] /\left(X^{3}\right)$, then $\operatorname{diam}(\mathbb{E} \mathbb{A} \mathbb{G}(R))=2$.


Figure $2.5 \mathbb{E} \mathbb{A} \mathbb{G}(R)$
Theorem 3.7. Let $R$ be a reduced ring such that $Z(R)$ is not an ideal of $R$. Then $\operatorname{diam}(\mathbb{E} \mathbb{A} \mathbb{G}(R))=2$ if and only if $R$ has exactly two minimal prime ideals and at least three nonzero annihilating-ideals.

Proof. The result is obviously true from the Theorem 2.1 and [9, Theorem 1.2].

Theorem 3.8. Let $R \cong R_{1} \times R_{2}$ where $R_{1}$ is an integral domain and $R_{2}$ is a ring with unique nonzero proper ideal. Then $\operatorname{diam}(\mathbb{E} \mathbb{A}(R))=2$.

Proof. By Theorem 2.5, $\operatorname{diam}(\mathbb{E} \mathbb{A}(R))=2$.
Open question 3.9. Determine $\operatorname{diam}(\mathbb{E} \mathbb{A}(R))$ for $R \cong R_{1} \times R_{2}$ where $R_{1}$ is an integral domain and $R_{2}$ is a ring with more than one nonzero proper ideals and not a PIR.

In the next theorem, we state necessary and sufficient conditions to have $\operatorname{diam}(\mathbb{E A G}(R))=3$ for artinian PIR.

Theorem 3.10. Let $R$ be an artinian PIR such that $Z(R)$ is not an ideal of $R$. Then $\operatorname{diam}(\mathbb{E} \mathbb{A}(R))=3$ if and only if $R$ is a ring with more than two minimal prime ideals.

Proof. Since $R$ is an artinian PIR and $Z(R)$ is not an ideal of $R, R \cong \prod_{i=1}^{n} R_{i}$ where $R_{i}^{\prime} s$ are SPRs for all $i=1$ to $n$ and $n \geq 2$. Assume that $\operatorname{diam}(\mathbb{E A G}(R))=$ 3. Since $\mathbb{A} \mathbb{G}(R)$ is a spanning subgraph of $\mathbb{E} \mathbb{A}(G), \operatorname{diam}(\mathbb{A} \mathbb{G}(R))=3$. Then by [9, Theorem 1.4], $R$ is a reduced ring with more than two minimal prime ideals or $R$ is a non-reduced ring. Consider the case $R$ is non-reduced. For $n=2$ and $R \cong R_{1} \times R_{2}$, where $R_{1}$ is not a field, then as noted in the proof of Theorem 3.5, $\operatorname{diam}(\mathbb{E} \mathbb{A}(R))=2$. Thus $n \geq 3$. Also note that in a commutative artinian ring, every maximal ideal is a minimal prime ideal. From this, $R$ is a non-reduced ring with more than two minimal prime ideals.

Conversely, assume that $R$ is a ring with more than two minimal prime ideals. Consider $R$ is a reduced ring with more than two minimal prime ideals. Then by Theorem 2.1 and $[9$, Theorem 1.4], $\operatorname{diam}(\mathbb{E A} \mathbb{G}(R))=3$. Suppose that $R$ is a nonreduced ring with more than two minimal prime ideals and $\operatorname{diam}(\mathbb{E A} \mathbb{G}(R)) \neq 3$. If $\operatorname{diam}(\mathbb{E} \mathbb{A}(R))=1$, then by Theorem $3.3, R$ is a SPR with at least two nonzero proper ideals. Here $Z(R)$ is an ideal of $R$, a contradiction. If $\operatorname{diam}(\mathbb{E} \mathbb{A}(R))=2$, then by Theorem 3.5, $R$ does not have more than two minimal prime ideals. Hence $\operatorname{diam}(\mathbb{E} \mathbb{A}(R))=3$.

Open question 3.11. Classify the diameter of $\mathbb{E} \mathbb{A}(R)$ for all artinian rings which are not PIRs.

## 4. Girth of $\mathbb{E A} \mathbb{A}(R)$

In this Section we discuss the girth of $\mathbb{E} \mathbb{A}(R)$ and also compare the girth of $\mathbb{A} \mathbb{G}(R)$ with $\mathbb{E} \mathbb{A}(R)$. As Theorem 3.1, $\operatorname{gr}(\mathbb{E} \mathbb{A} \mathbb{G}(R)) \leq 4$. Here we characterize the rings for which $\operatorname{gr}(\mathbb{E} \mathbb{A}(R))=3,4$ or $\infty$.

Theorem 4.1. Let $R \cong R_{1} \times R_{2}$ be a ring. If any one of $R_{i}$ is a ring with more than one nonzero proper ideals for $i=1,2$, then $\operatorname{gr}(\mathbb{E} \mathbb{A}(R))=3$.

Proof. Suppose that $R_{1}$ is a ring with more than one nonzero proper ideals. Since $\mathbb{E} \mathbb{G}\left(R_{1}\right)$ is connected, there exist two nonzero proper ideals $I_{1}$ and $I_{2}$ in $R_{1}$ such that $I_{1}{ }^{n} I_{2}{ }^{m}=(0)$ with $I_{1}{ }^{n} \neq(0), I_{2}{ }^{m} \neq(0)$ for some positive integers $n, m$. Then we have the following cycle of length 3 in $\mathbb{E A} \mathbb{G}(R), I_{1} \times(0)-I_{2} \times$ $(0)-(0) \times R_{2}-I_{1} \times(0)$. Thus $g r(\mathbb{E A G}(R))=3$.

Theorem 4.2. Let $R \cong \prod_{i=1}^{n} R_{i}$ where $R_{i}^{\prime}$ s are rings for every $i$ with $n \geq 2$. Then the following hold.
(i) For $n=2, \operatorname{gr}(\mathbb{E} \mathbb{A}(R))=\infty$ if and only if $R \cong R_{1} \times R_{2}$, where $R_{1}$ is a field and $R_{2}$ is an integral domain.
(ii) $\operatorname{gr}(\mathbb{E} \mathbb{A}(R))=3$ if and only if one of the following statements hold.
(a) When $n \geq 3$
(b) For $n=2$, both $R_{i}^{\prime}$ s are not integral domains.
(c) For $n=2, R_{1}$ is an integral domain and $R_{2}$ is a ring with more than one nonzero proper ideals.
(iii) For $n=2 \operatorname{gr}(\mathbb{E} \mathbb{A}(R))=4$ if and only if either $R_{1}$ and $R_{2}$ are integral domains which are not fields or $R_{1}$ is an integral domain and $R_{2}$ is a ring with unique nonzero proper ideal.

Proof. (i) For $n=2$ and assume that $\operatorname{gr}(\mathbb{E} \mathbb{A}(R))=\infty$. Since $\mathbb{A} \mathbb{G}(R)$ is a spanning subgraph of $\mathbb{E} \mathbb{A}(R), \mathbb{E} \mathbb{A}(R) \cong \mathbb{A} \mathbb{G}(R)$. Then by Theorem 2.3, $R_{i}^{\prime} s$ are reduced for all $i=1,2$. Also by [3, Theorem 3.1], $R \cong R_{1} \times R_{2}$ where $R_{1}$ is a field and $R_{2}$ is an integral domain. Conversely, assume that $R \cong R_{1} \times R_{2}$ where $R_{1}$ is a field and $R_{2}$ is an integral domain, then $R_{1}$ and $R_{2}$ are reduced. By Theorem 2.3 and Corollary 1.2, $\mathbb{E} \mathbb{A}(R)$ is a star graph. Therefore $\operatorname{gr}(\mathbb{E} \mathbb{A}(R))=\infty$.
(ii) (a) Assume that $\operatorname{gr}(\mathbb{E} \mathbb{A}(R))=3$. Then from the following cycle $R_{1} \times$ (0) $\times \cdots \times(0)-(0) \times R_{2} \times(0) \times \cdots \times(0)-(0) \times(0) \times R_{3} \times(0) \times \cdots \times(0)-$ $R_{1} \times(0) \times \cdots \times(0), n \geq 3$ is true. Conversely, when $n \geq 3$, the result is obviously true.
(b) Now consider $n=2$ and assume that $\operatorname{gr}(\mathbb{E} \mathbb{A}(R))=3$. Suppose that $R_{1}$ and $R_{2}$ are integral domains. Since $R_{1}$ and $R_{2}$ are reduced, by Theorem 2.3, Corollary 1.2 and [3, Corollary 2.5], $\operatorname{gr}(\mathbb{E} \mathbb{A}(R))=4$ or $\infty$, a contradiction. From this case $R_{i}^{\prime} s$ are not integral domains. Conversely, assume that $R_{1}$ and $R_{2}$ are not integral domains. Consider $R_{1}$ and $R_{2}$ are rings with unique nonzero proper ideals $I$ and $J$ in $R_{1}$ and $R_{2}$ respectively such that $I^{2}=(0)$ and $J^{2}=$ (0). Then in $\mathbb{E A} \mathbb{G}(R), I \times(0)-I \times J-(0) \times J-I \times(0)$ is a cycle of length 3 so that $\operatorname{gr}(\mathbb{E A} \mathbb{G}(R))=3$. Also consider $R_{1}$ is a ring with more than one nonzero proper ideals. Then by Theorem 4.1, $\operatorname{gr}(\mathbb{E} \mathbb{A}(R))=3$. From these cases, $\operatorname{gr}(\mathbb{E} \mathbb{A}(R))=3$.
(c) For $n=2$, assume that $\operatorname{gr}(\mathbb{E} \mathbb{A}(R))=3$. If $R_{1}$ is an integral domain and $R_{2}$ is a ring with unique nonzero proper ideal, then by Theorem 2.5, $\operatorname{gr}(\mathbb{E} \mathbb{A}(R))=4$, a contradiction. Thus $R_{1}$ is an integral domain and $R_{2}$ is a ring with more than one nonzero proper ideals. Converse part follows from Theorem 4.1.
(iii) Proof follows from (i) and (ii).

We next characterize when $\operatorname{gr}\left(\mathbb{E A} \mathbb{G}\left(\mathbb{Z}_{n}\right)\right)$ is 3,4 or $\infty$.
Theorem 4.3. For $n \in \mathbb{N}$, let $n=\prod_{i=1}^{k} p_{i}^{\alpha_{i}}$ be the distinct prime factorization of $n$. Then the following assertions are true.
(i) $\operatorname{gr}\left(\mathbb{E} \mathbb{A}\left(\mathbb{Z}_{n}\right)\right)=3$ if and only if one of the following assertions must occur.
(a) When $k \geq 3$.
(b) When $k=1$ and $\alpha_{1} \geq 4$.
(c) When $k=2$, either $\alpha_{1}=1, \alpha_{2}>2$ or $\alpha_{1}, \alpha_{2} \geq 2$.
(ii) $\operatorname{gr}\left(\mathbb{E} \mathbb{A}\left(\mathbb{Z}_{n}\right)\right)=4$ if and only if $k=2$ with $\alpha_{1}=1, \alpha_{2}=2$.
(iii) $\operatorname{gr}\left(\mathbb{E} \mathbb{A}\left(\mathbb{Z}_{n}\right)\right)=\infty$ if and only if either $k=1$ with $\alpha_{1}=2$ or 3 or $k=2$ with $\alpha_{1}=1, \alpha_{2}=1$.

Proof. When $k \geq 2$, the result holds by Theorem 4.2. It remains to consider the case $k=1$. Assume that $\operatorname{gr}\left(\mathbb{E} \mathbb{G}\left(\mathbb{Z}_{n}\right)\right)=3$ and $\alpha_{1}<4$. Then $\left|\mathbb{A}(R)^{*}\right|=0,1$ or 2
and hence $\operatorname{gr}\left(\mathbb{E} \mathbb{A} \mathbb{G}\left(\mathbb{Z}_{n}\right)\right)=\infty$. This shows that $\alpha_{1} \geq 4$. Conversely, assume that $\alpha_{1} \geq 4$, then $\mathbb{Z}_{n}$ is a local ring with maximal ideal $\left(p_{1}\right)$ and $\left|\mathbb{A}(R)^{*}\right|=\alpha_{1}-1$. Since $\mathbb{Z}_{n}$ is a SPR, by Theorem 2.4, $\operatorname{gr}\left(\mathbb{E} \mathbb{A} \mathbb{G}\left(\mathbb{Z}_{n}\right)\right)=3$. By above, (iii) holds for the case $k=1$.

We conclude this paper with the following theorem to have a better comparison of the girth between $\mathbb{E} \mathbb{A} \mathbb{G}(R)$ and $\mathbb{A} \mathbb{G}(R)$.
Theorem 4.4. Let $R$ be a ring. Then the following hold.
(i) If $\operatorname{gr}(\mathbb{A} \mathbb{G}(R))=3$, then $\operatorname{gr}(\mathbb{E} \mathbb{A} \mathbb{G}(R))=3$.
(ii) If $\operatorname{gr}(\mathbb{A} \mathbb{G}(R))=4$, then $\operatorname{gr}(\mathbb{E} \mathbb{A} \mathbb{G}(R))=4$.
(iii) If $\operatorname{gr}(\mathbb{A} \mathbb{G}(R))=\infty$, then $\operatorname{gr}(\mathbb{E} \mathbb{A} \mathbb{G}(R))=3,4$ or $\infty$.
(iv) If $\operatorname{gr}(\mathbb{E} \mathbb{A} \mathbb{G}(R))=3$, then $\operatorname{gr}(\mathbb{A} \mathbb{G}(R))=3$ or $\infty$.
(v) If $\operatorname{gr}(\mathbb{E} \mathbb{A} \mathbb{G}(R))=4$, then $\operatorname{gr}(\mathbb{A} \mathbb{G}(R))=4$ or $\infty$.
(vi) If $\operatorname{gr}(\mathbb{E} \mathbb{A} \mathbb{G}(R))=\infty$, then $\operatorname{gr}(\mathbb{A} \mathbb{G}(R))=\infty$.

Proof. Since $\mathbb{A} \mathbb{G}(R)$ is a spanning subgraph of $\mathbb{E} \mathbb{A}(R)$, (i), (iii), (v), (vi) are obviously true.
(ii) Assume that $\operatorname{gr}(\mathbb{A} \mathbb{G}(R))=4$. This shows that $\mathbb{A} \mathbb{G}(R)$ is a trianglefree graph. Then by [2, Lemma 1], $R \cong R_{1} \times R_{2}$ where either $R_{1}$ and $R_{2}$ are integral domains which are not fields or $R_{1}$ is an integral domain which is not a field and $R_{2}$ is a ring with unique nonzero proper ideal. By Theorem 4.2(iii), $\operatorname{gr}(\mathbb{E} \mathbb{A} \mathbb{G}(R))=4$.
(iv) Assume that $\operatorname{gr}(\mathbb{E} \mathbb{A}(R))=3$. Since $\mathbb{A} \mathbb{G}(R)$ is a spanning subgraph of $\mathbb{E} \mathbb{A} \mathbb{G}(R), \operatorname{gr}(\mathbb{A} \mathbb{G}(R))=3,4$ or $\infty$. By part (ii), $\operatorname{gr}(\mathbb{A} \mathbb{G}(R)) \neq 4$ and so $\operatorname{gr}(\mathbb{A} \mathbb{G}(R))=3$ or $\infty$.

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Received 27 June 2020 Revised 4 February 2021 Accepted 22 March 2022

