Discussiones Mathematicae
General Algebra and Applications 42 (2022) 255–277
https://doi.org/10.7151/dmgaa.1389

FUZZY WEAKLY 2-ABSORBING IDEALS OF A LATTICE

Shriram K. Nimbhorkar and Yogita S. Patil

Department of Mathematics
Dr. Babasaheb Ambedkar Marathwada University
Aurangabad 431004, India

e-mail: sknimbhorkar@gmail.com
saharshyog.143@rediffmail.com

Abstract

As a generalization of the concept of a weakly prime ideal, we introduce the concepts of a fuzzy weak prime ideal, a fuzzy weakly 2-absorbing ideal of a lattice. Some results of fuzzy weakly 2-absorbing ideals and fuzzy weakly primary ideals are proved. We also introduce and study fuzzy weakly 2-absorbing ideals in a product of lattices.

Keywords: lattice, fuzzy sublattice, fuzzy ideal, fuzzy weakly prime ideal, weakly 2-absorbing fuzzy ideal.

2010 Mathematics Subject Classification: 06D72.

1. Introduction

Anderson and Smith [3] introduced the concept of a weakly prime ideal in a commutative ring. Badawi [4] introduced the concept of a 2-absorbing ideal and a weakly 2-absorbing ideal of a commutative ring. A proper ideal I of a commutative ring R is said to be weakly 2-absorbing, if whenever $a, b, c \in R$, $0 \neq abc \in I$, then either $ab \in I$ or $ac \in I$ or $bc \in I$. Anderson and Badawi [2] introduced and studied n-absorbing ideals in a commutative ring. Payrovi and Babaei [10], Badawi and Darani [5] have studied 2-absorbing ideals in commutative ring. Wasadikar and Gaikwad [11] introduced the concept of a 2-absorbing ideal in a lattice.

Zadeh [13] developed the concept of a fuzzy set. Ajmal and Thomas [1] defined a fuzzy lattice and fuzzy sublattice as a fuzzy algebra. Koguep *et al.* [8] have studied fuzzy prime ideals in lattices.

In this paper, we introduce the concept of a fuzzy weakly 2-absorbing ideal of a lattice. This is a generalization of the concept of a fuzzy prime ideal of a lattice. Also we define a weakly fuzzy primary ideal and the fuzzy radical of a fuzzy ideal of a lattice. Some properties of fuzzy weakly 2-absorbing ideals and fuzzy weakly primary ideals are proved. We also study these concepts in a product of lattices.

2. PRELIMINARIES

Throughout in this paper, $L = (L, \wedge, \vee)$ denotes a lattice with 0. We recall some known concepts and results.

Definition 2.1. A fuzzy subset μ of L is a function $\mu : L \to [0, 1]$.

Definition 2.2. A fuzzy subset μ of L is called proper if it is a non-constant function.

Definition 2.3 [8]**.** For any $\alpha \in [0, 1]$ the set $\mu_\alpha = \{x \in X/\mu(x) \geq \alpha\}$ is called the α-cut of μ or α-level set.

Definition 2.4 [8]**.** A fuzzy subset μ of L is called a fuzzy sublattice of L if $\mu(x \wedge y) \wedge \mu(x \vee y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in L$.

Definition 2.5 [8]**.** A fuzzy sublattice μ of L is called a fuzzy ideal of L if $\mu(x \vee y) = \mu(x) \wedge \mu(y)$ for all $x, y \in L$.

Definition 2.6. For fuzzy subsets μ, η of L, $\mu \subseteq \eta$ means $\mu(x) \leq \eta(x)$ for all $x \in L$.

The following result is well-known.

Lemma 2.1. *Let μ be a fuzzy sublattice of L. Then μ is a fuzzy ideal of L if and only if $\mu(x) \leq \mu(y)$, whenever, $x \geq y$ for all $x, y \in L$.*

3. FUZZY PRIME IDEALS OF A LATTICE

The following concept is well-known in lattice theory, see Grätzer [7].

Definition 3.1. A nonempty subset I of a lattice L is called an ideal, if for $a, b \in L$, the following conditions hold.

(i) If $a, b \in I$, then $a \vee b \in I$ and

(ii) if $a \leq b$ and $b \in I$, then $a \in I$.

A proper ideal I (i.e., $I \neq L$) is called a prime ideal, if $a \wedge b \in I$ implies that either $a \in I$ or $b \in I$.

Wasadikar and Gaikwad [11] have introduced the concept of a weakly prime ideal in a lattice.

Definition 3.2. A proper ideal I of a lattice L is called a weakly prime ideal, if for $a, b \in L$, $a \wedge b \neq 0$, $a \wedge b \in I$ implies that either $a \in I$ or $b \in I$.

Koguep *et al.* [8], have defined a fuzzy prime ideal as follows.

Definition 3.3. A proper fuzzy ideal μ of L is called a fuzzy prime ideal, if for all $a, b \in L$, $\mu(a \wedge b) \leq \mu(a) \vee \mu(b)$.

In fact, a proper fuzzy ideal μ of L is fuzzy prime if and only if for all $a, b \in L$, $\mu(a \wedge b) = \mu(a) \vee \mu(b)$.

We define a fuzzy weakly prime ideal as follows.

Definition 3.4. A proper fuzzy ideal μ of L is called a fuzzy weakly prime ideal, if $a, b \in L$, $a \wedge b \neq 0$, then $\mu(a \wedge b) \leq \mu(a) \vee \mu(b)$.

We have the following theorem.

Theorem 3.1. *An ideal I of L is a weakly prime ideal if and only if the characteristic function χ_I of I is a fuzzy weakly prime ideal of L.*

Proof. Clearly, χ_I is a fuzzy ideal of L. Suppose that I is a weakly prime ideal of L. Let $a, b \in L$ be such that $a \wedge b \neq 0$. If $a \wedge b \in I$, then as I is weakly prime, either $a \in I$ or $b \in I$. Hence we have

$$\chi_I(a \wedge b) = 1 = \chi_I(a) \vee \chi_I(b).$$

If $a \wedge b \notin I$, then neither $a \in I$, nor $b \in I$ and we have

$$\chi_I(a \wedge b) = 0 = 0 \vee 0 = \chi_I(a) \vee \chi_I(b).$$

Thus χ_I is fuzzy weakly prime.

Conversely, suppose that χ_I is a fuzzy weakly prime ideal of L. Let $a, b \in L$, $a \wedge b \neq 0$. Suppose that $a \wedge b \in I$. If none of $a, b \in I$, then

$$\chi_I(a \wedge b) = 1 \neq 0 = 0 \vee 0 = \chi_I(a) \vee \chi_I(b).$$

This contradicts the assumption that χ_I is weakly prime. Hence either $a \in I$ or $b \in I$. Thus I must be weakly prime. ■

The following example shows that the condition of "weakly prime"in Theorem 3.1 is necessary.

Example 3.1. Consider the lattice L shown in Figure 1. We note that the ideal $I = (a]$ is not a weakly prime ideal of L, as $d \wedge e = a \in I$ but neither $d \in I$, nor $e \in I$.

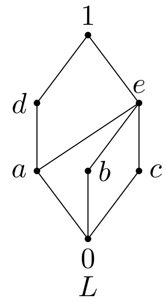


Figure 1

We have $d \wedge e = a \in I$. Hence $\chi_I(d \wedge e) = 1$. But $\chi_I(d) = \chi_I(e) = 0$. Thus $\chi_I(d \wedge e) \nleq \chi_I(d) \vee \chi_I(e) = 0$. Hence χ_I is not a fuzzy weakly prime ideal of L.

The following result is from Nimbhorkar and Patil [9].

Theorem 3.2. *An ideal I of a lattice L is a prime ideal if and only if χ_I, the characteristic function of I is a fuzzy prime ideal of L.*

The proof of the following lemma follows from the definition of a fuzzy prime ideal.

Lemma 3.1. *If μ is a fuzzy prime ideal of L, then μ is a fuzzy weakly prime ideal of L.*

The following example shows that the converse of Lemma 3.1 does not hold.

Example 3.2. Consider the lattice L shown in Figure 1. We note that the ideal $I = (c]$ is a weakly prime ideal of L but is not a prime ideal. Hence by Theorem 3.1, χ_I is a fuzzy weakly prime ideal and by Theorem 3.2, χ_I is not fuzzy prime.

4. FUZZY WEAKLY 2-ABSORBING IDEALS

The following definition is from Wasadikar and Gaikwad [11].

Definition 4.1. An ideal I of L is called a weakly 2-absorbing ideal, if for $a, b, c \in L$, $a \wedge b \wedge c \neq 0$, $a \wedge b \wedge c \in I$ implies that either $a \wedge b \in I$ or $b \wedge c \in I$ or $c \wedge a \in I$.

We extend the concept of a weakly 2-absorbing ideal, in the context of a fuzzy ideal of a lattice and prove some properties of fuzzy weakly 2-absorbing ideals. We denote by $FI(L)$, the set of all fuzzy ideals of L.

The following definition and result are from Nimbhorkar and Patil [9].

Definition 4.2. A proper fuzzy ideal μ of L is called a fuzzy 2-absorbing ideal of L if for all $a, b, c \in L$,

$$\mu(a \wedge b \wedge c) \leq \max\{\mu(a \wedge b), \mu(b \wedge c), \mu(c \wedge a)\}.$$

Lemma 4.1. *An ideal I of L is a 2-absorbing ideal if and only if χ_I is a fuzzy 2-absorbing ideal of L.*

We define a fuzzy weakly 2-absorbing ideal as follows.

Definition 4.3. A proper fuzzy ideal μ of L is called a fuzzy weakly 2-absorbing ideal of L if for all $a, b, c \in L$, $a \wedge b \wedge c \neq 0$, then

$$\mu(a \wedge b \wedge c) \leq \max\{\mu(a \wedge b), \mu(b \wedge c), \mu(c \wedge a)\}.$$

Since $\mu(a \wedge b), \mu(b \wedge c), \mu(c \wedge a)$ are nonnegative real numbers, the definition of a fuzzy weakly 2-absorbing ideal is equivalent to μ is a fuzzy weakly 2-absorbing ideal iff for all $a, b, c \in L$, $a \wedge b \wedge c \neq 0$, implies that

$$\mu(a \wedge b \wedge c) \leq \mu(a \wedge b) \vee \mu(b \wedge c) \vee \mu(c \wedge a).$$

Lemma 4.2. *An ideal I of L is a weakly 2-absorbing ideal if and only if χ_I is a fuzzy weakly 2-absorbing ideal of L.*

Proof. Suppose that I is a weakly 2-absorbing ideal of L. Let $a, b, c \in L$ be such that $a \wedge b \wedge c \neq 0$. If $a \wedge b \wedge c \in I$, then as I is weakly 2-absorbing, either $a \wedge b \in I$ or $b \wedge c \in I$ or $c \wedge a \in I$. Thus in this case,

$$\chi_I(a \wedge b \wedge c) \leq \chi_I(a \wedge b) \vee \chi_I(b \wedge c) \vee \chi_I(c \wedge a).$$

If $a \wedge b \wedge c \notin I$, then clearly, $a \wedge b \notin I$, $b \wedge c \notin I$ and $c \wedge a \notin I$.

Thus in this case also,

$$\chi_I(a \wedge b \wedge c) \leq \chi_I(a \wedge b) \vee \chi_I(b \wedge c) \vee \chi_I(c \wedge a).$$

Hence χ_I is a fuzzy 2-absorbing ideal of L.

Conversely, suppose that χ_I is a fuzzy 2-absorbing ideal of L. Let $a, b, c \in L$ be such that $a \wedge b \wedge c \neq 0$. Suppose that $a \wedge b \wedge c \in I$, but $a \wedge b \notin I$, $b \wedge c \notin I$ and $c \wedge a \notin I$. This implies that

$$\chi_I(a \wedge b \wedge c) = 1 \text{ and } \chi_I(a \wedge b) = \chi_I(b \wedge c) = \chi_I(c \wedge a) = 0.$$

Hence

$$\chi_I(a \wedge b \wedge c) \nleq \chi_I(a \wedge b) \vee \chi_I(b \wedge c) \vee \chi_I(c \wedge a),$$

a contradiction. ∎

It is obvious that every fuzzy 2-absorbing ideal of L is a fuzzy weakly 2-absorbing ideal of L. The following example shows that a fuzzy weakly 2-absorbing ideal of L need not be a fuzzy 2-absorbing ideal of L.

Example 4.1. Consider the lattice shown in Figure 2.

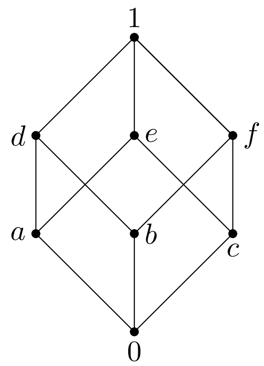


Figure 2

The ideal $I = (0]$ is trivially a weakly 2-absorbing ideal. It is not 2-absorbing as $d \wedge e \wedge f = 0 \in I$ but neither $d \wedge e = a \in I$, nor $e \wedge f = c \in I$ nor $f \wedge d = b \in I$. It follows from Lemma 4.2 that χ_I is a fuzzy weakly 2-absorbing ideal and by Lemma 4.1, χ_I is not a fuzzy 2-absorbing ideal.

The following lemma shows that any level set of a fuzzy weakly 2-absorbing ideal of L is a weakly 2-absorbing ideal of L.

Lemma 4.3. *Let μ be a fuzzy weakly 2-absorbing ideal of L. Then the level ideal μ_t is a weakly 2-absorbing ideal of L for each $t \in Image(\mu)$.*

Conversely, if each level ideal μ_t, for $t \in Image(\mu)$ is a weakly 2-absorbing ideal of L, then μ is a fuzzy weakly 2-absorbing ideal of L.

Proof. Let μ be a fuzzy weakly 2-absorbing ideal of L. Let $t \in Image(\mu)$. Let $a, b, c \in L$ be such that $a \wedge b \wedge c \neq 0$ and $a \wedge b \wedge c \in \mu_t$. Then $t \leq \mu(a \wedge b \wedge c)$. Since μ is a fuzzy weakly 2-absorbing ideal,

$$t \leq \mu(a \wedge b \wedge c) \leq \mu(a \wedge b) \vee \mu(b \wedge c) \vee \mu(c \wedge a). \tag{4.1}$$

Since $t, \mu(a \wedge b), \mu(b \wedge c), \mu(c \wedge a)$ are nonnegative real numbers, $\mu(a \wedge b) < t$, $\mu(b \wedge c) < t$ and $\mu(c \wedge a) < t$, will imply that

$$\mu(a \wedge b \wedge c) \leq \mu(a \wedge b) \vee \mu(b \wedge c) \vee \mu(c \wedge a) < t. \tag{4.2}$$

Then (4.1) and (4.2) lead to $t < t$, which is not possible. Hence $t \leq \mu(a \wedge b)$ or $t \leq \mu(b \wedge c)$ or $t \leq \mu(c \wedge a)$. Thus either $a \wedge b$ or $b \wedge c$ or $c \wedge a \in \mu_t$; i.e., μ_t is a weakly 2-absorbing ideal of L.

Conversely, assume that μ_t is a weakly 2-absorbing ideal of L for each $t \in Image(\mu)$. Let $a, b, c \in L$, $a \wedge b \wedge c \neq 0$ and $\mu(a \wedge b \wedge c) = t$. Then $a \wedge b \wedge c \in \mu_t$. Since μ_t is a weakly 2-absorbing ideal of L, either $a \wedge b$ or $b \wedge c$ or $c \wedge a \in \mu_t$. Thus either $\mu(a \wedge b) \geq t$ or $\mu(b \wedge c) \geq t$ or $\mu(c \wedge a) \geq t$. This implies that

$$t = \mu(a \wedge b \wedge c) \leq \mu(a \wedge b) \vee \mu(b \wedge c) \vee \mu(c \wedge a).$$

Thus μ is a fuzzy weakly 2-absorbing ideal of L. ∎

Now we show that every fuzzy weakly prime ideal of L is a fuzzy weakly 2-absorbing ideal.

Lemma 4.4. *Let μ be a fuzzy weakly prime ideal of L. Then μ is a fuzzy weakly 2-absorbing ideal of L.*

Proof. Let μ be a fuzzy weakly prime ideal of L. Then for all $a, b \in L$, $a \wedge b \neq 0$ implies that,

$$\mu(a \wedge b) \leq \mu(a) \vee \mu(b).$$

Hence for all $a, b, c \in L$, for which $a \wedge b \wedge c \neq 0$, we have

$$\begin{aligned} \mu(a \wedge b \wedge c) &\leq \mu(a \wedge b) \vee \mu(c), \\ \mu(a \wedge b \wedge c) &\leq \mu(b \wedge c) \vee \mu(a), \\ \mu(a \wedge b \wedge c) &\leq \mu(c \wedge a) \vee \mu(b). \end{aligned}$$

Hence

$$\mu(a \wedge b \wedge c) \leq \mu(a \wedge b) \vee \mu(c) \vee \mu(b \wedge c) \vee \mu(a) \vee \mu(c \wedge a) \vee \mu(b). \tag{4.3}$$

By the definition of a fuzzy ideal, (see Koguep *et al.* [8]), it follows that for any $a, b \in L$, $\mu(a) \leq \mu(a \wedge b)$. Hence (4.3) reduces to

$$\mu(a \wedge b \wedge c) \leq \mu(a \wedge b) \vee \mu(b \wedge c) \vee \mu(c \wedge a).$$

Thus μ is a fuzzy weakly 2-absorbing ideal of L. ∎

The following example shows that the converse of Lemma 4.4 does not hold.

Example 4.2. Consider the lattice L shown in Figure 3. We note that $I = (c]$ is a weakly 2-absorbing ideal but it is neither prime, nor weakly prime as $f \wedge g = c \in I$ but $f \notin I$, $g \notin I$. Hence by Lemma 4.2, χ_I is fuzzy weakly 2-absorbing and by Theorem 3.1, χ_I is not fuzzy weakly prime.

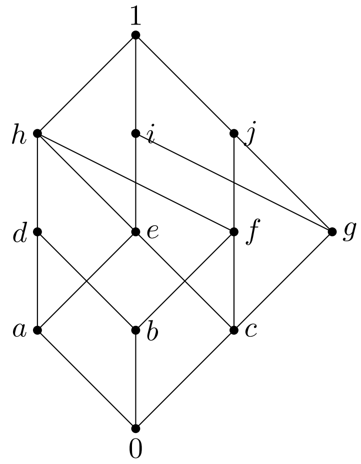


Figure 3

Lemma 4.5. *The intersection of any two distinct fuzzy weakly prime ideals of L is a fuzzy weakly 2-absorbing ideal of L.*

Proof. Let μ, θ be two distinct fuzzy weakly prime ideals of L. We know that for any $a \in L$, $(\mu \cap \theta)(a) = \mu(a) \wedge \theta(a)$. Let $a, b, c \in L$, $a \wedge b \wedge c \neq 0$. We have

$$(\mu \cap \theta)(a \wedge b \wedge c) = \mu(a \wedge b \wedge c) \wedge \theta(a \wedge b \wedge c) \tag{4.4}$$

By Lemma 4.4, every fuzzy weakly prime ideal is fuzzy weakly 2-absorbing. Hence from (4.4), we get

$$\begin{aligned} &(\mu \cap \theta)(a \wedge b \wedge c) \\ &\leq [\mu(a \wedge b) \vee \mu(b \wedge c) \vee \mu(c \wedge a)] \wedge [\theta(a \wedge b) \vee \theta(b \wedge c) \vee \theta(c \wedge a)]. \end{aligned} \tag{4.5}$$

Since μ and θ are fuzzy weakly prime ideals, we can write

$$\mu(a \wedge b) \vee \mu(b \wedge c) \vee \mu(c \wedge a) \leq \mu(a) \vee \mu(b) \vee \mu(c)$$

and

$$\theta(a \wedge b) \vee \theta(b \wedge c) \vee \theta(c \wedge a) \leq \theta(a) \vee \theta(b) \vee \theta(c).$$

We note that all the terms on the right hand side of (4.5) belong to the distributive lattice $[0, 1]$. Hence we can write

$$\begin{aligned} (\mu \cap \theta)(a \wedge b \wedge c) &\leq [\mu(a) \vee \mu(b) \vee \mu(c)] \wedge [\theta(a) \vee \theta(b) \vee \theta(c)] \\ &= [\mu(a) \wedge \theta(a)] \vee [\mu(a) \wedge \theta(b)] \vee [\mu(a) \wedge \theta(c)] \\ &\vee\ [\mu(b) \wedge \theta(a)] \vee [\mu(b) \wedge \theta(b)] \vee [\mu(b) \wedge \theta(c)] \\ &\vee\ [\mu(c) \wedge \theta(a)] \vee [\mu(c) \wedge \theta(b) \vee [\mu(c) \wedge \theta(c)]. \end{aligned} \tag{4.6}$$

For any fuzzy ideal σ, we have $\sigma(x) \leq \sigma(x \wedge y)$, for all $x, y \in L$. Hence $\mu(x) \leq \mu(x \wedge y)$ and $\theta(y) \leq \theta(x \wedge y)$ for all $x, y \in L$. This implies

$$\mu(x) \wedge \theta(y) \leq \mu(x \wedge y) \wedge \theta(x \wedge y) = (\mu \cap \theta)(x \wedge y).$$

Applying this to the R. H. S. of (4.6), we get

$$\begin{aligned}(\mu \cap \theta)(a \wedge b \wedge c) \leq{}& (\mu \cap \theta)(a) \vee (\mu \cap \theta)(a \wedge b) \vee (\mu \cap \theta)(b \wedge c) \\ & \vee (\mu \cap \theta)(c \wedge a) \vee (\mu \cap \theta)(b) \vee (\mu \cap \theta)(c).\end{aligned} \tag{4.7}$$

Since $\mu \cap \theta$ is a fuzzy ideal, for all $x, y \in L$, we have

$$(\mu \cap \theta)(x) \leq (\mu \cap \theta)(x \wedge y).$$

Applying this to the R. H. S. of (4.7), we get

$$(\mu \cap \theta)(a \wedge b \wedge c) \leq (\mu \cap \theta)(a \wedge b) \vee (\mu \cap \theta)(b \wedge c) \vee (\mu \cap \theta)(c \wedge a).$$

Thus $\mu \cap \theta$ is a fuzzy 2-absorbing ideal of L. ■

The following example shows that the condition of "weakly primeness"in Lemma 4.5 is necessary. This example also shows that in general the intersection of two fuzzy weakly 2-absorbing ideals need not be a fuzzy weakly 2-absorbing ideal.

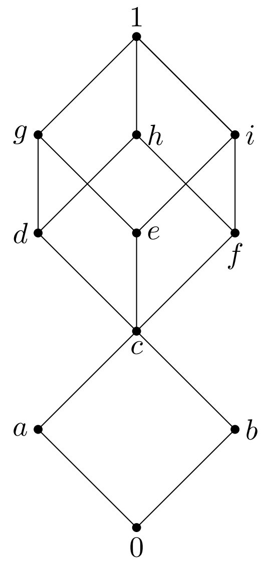


Figure 4

Example 4.3. Consider the lattice shown in Figure 4. We know that for any two ideals I, J of L, $\chi_I \cap \chi_J = \chi_{I \cap J}$. We note that the ideals $I = (d]$ and $J = (f]$ are not weakly prime and $I \cap J = (c]$ is not 2-absorbing. By Theorem 3.1, χ_I and χ_J are not fuzzy weakly prime and by Lemma 4.2, $\chi_{(c]}$ is not fuzzy 2-absorbing.

5. Fuzzy primary ideals

The following definition is from Wasadikar and Gaikwad [11].

Definition 5.1. Let L be a lattice with 0. An ideal I of L is called a weakly primary ideal, if for $a, b \in L$, $a \wedge b \neq 0$, $a \wedge b \in I$ implies that either $a \in I$ or $b \in \sqrt{I}$, where $\sqrt{I}$ denotes the radical of I (i.e., the intersection of all prime ideals containing I). If there does not exist a prime ideal containing an ideal I in a lattice L, then we define $\sqrt{I} = L$.

We note the following.
Let I be an ideal of L. Let $\mathbb{A}$ denote the set of all prime ideals P of L such that $I \subseteq P$. Let $\mathbb{B}$ denote the set of all weakly prime ideals Q of L such that $I \subseteq Q$. Since every prime ideal is a weakly prime ideal, but not conversely, it follows that $\mathbb{A} \subseteq \mathbb{B}$. Let $a \in \bigcap(Q|Q \in \mathbb{B})$. We note that if P is a prime ideal containing I, then $a \in P$. Thus $a \in \bigcap \mathbb{A}$. Hence $\bigcap\{P \in \mathbb{B}\} \subseteq \bigcap\{P \in \mathbb{A}\}$.
This motivates us to define the weakly prime radical of I as follows.

Definition 5.2. Let I be an ideal of L. We define the weakly prime radical of I as the intersection of all weakly prime ideals of L containing I. We denote it by $\sqrt{_w I}$. If there does not exist any weakly prime ideal containing I, we define $\sqrt{_w I}$ as L.

Example 5.1. Consider the lattice shown in Figure 5. We note that $I = (c]$ is not a weakly prime ideal and so is not a prime ideal. The ideal $J = (e]$ is a weakly prime ideal and not a prime ideal. We note that the weakly prime radical of I i.e., $\sqrt{_w I} = J$. Since there is no proper prime ideal containing I, the prime radical of I, i.e., $\sqrt{I} = L$. Thus the concepts of the prime radical and the weakly prime radical of an ideal are different.

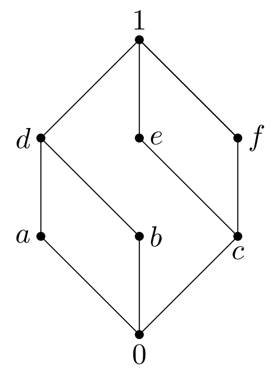


Figure 5

In this section we define the fuzzy weakly radical of a fuzzy ideal.

Definition 5.3. Let μ be a fuzzy ideal of L. We define the fuzzy weakly radical of μ as the intersection of all fuzzy weakly prime ideals containing μ and we denote it by $\sqrt{_w\mu}$.

We note that for a fuzzy ideal μ of L always $\mu \subseteq \sqrt{_w\mu}$. It can be shown that for an ideal I of L, $\sqrt{_w\chi_I} = \chi_{\sqrt{_wI}}$.

Definition 5.4. A proper fuzzy ideal μ of L is called a fuzzy weakly primary ideal if for $a, b \in L$, $a \wedge b \neq 0$ implies that $\mu(a \wedge b) \leq \mu(a) \vee \sqrt{_w\mu}(b)$.

Lemma 5.1. *A proper ideal I of L is a weakly primary ideal of L if and only if χ_I is a fuzzy weakly primary ideal of L.*

Proof. Suppose that I is a weakly primary ideal of L. Let $a, b \in L$, $a \wedge b \neq 0$.

(i) If $a \wedge b \in I$, then as I is a weakly primary ideal of L, either $a \in I$ or $b \in \sqrt{_wI}$. Hence $\chi_I(a \wedge b) \leq \chi_I(a) \vee \sqrt{_w\chi_I}(b)$.

(ii) If $a \wedge b \notin I$, then clearly $a \notin I$ and $b \notin I$. In this case also $\chi_I(a \wedge b) \leq \chi_I(a) \vee \sqrt{_w\chi_I}(b)$.

Thus χ_I is a fuzzy weakly primary ideal of L.

Conversely, suppose that χ_I is a fuzzy weakly primary ideal of L. Let $a \wedge b \in I$, $a \wedge b \neq 0$. Then $1 = \chi_I(a \wedge b) \leq \chi_I(a) \vee \sqrt{_w\chi_I}(b)$, implies that either $\chi_I(a) = 1$ or $\sqrt{_w\chi_I}(b) = 1$. Thus either $a \in I$ or $b \in \sqrt{_wI}$. ■

Now we give a relationship between a fuzzy weakly prime ideal and a fuzzy weakly primary ideal.

Lemma 5.2. *If μ is a fuzzy weakly prime ideal of L, then μ is a fuzzy weakly primary ideal of L.*

Proof. Let μ be a fuzzy weakly prime ideal of L. Let $a, b \in L$, $a \wedge b \neq 0$. We have $\mu(a \wedge b) \leq \mu(a) \vee \mu(b)$. Since $\mu \subseteq \sqrt{_w\mu}$, we get the result. ■

The following example shows that the converse of Lemma 5.2 does not hold.

Example 5.2. Consider the ideal $I = (c]$ of the lattice shown in Figure 5. We have noted that $\sqrt{_wI} = J = (e]$. We know that for any ideal A of L, $\sqrt{_w\chi_A} = \chi_{\sqrt{_wA}}$. Hence $\sqrt{_w\chi_I} = \chi_{\sqrt{_wI}} = \chi_J$. Since J is a weakly prime ideal, χ_J is a fuzzy weakly prime ideal and so χ_I is a fuzzy weakly primary ideal. We have $\chi_I(e \wedge f) = \chi_I(c) = 1$ but $\chi_I(e) \vee \chi_I(f) = 0$ as $e, f \notin I$. Thus χ_I is not fuzzy weakly prime.

Theorem 5.1. *A fuzzy ideal μ of L is fuzzy weakly primary if and only if the level set μ_t, $t \in Image(\mu)$ is a weakly primary ideal of L.*

Proof. Suppose that μ is a fuzzy weakly primary ideal of L. Let $a, b \in L$ be such that $a \wedge b \neq 0$, $a \wedge b \in \mu_t$ and $a \notin \mu_t$, $b \notin \sqrt{_w\mu_t}$. Then we have

$$t \leq \mu(a \wedge b),\ \mu(a) < t,\ \sqrt{_w\mu}(b) < t.$$

Since μ is fuzzy primary, we have

$$\mu(a \wedge b) \leq \mu(a) \vee \sqrt{_w\mu}(b).$$

Thus we get $t < t$, which is not possible. Hence μ_t is a weakly primary ideal of L.

Conversely, suppose that μ_t is a weakly primary ideal of L. Let $a, b \in L$ be such that $a \wedge b \neq 0$ and $\mu(a \wedge b) \nleq \mu(a) \vee \sqrt{_w\mu}(b)$. Let $\mu(a \wedge b) = t$. Then $\mu(a) < t$ and $\sqrt{_w\mu}(b) < t$. Since μ_t is a weakly primary ideal, $a \wedge b \in \mu_t$ implies that either $a \in \mu_t$ or $b \in \sqrt{_w\mu}_t$, i.e. either $\mu(a) \geq t$ or $\sqrt{_w\mu}(b) \geq t$, a contradiction. ■

Definition 5.5. A proper fuzzy ideal μ of L is called a fuzzy weakly 2-absorbing primary ideal of L, if for $a, b, c \in L$, $a \wedge b \wedge c \neq 0$, then

$$\mu(a \wedge b \wedge c) \leq \mu(a \wedge b) \vee \sqrt{_w\mu}(b \wedge c) \vee \sqrt{_w\mu}(c \wedge a).$$

Lemma 5.3. *A proper ideal I of L is a weakly 2-absorbing primary ideal, if and only if χ_I is a fuzzy weakly 2-absorbing primary ideal of L.*

Proof. Suppose that I is a weakly 2-absorbing primary ideal of L. Let $a, b, c \in L$, $a \wedge b \wedge c \neq 0$. Consider $\chi_I(a \wedge b \wedge c)$. If $a \wedge b \wedge c \in I$, then $\chi_I(a \wedge b \wedge c) = 1$. As I is weakly 2-absorbing primary, we have either $a \wedge b \in I$ or $b \wedge c \in \sqrt{_wI}$ or $c \wedge a \in \sqrt{_wI}$. Hence either $\chi_I(a \wedge b) = 1$ or $\chi_{\sqrt{_wI}}(b \wedge c) = \sqrt{_w\chi_I}(b \wedge c) = 1$ or $\chi_{\sqrt{_wI}}(c \wedge a) = \sqrt{_w\chi_I}(c \wedge a) = 1$. Thus

$$\chi_I(a \wedge b \wedge c) \leq \chi_I(a \wedge b) \vee \chi_{\sqrt{_wI}}(b \wedge c) \vee \chi_{\sqrt{_wI}}(c \wedge a).$$

If $a \wedge b \wedge c \notin I$, then $\chi_(a \wedge b \wedge c) = 0$. Clearly, $a \wedge b \notin I$.

Hence $\chi_I(a \wedge b \wedge c) \leq \chi_I(a \wedge b) \vee \chi_{\sqrt{_wI}}(b \wedge c) \vee \chi_{\sqrt{_wI}}(c \wedge a)$. Thus χ_I is a fuzzy weakly 2-absorbing primary ideal.

Conversely, suppose that χ_I is a fuzzy weakly 2-absorbing primary ideal. Let $a \wedge b \wedge c \in I$, $a \wedge b \wedge c \neq 0$. Then $\chi_I(a \wedge b \wedge c) = 1$. Suppose that $a \wedge b \notin I$, $b \wedge c \notin \sqrt{_wI}$ and $c \wedge a \notin \sqrt{_wI}$. Since χ_I is a fuzzy 2-absorbing primary ideal, we have

$$1 = \chi_I(a \wedge b \wedge c) \leq \chi_I(a \wedge b) \vee \chi_{\sqrt{_wI}}(b \wedge c) \vee \chi_{\sqrt{_wI}}(c \wedge a).$$

Since each of $\chi_I(a \wedge b)$, $\chi_{\sqrt{_wI}}(b \wedge c)$, $\chi_{\sqrt{_wI}}(c \wedge a)$ belongs to $[0, 1]$, at least one of these numbers must be 1. This implies that either $a \wedge b \in I$ or $b \wedge c \in \sqrt{_wI}$ or $c \wedge a \in \sqrt{_wI}$.

Thus I is a weakly 2-absorbing primary ideal. ■

Lemma 5.4. *If μ is a fuzzy weakly primary ideal of L, then μ is a fuzzy weakly 2-absorbing primary ideal of L.*

Proof. Let μ be a fuzzy weakly primary fuzzy ideal of L. Let $a, b, c \in L$, $a \wedge b \wedge c \neq 0$. As μ is a fuzzy weakly primary ideal, we have

$$\begin{aligned}\mu(a \wedge b \wedge c) &= \mu(a \wedge b \wedge b \wedge c)\\ &\leq \mu(a \wedge b) \vee \sqrt{{}_w\mu}(b \wedge c)\\ &\leq \mu(a \wedge b) \vee \sqrt{{}_w\mu}(b \wedge c) \vee \sqrt{{}_w\mu}(c \wedge a).\end{aligned}$$

Thus μ is a fuzzy weakly 2-absorbing primary ideal. ■

The following example shows that a fuzzy weakly 2-absorbing primary ideal of L need not be a fuzzy weakly primary ideal.

Example 5.3. Consider the ideal $I = (a]$ of the lattice L shown in Figure 6. We note that $(h] = \{0, a, b, c, e, f, g, h\}$ is the only prime ideal containing I and $(e], (f]$ are the only weakly prime ideals of L containing I. Hence $\sqrt{{}_wI} = (h] \cap (e] \cap (f] = (a] = I$. We note that I is a weakly 2-absorbing primary ideal as for any $x, y, z \in L$, $x \wedge y \wedge z \in I$ implies that either $x \wedge y \in I$ or $y \wedge z \in \sqrt{{}_wI}$ or $z \wedge x \in \sqrt{{}_wI}$. Hence by Lemma 5.3, χ_I is a fuzzy weakly 2-absorbing primary ideal of L. We note that $\chi_I(e \wedge f) = \chi_I(a) = 1$ but $\chi_I(e) = 0$ as well as $\sqrt{{}_wI}(f) = 0$. Thus $\chi_I(e \wedge f) \nleq \chi_I(e) \vee \sqrt{{}_wI}(f)$. Hence χ_I is not a fuzzy weakly primary ideal of L.

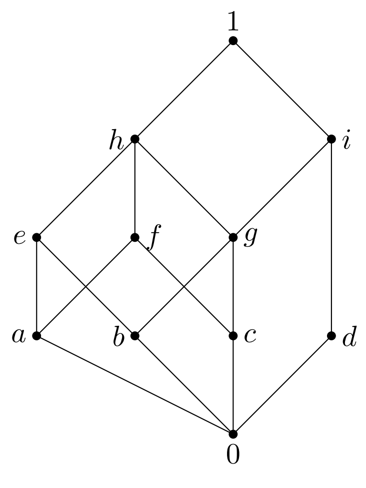


Figure 6

Lemma 5.5. *If μ is a fuzzy weakly 2-absorbing ideal of L, then μ is a fuzzy weakly 2-absorbing primary ideal of L.*

Proof. Let μ be a fuzzy weakly 2-absorbing ideal of L. Let $a, b, c \in L$, $a \wedge b \wedge c \neq 0$. Since μ is a fuzzy weakly 2-absorbing ideal, we get

$$\mu(a \wedge b \wedge c) \leq \mu(a \wedge b) \vee \mu(b \wedge c) \vee \mu(c \wedge a).$$

Since $\mu \subseteq \sqrt[w]{\mu}$, we get the result. ∎

The following figure is from Gaikwad [6, p. 91]. We use it to show that the converse of Lemma 5.5 need not hold.

Example 5.4. Consider the lattice shown in Figure 7. The only weakly prime ideal (in fact the only prime ideal) of L containing the ideal $I = (f]$ is $J = (p]$. We have $\sqrt[w]{I} = (p]$. Also $\sqrt[w]{\chi_I} = \chi_{\sqrt[w]{I}} = \chi_J$. We note that I is a 2-absorbing primary ideal of L. Hence by Lemma 5.3, χ_I is a fuzzy 2-absorbing primary ideal of L. We note that I is not a weakly 2-absorbing ideal of L, as $n \wedge o \wedge p = a \in I$, but $n \wedge 0 = j \notin I$, $n \wedge p = e \notin I$ and $o \wedge p = l \notin I$. We have

$$\chi_I(n \wedge o \wedge p) = 1 \nleq \chi_I(n \wedge o) \vee \chi_I(n \wedge p) \vee \chi_I(o \wedge p) = 0.$$

Thus χ_I is not a fuzzy weakly 2-absorbing ideal of L.

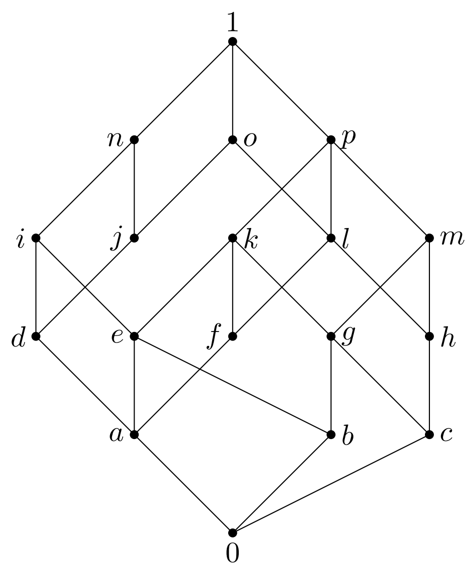


Figure 7

Lemma 5.6. *If for a fuzzy ideal μ of L, $\sqrt[w]{\mu}$ is a fuzzy weakly prime ideal, then μ is a weakly 2-absorbing primary ideal.*

Proof. Suppose that $\sqrt{_w\mu}$ is a fuzzy weakly prime ideal for some fuzzy ideal μ of L. If μ is not a weakly 2-absorbing primary ideal, then there exist $a, b, c \in L$, $a \wedge b \wedge c \neq 0$ and

$$\mu(a \wedge b \wedge c) \nleq \mu(a \wedge b) \vee \sqrt{_w\mu}(b \wedge c) \vee \sqrt{_w\mu}(a \wedge c). \tag{5.1}$$

This implies that

$$\mu(a \wedge b) \vee \sqrt{_w\mu}(b \wedge c) \vee \sqrt{_w\mu}(a \wedge c) < \mu(a \wedge b \wedge c).$$

Since $\sqrt{_w\mu}$ is fuzzy weakly prime, we have

$$\sqrt{_w\mu}(a \wedge b \wedge c) = \sqrt{_w\mu}(b \wedge c) \vee \sqrt{_w\mu}(a) = \sqrt{_w\mu}(a \wedge c) \vee \sqrt{_w\mu}(b).$$

Hence

$$\begin{aligned}\sqrt{_w\mu}(b \wedge c) \vee \sqrt{_w\mu}(a \wedge c) &= \sqrt{_w\mu}(b \wedge c) \vee \sqrt{_w\mu}(a) \vee \sqrt{_w\mu}(c)\\ &= \sqrt{_w\mu}(a \wedge b \wedge c) \vee \sqrt{_w\mu}(c)\end{aligned}$$

Thus from (5.1) we get,

$$\mu(a \wedge b) \vee \sqrt{_w\mu}(a \wedge b \wedge c) \vee \sqrt{_w\mu}(c) < \mu(a \wedge b \wedge c).$$

This implies that

$$\sqrt{_w\mu}(a \wedge b \wedge c) < \mu(a \wedge b \wedge c),$$

which is not possible. Hence μ is weakly 2-absorbing primary. ∎

The following example shows that the converse of Lemma 5.6 does not hold.

Example 5.5. Consider the lattice L shown in Figure 8. The only weakly prime ideals of L containing the ideal $I = (c]$ are $(h]$ and $(i]$. Hence $\sqrt{_wI} = (h] \cap (i] = (f]$. For any $x, y, z \in I$, $x \wedge y \wedge z \neq 0$, $x \wedge y \wedge z \in I$ implies that either $x \wedge y \in I$ or $y \wedge z \in \sqrt{_wI}$ or $x \wedge z \in \sqrt{_wI}$. Hence I is a weakly 2-absorbing primary ideal and so by Lemma 5.3, χ_I is a fuzzy weakly 2-absorbing primary ideal. We note that $d \wedge e = a \in \sqrt{_wI}$ but $d \notin \sqrt{_wI}$ and $e \notin \sqrt{_wI}$. Thus $\sqrt{_wI}$ is not a weakly prime ideal of L. Hence by Theorem 3.1, $\sqrt{_w\chi_I} = \chi_{\sqrt{_wI}}$ is not a fuzzy weakly prime ideal of L.

We omit the easy proof of the following lemma.

Lemma 5.7. *Let μ be a fuzzy ideal of L. Then $\sqrt{_w\mu} = \sqrt{_w\sqrt{_w\mu}}$.*

Theorem 5.2. *Let μ be a fuzzy ideal of L. Then $\sqrt{_w\mu}$ is fuzzy weakly prime if and only if $\sqrt{_w\mu}$ is fuzzy weakly primary.*

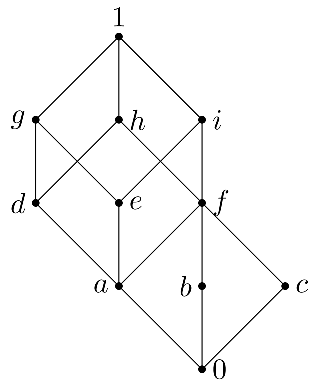


Figure 8

Proof. It follows from Lemma 5.2, that if $\sqrt{_w\mu}$ is fuzzy weakly prime, then $\sqrt{_w\mu}$ is fuzzy weakly primary.

The converse follows form the definition of a fuzzy weakly primary ideal and Lemma 5.7. ■

The proof of the following theorem follows from the definition of a fuzzy weakly 2-absorbing ideal, a fuzzy weakly 2-absorbing primary ideal and Lemma 5.7.

Theorem 5.3. *Let μ be a fuzzy ideal of L. Then $\sqrt{_w\mu}$ is fuzzy weakly 2-absorbing if and only if $\sqrt{_w\mu}$ is fuzzy weakly 2-absorbing primary.*

6. FUZZY IDEALS IN A DIRECT PRODUCT OF LATTICES

In this section, we consider fuzzy ideals in a direct product of lattices. It is known that if $L_1, \ldots, L_k$ are lattices, then their Cartesian product $L = L_1 \times L_2 \times \cdots \times L_k$ is a lattice under componentwise operations of meet and join and if $a = (a_1, \ldots, a_k)$, $b = (b_1, \ldots, b_k)$, then $a \leq b$ iff $a_i \leq b_i$ for $i = 1, \ldots, k$.

Definition 6.1. Let $L = L_1 \times L_2 \times \cdots \times L_k$ be a direct product of lattices $L_1, \ldots, L_k$. A mapping $\mu : L \to [0,1]$ is called a fuzzy set of L.

We note the following.

Theorem 6.1. *Let $L = L_1 \times L_2 \times \cdots \times L_k$ be a direct product of lattices $L_1, \ldots, L_k$. If $\mu_i, 1 \leq i \leq k$ are fuzzy ideals of L_i respectively, then $\mu : L \to [0,1]$ defined by $\mu(a_1, \ldots, a_k) = \mu_1(a_1) \wedge \cdots \wedge \mu_k(a_k)$ is a fuzzy ideal of L.*

Proof. The proof follows from the definition of the lattice operations in a direct product of lattices and that of μ. ■

Notation. We call the fuzzy set μ in Theorem 6.1 as a product of the fuzzy sets μ_i, $1 \leq i \leq k$ and write $\mu = \mu_1 \times \cdots \times \mu_k$.

The following theorem is from Nimbhorkar and Patil [9].

Theorem 6.2. *Let $L = L_1 \times L_2$ be a direct product of lattices L_1, L_2. If $\mu : L \to [0,1]$ is a fuzzy ideal of L, then there exist fuzzy ideals μ_1, μ_2 of L_1 and L_2 respectively, such that $\mu = \mu_1 \times \mu_2$. Moreover, if μ is fuzzy prime, then so are μ_1 and μ_2.*

The following lemma shows that Theorem 6.2 holds for fuzzy weakly prime ideals also.

Lemma 6.1. *Let $L = L_1 \times L_2$ be a direct product of lattices L_1, L_2. If μ is a fuzzy weakly prime ideal of L, then the fuzzy ideals μ_1, μ_2 in Theorem 6.2 are weakly prime.*

Proof. Let $x, y \in L_1$, $x \wedge y \neq 0$. We have

$$\begin{aligned}\mu_1(x \wedge y) &= \mu(x \wedge y, 0)\\ &= \mu[(x,0) \wedge (y,0)]\\ &\leq \mu[(x,0)] \wedge \mu[(y,0)]\\ &= \mu_1(x) \wedge \mu_1(y).\end{aligned}$$

Thus μ_1 is a fuzzy weakly prime ideal of L_1.

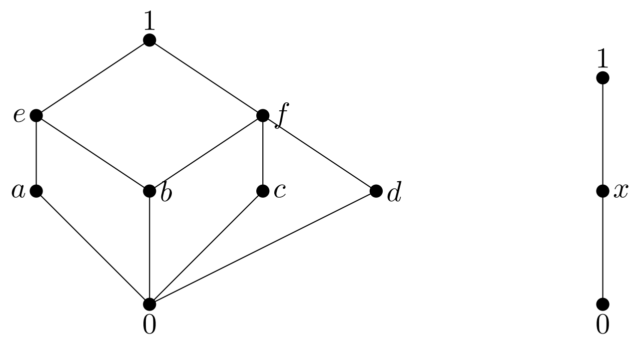
Similarly, we can show that μ_2 is a fuzzy weakly prime ideal of L_2. ■

The following example shows that a product of fuzzy weakly prime ideals need not be fuzzy weakly prime.

Example 6.1. Let $L = L_1 \times L_2$ be a direct product of lattices L_1, L_2. Let μ_1, μ_2 be fuzzy weakly prime ideals of L_1 and L_2 respectively. Then $\mu = \mu_1 \times \mu_2$ need not be a fuzzy weakly prime ideal of L. Consider the lattices L_1 and L_2 as shown in Figure 9. The ideal $I = (e]$ is a weakly prime ideal of L_1 and $J = (x]$ that of L_2. However, the ideal $I \times J = ((e,x)]$ is not a weakly prime ideal of L as $(c,x) \wedge (d,x) = (0,x) \in I \times J$ but $(c,x) \notin I \times J$ and $(d,x) \notin I \times J$.

Remark 6.1. From Example 6.1, we conclude that in general, $\sqrt{_w(\mu \times \theta)} \neq \sqrt{_w\mu} \times \sqrt{_w\theta}$.

Theorem 6.3. *Let $L = L_1 \times L_2$ be a direct product of lattices L_1, L_2. Let μ_1, μ_2 be fuzzy ideals of L_1 and L_2 respectively. Suppose that $\mu_1(0_1) = \mu_2(0_2) = 1$, where 0_1 is the least element of L_1 and 0_2 that of L_2. If $\mu = \mu_1 \times \mu_2$ is a fuzzy weakly 2-absorbing ideal of L, then μ_1 is a fuzzy weakly 2-absorbing ideal of L_1 and μ_2 that of L_2.*



L_1 L_2

Figure 9

Proof. Let $a, b, c \in L_1$, $a \wedge b \wedge c \neq 0$. Since μ is a fuzzy weakly 2-absorbing ideal of L, we have

$$\mu(a \wedge b \wedge c, 0_2) \leq \mu(a \wedge b, 0_2) \vee \mu(b \wedge c, 0_2) \vee \mu(a \wedge c, 0_2). \tag{6.1}$$

By the definition of μ, we can write (6.1) as

$$\begin{aligned}&\mu_1(a \wedge b \wedge c) \wedge \mu_2(0_2)\\ &\leq [\mu_1(a \wedge b) \wedge \mu_2(0_2)] \vee [\mu_1(b \wedge c) \wedge \mu_2(0_2)] \vee [\mu_1(a \wedge c) \wedge \mu_2(0_2).\end{aligned}$$

By using $\mu_2(0_2) = 1$, we get

$$\mu_1(a \wedge b \wedge c) \leq \mu_1(a \wedge b) \vee \mu_1(b \wedge c) \vee \mu_1(a \wedge c).$$

Thus μ_1 is a fuzzy weakly 2-absorbing ideal of L_1. Similarly, we can prove that μ_2 is a fuzzy weakly 2-absorbing ideal of L_2. ∎

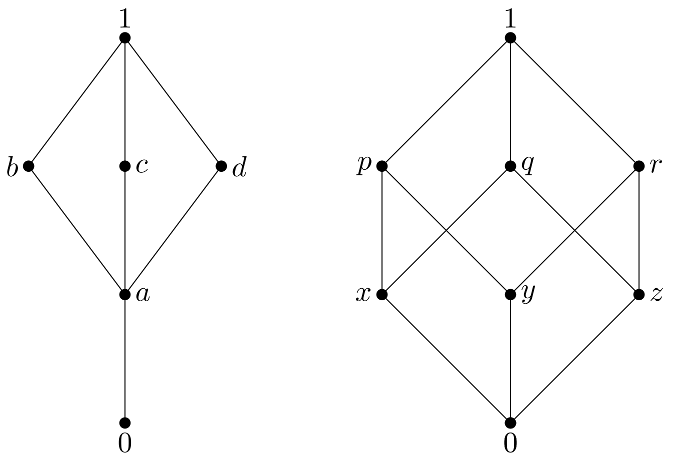
By using similar steps, we can prove the following theorem.

Theorem 6.4. *Let $L = L_1 \times L_2 \times \cdots \times L_k$ be a direct product of lattices $L_1, \ldots, L_k$. Let $\mu_i, 1 \leq i \leq k$ be fuzzy ideals of L_i, respectively. Suppose that for each $i = 1, \ldots, k$, $\mu_i(0_i) = 1$, where 0_i is the least element of L_i. If $\mu = \mu_1 \times \cdots \times \mu_k$ is a fuzzy weakly 2-absorbing ideal of L, then μ_i, is a fuzzy weakly 2-absorbing ideal of L_i, $i = 1, \ldots, k$.*

The following example shows that the converse of Theorem 6.3 need not hold.

Example 6.2. Consider the lattices L_1, L_2 and $L = L_1 \times L_2$ as shown in Figure 10. We note that the ideals $I = (a]$ and $J = (x]$ are weakly 2-absorbing ideals.

However, the ideal $K = I \times J$ is not a weakly 2-absorbing ideal of L, as $(b, 1) \wedge (c, y) \wedge (1, x) = (a, 0) \in K$ but $(b, 1) \wedge (c, y) = (a, y) \notin K$, $(c, y) \wedge (1, x) = (c, 0) \notin K$ and $(b, 1) \wedge (1, x) = (b, x) \notin K$. By Lemma 5.3, χ_I and χ_J are fuzzy 2-absorbing and $\chi_K = \chi_I \times \chi_J$ is not fuzzy weakly 2-absorbing.



L_1 L_2

Figure 10

Theorem 6.5. *Let $L = L_1 \times L_2$ be a direct product of lattices L_1, L_2. Let μ_1, μ_2 be fuzzy ideals of L_1 and L_2 respectively. Suppose that* (i) $\mu_1(0_1) = \mu_2(0_2) = 1$, *where 0_1 is the least element of L_1 and 0_2 that of L_2 and* (ii) $\mu_1(1_1) = \mu_2(1_2) = 0$, *where 1_1 is the greatest element of L_1 and 1_2 that of L_2. If $\mu = \mu_1 \times \mu_2$ is a fuzzy weakly 2-absorbing ideal of L, then μ_1 is a fuzzy weakly prime ideal of L_1 and μ_2 that of L_2.*

Proof. Suppose that μ_1 is not a fuzzy weakly prime ideal of L_1. Then there exist $a, b \in L_1$, $a \wedge b \neq 0$ such that $\mu(a \wedge b) \nleq \mu(a) \vee \mu(b)$. Consider the elements $x = (a, 1)$, $y = (1, 0)$, $z = (b, 1)$ from L. Then $x \wedge y \wedge z \neq 0$. We note the following.

$\mu(x \wedge y \wedge z) = \mu(a \wedge b, 0) = \mu_1(a \wedge b) \wedge \mu_2(0) = \mu_1(a \wedge b)$.

$\mu(x \wedge y) = \mu(a, 0) = \mu_1(a) \wedge \mu_2(0) = \mu_1(a)$.

$\mu(y \wedge z) = \mu(b, 0) = \mu_1(b) \wedge \mu_2(0) = \mu_1(b)$.

$\mu(z \wedge x) = \mu(a \wedge b, 1) = \mu_1(a \wedge b) \wedge \mu_2(1) = 0$.

Since μ is a fuzzy weakly 2-absorbing ideal, we have

$\mu(x \wedge y \wedge z) \leq \mu(x \wedge y) \vee \mu(y \wedge z) \vee \mu(z \wedge x)$,

i.e., $\mu_1(a \wedge b) \leq \mu_1(a) \vee \mu_1(b) \vee 0 = \mu_1(a) \vee \mu_1(b)$, a contradiction. Hence μ_1 is a fuzzy weakly prime ideal. Similarly, we can show that μ_2 is a fuzzy weakly prime ideal. ■

Theorem 6.6. *Let $L = L_1 \times L_2$ be a direct product of lattices L_1, L_2. Let μ_1, μ_2 be fuzzy weakly prime ideals of L_1 and L_2 respectively and $\mu = \mu_1 \times \mu_2$. Then μ is a fuzzy weakly 2-absorbing ideal of L.*

Proof. Let $x_1 = (a, x)$, $x_2 = (b, y)$, $x_3 = (c, z)$ be elements in L such that $x_1 \wedge x_2 \wedge x_3 \neq 0$. To show μ is fuzzy weakly 2-absorbing, we need to show that

$$\mu[(a, x) \wedge (b, y) \wedge (c, z)] \leq \mu[(a, x) \wedge (b, y)] \vee \mu[(b, y) \wedge (c, z)] \vee \mu[(a, x) \wedge (c, z)],$$

i.e., to show that

$$\mu(a \wedge b \wedge c, x \wedge y \wedge z) \leq \mu(a \wedge b, x \wedge y) \vee \mu(b \wedge c, y \wedge z) \vee \mu(a \wedge c, x \wedge z) \tag{6.2}$$

We have

$$\mu(a \wedge b \wedge c, x \wedge y \wedge z) = \mu_1(a \wedge b \wedge c) \wedge \mu_2(x \wedge y \wedge z).$$

As μ_1, μ_2 are fuzzy weakly prime ideals, we can write

$$\mu_1(a \wedge b \wedge c) = \mu_1(a) \vee \mu_1(b) \vee \mu_1(c)$$

and

$$\mu_2(x \wedge y \wedge z) = \mu_2(x) \vee \mu_2(y) \vee \mu_2(z).$$

Also we have

$$\begin{aligned} &\mu(a \wedge b, x \wedge y) \vee \mu(b \wedge c, y \wedge z) \vee \mu(a \wedge c, x \wedge z) \\ &= [\mu_1(a \wedge b) \wedge \mu_2(x \wedge y)] \vee [\mu_1(b \wedge c) \wedge \mu_2(y \wedge z)] \vee [\mu_1(a \wedge c) \wedge \mu_2(x \wedge z). \end{aligned} \tag{6.3}$$

Since μ_1, μ_2 are fuzzy weakly prime ideals, we can write the R. H. S. of (6.3) as

$$\begin{aligned} &\{[\mu_1(a) \vee \mu_1(b)] \wedge [\mu_2(x) \vee \mu_2(y)]\} \vee \{[\mu_1(b) \vee \mu_1(c)] \wedge [\mu_2(y) \vee \mu_2(z)]\} \\ &\vee \{[\mu_1(a) \vee \mu_1(c)] \wedge [\mu_2(x) \vee \mu_2(z)]\}. \end{aligned} \tag{6.4}$$

By applying distributivity, (6.4) can be written as

$$[\mu_1(a) \vee \mu_1(b) \vee \mu_1(c)] \wedge [\mu_2(x) \vee \mu_2(y) \vee \mu_2(z)]. \tag{6.5}$$

Thus (6.2) holds and μ is fuzzy weakly 2-absorbing. ■

Definition 6.2. A lattice L is called an integral lattice, if for nonzero $x, y \in L$, $x \wedge y \neq 0$.

Theorem 6.7. *Let $L = L_1 \times L_2$ be a direct product of lattices L_1, L_2 with 0. Suppose that L_1 is an integral lattice and μ is a nonconstant fuzzy ideal of L_1 such that $\mu(0) = 1$. The following statements are equivalent.*

(1) *$\mu \times \chi_{L_2}$ is a fuzzy 2-absorbing ideal of L.*

(2) *$\mu \times \chi_{L_2}$ is a fuzzy weakly 2-absorbing ideal of L.*

(3) *$\mu \times \chi_{L_2}$ is a fuzzy 2-absorbing ideal of L.*

(4) *μ is a fuzzy 2-absorbing ideal of L.*

(5) *μ is a fuzzy weakly 2-absorbing ideal of L.*

Proof. (1)⇒(2): Obvious.
(2)⇒(3): Let $x = (a_1, b_1)$, $y = (a_2, b_2)$ and $z = (a_3, b_3)$ be elements of L.

Case 1. Suppose that $x \wedge y \wedge z \neq 0$. Then clearly,

$$(\mu \times \chi_{L_2})(x \wedge y \wedge z) \leq (\mu \times \chi_{L_2})(x \wedge y) \vee (\mu \times \chi_{L_2})(y \wedge z) \vee (\mu \times \chi_{L_2})(x \wedge z).$$

Case 2. Suppose that $x \wedge y \wedge z = 0$. Then $a_1 \wedge a_2 \wedge a_3 = 0$. Since L_1 is an integral lattice, at least one of a_i must be 0. Without loss of generality, we may assume that $a_1 = 0$. We have to show that

$$(\mu \times \chi_{L_2})(x \wedge y \wedge z) \leq (\mu \times \chi_{L_2})(x \wedge y) \vee (\mu \times \chi_{L_2})(y \wedge z) \vee (\mu \times \chi_{L_2})(x \wedge z). \tag{6.6}$$

The L. H. S. of (6.6) can be written as

$$\begin{aligned}(\mu \times \chi_{L_2})(x \wedge y \wedge z) &= (\mu \times \chi_{L_2})(a_1 \wedge a_2 \wedge a_3, b_1 \wedge b_2 \wedge b_3) \\ &= \mu(a_1 \wedge a_2 \wedge a_3) \wedge \chi_{L_2}(b_1 \wedge b_2 \wedge b_3) \\ &= 1.\end{aligned} \tag{6.7}$$

Since we have assumed that $a_1 = 0$, we have

$$(\mu \times \chi_{L_2})(x \wedge y) = (\mu \times \chi_{L_2})(0, b_1 \wedge b_2) = \mu(0) \wedge \chi_{L_2}(b_1 \wedge b_2) = 1. \tag{6.8}$$

Hence the R. H. S. of (6.6) is equal to 1. Thus from (6.7), it follows that (6.6) holds.
(3)⇒(4): Let $a_1, a_2, a_3 \in L_1$. Let $b \in L_2$. We have

$$\begin{aligned}&\mu(a_1 \wedge a_2 \wedge a_3) = \mu(a_1 \wedge a_2 \wedge a_3) \wedge 1 \\ &= \mu(a_1 \wedge a_2 \wedge a_3) \wedge \chi_{L_2}(b) \\ &= (\mu \times \chi_{L_2})((a_1, b) \wedge (a_2, b) \wedge (a_3, b) \\ &\leq (\mu \times \chi_{L_2})((a_1, b) \wedge (a_2, b)) \vee (\mu \times \chi_{L_2})((a_2, b) \wedge (a_3, b) \\ &\vee\ (\mu \times \chi_{L_2})((a_1, b) \wedge (a_3, b) \\ &= [\mu(a_1 \wedge a_2) \wedge \chi_{L_2}(b)] \vee [\mu(a_2 \wedge a_3) \wedge \chi_{L_2}(b)] \vee [\mu(a_1 \wedge a_3) \wedge \chi_{L_2}(b) \\ &= \mu(a_1 \wedge a_2) \vee \mu(a_2 \wedge a_3) \vee \mu(a_1 \wedge a_3) \text{ as } \chi(b) = 1.\end{aligned}$$

Thus μ is fuzzy 2-absorbing.

(4)⇒(5): Obvious.

(5)⇒(1): To show that $\mu \times \chi_{L_2}$ is fuzzy 2-absorbing. Let $(a_1, b_1), (a_2, b_2), (a_3, b_3) \in L$.

Case 1. Suppose that $a_1 \wedge a_2 \wedge a_3 \neq 0$. We have

$$\begin{aligned}
&(\mu \times \chi_{L_2})((a_1, b_1) \wedge (a_2, b_2) \wedge (a_3, b_3)) \\
&= (\mu \times \chi_{L_2})(a_1 \wedge a_2 \wedge a_3, b_1 \wedge b_2 \wedge b_3) \\
&= \mu(a_1 \wedge a_2 \wedge a_3) \wedge \chi L_2(b_1 \wedge b_2 \wedge b_3) \\
&= \mu(a_1 \wedge a_2 \wedge a_3) \text{ as } \chi L_2(b_1 \wedge b_2 \wedge b_3) = 1 \\
&\leq \mu(a_1 \wedge a_2) \vee \mu(a_2 \wedge a_3) \vee \mu(a_1 \wedge a_3) \text{ as } \mu \text{ is fuzzy 2-absorbing} \\
&\leq [\mu(a_1 \wedge a_2) \wedge \chi_{L_2}(b_1 \wedge b_2)] \vee [\mu(a_2 \wedge a_3) \wedge \chi_{L_2}(b_2 \wedge b_3)] \\
&\vee\ [\mu(a_1 \wedge a_3) \wedge \chi_{L_2}(b_1 \wedge b_3)] \\
&= (\mu \times \chi_{L_2})((a_1, b_1) \wedge (a_2, b_2)) \vee (\mu \times \chi_{L_2})((a_2, b_2) \wedge (a_3, b_3)) \\
&\vee\ (\mu \times \chi_{L_2})((a_1, b_1) \wedge (a_3, b_3)).
\end{aligned}$$

Case 2. Suppose that $a_1 \wedge a_2 \wedge a_3 = 0$. Since L is an integral lattice, at least one $a_i = 0$. Without loss of generality, we assume that $a_1 = 0$. We have

$$\begin{aligned}
&(\mu \times \chi_{L_2})((a_1, b_1) \wedge (a_2, b_2) \wedge (a_3, b_3)) \\
&= (\mu \times \chi_{L_2})(a_1 \wedge a_2 \wedge a_3, b_1 \wedge b_2 \wedge b_3) \\
&= (\mu \times \chi_{L_2})(0, b_1 \wedge b_2 \wedge b_3) \\
&= \mu(0) \wedge \chi L_2(b_1 \wedge b_2 \wedge b_3) \\
&= 1.
\end{aligned}$$

We have

$$\begin{aligned}
(\mu \times \chi_{L_2})((a_1, b_1) \wedge (a_2, b_2)) &= (\mu \times \chi_{L_2})((a_1 \wedge a_2, b_1 \wedge b_2)) \\
&= (\mu \times \chi_{L_2})((0, b_1 \wedge b_2)) \\
&= \mu(0) \wedge \chi L_2(b_1 \wedge b_2) \\
&= 1.
\end{aligned}$$

Hence

$$(\mu \times \chi_{L_2})((a_1, b_1) \wedge (a_2, b_2)) \vee (\mu \times \chi_{L_2})((a_2, b_2) \wedge (a_3, b_3)) = 1.$$

Thus

$$\begin{aligned}
&(\mu \times \chi_{L_2})((a_1, b_1) \wedge (a_2, b_2) \wedge (a_3, b_3)) \\
&\leq (\mu \times \chi_{L_2})((a_1, b_1) \wedge (a_2, b_2)) \vee (\mu \times \chi_{L_2})((a_2, b_2) \wedge (a_3, b_3)).
\end{aligned}$$

Thus $\mu \times \chi_{L_2}$ is fuzzy 2-absorbing. ■

Acknowledgement

The authors are thankful to the referee for helpful comments, which improved the paper.

References

[1] N. Ajmal and K.V. Thomas, *Fuzzy lattices*, Infor. Sc. **79** (1994) 271–291. https://doi.org/10.1016/0020-0255(94)90124-4

[2] D.F. Anderson and A. Badawi, *On n-absorbing ideals of commutative rings*, Comm. Algebra **39** (2011) 1646–1672. https://doi.org/10.1080/00927871003738998

[3] D.D. Anderson and E. Smith, *Weakly prime ideals*, Houston J. Math. **29** (4) (2003) 831–840.

[4] A. Badawi, *On 2-absorbing ideals of commutative rings*, Bull. Austral. Math. Soc. **75** (2007) 417–429. https://doi.org/10.1017/S0004972700039344

[5] A. Badawi and A.Y. Darani, *On weakly 2-absorbing ideals of commutative rings*, Hoston J. Math. **39** (2) (2013) 441–452.

[6] K.T. Gaikwad, A Study of Ideals in Lattices and Ordered Structures, Ph.D. Thesis (Dr. Babasaheb Ambedkar Marathwada University, Aurangabad, India, 2018).

[7] G. Grätzer, Lattice Theory: First Concepts and Distributive Lattices (W.H. Freeman and Co. San Francisco, 1971).

[8] B.B.N. Koguep, C. Nkuimi and C. Lele, *On fuzzy prime ideals of lattice*, Samsa J. Pure and Appl. Math. **3** (2008) 1–11.

[9] S.K. Nimbhorkar and Y.S. Patil, *Generalizations of prime fuzzy ideals of a lattice*, J. Hyperstructures **9** (2) (2020) 1–33.

[10] S.H. Payrovi and S. Babaei, *On 2-absorbing ideals*, Inter. Math. Forum **7** (6) (2012) 265–271.

[11] M.P. Wasadikar and K.T. Gaikwad, *On 2-absorbing and weakly 2-absorbing ideals of lattices*, Math. Sci. Int. Res. J. **4** (2015) 82–85.

[12] M.P. Wasadikar and K.T. Gaikwad, *Some properties of 2-absorbing primary ideals in lattices*, AKCE Internat. J. Graphs and Combin. **16** (2019) 18–26. https://doi.org/10.1016/j.akcej.2018.01.015

[13] L.A. Zadeh, *Fuzzy sets*, Infor. and Cont. **8** (1965) 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X

Received 22 July 2020
Revised 11 January 2021
Accepted 19 March 2022