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k-SIMPLICITY OF LEAVITT PATH ALGEBRAS WITH COEFFICIENTS IN A k-SEMIFIELD

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Abstract

In this paper, we consider Leavitt path algebras having coefficients in a k-semifield. Concentrating on the aspect of k-simplicity, we find a set of necessary and sufficient conditions for the k-simplicity of the Leavitt path algebra $L_S(\Gamma)$ of a directed graph Γ over a non-zeroid k-semifield S.

Keywords: Leavitt path algebra, k-semifield, semiring, semifield, k-simplicity.

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1. Introduction

Since the last decade, Leavitt path algebras have drawn a great deal of interest in various disciplines. The Leavitt path algebras, in a way, trace their origin to the Leavitt algebras (introduced by Leavitt [10] in 1962, and denoted by $L(m, n)$ nowadays); which are a class of K-algebras (K being a field) universal with respect to an isomorphism property between finite-rank free modules. Then, the

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C^*-algebra $\mathcal{O}_n$ [5], the C^*-algebra $\mathcal{O}_A$ of a finite matrix A [6], the Cuntz-Krieger algebra $C^*(E)$ of a finite graph E, all paved the path for Leavitt path algebras, which involve both graphs and algebraic structures. Abrams and Aranda Pino introduced [1] the Leavitt path algebra $L_K(E)$ over a field K and a row-finite graph E in 2005. As they mentioned, the motivation was to complete the 'algebraic picture' of the interrelated fields of Leavitt Algebras, graph C^*-algebras and $\mathcal{CK}_A(K)$ (the algebraic analog of $\mathcal{O}_A$). Since then, Leavitt path algebras have been studied from both analytical and algebraic perspectives. The survey by Abrams [3] provides an extensive overview in this regard.

After initially considering row-finite graphs, Abrams and Aranda Pino defined Leavitt path algebras for any directed graph [2] in 2008. Later, Tomforde [14] considered Leavitt path algebras over commutative rings. In 2016, Katsov *et al.* [9] studied Leavitt path algebras in semiring setting.

Simplicity (i.e., absence of non-trivial ideals) is an important structural aspect of such algebraic studies. Abrams and Aranda Pino determined the condition for simplicity of $L_K(E)$ (in [1, 2]). Tomforde [14] studied *basically simple* Leavitt path algebras in the commutative ring setting. Likewise, Katsov *et al.* [9] found a set of necessary and sufficient conditions for simplicity of a Leavitt path algebra over any commutative semiring. As Katsov mentioned in [9], simple algebras are the building blocks in the structural theory of algebras. Unlike rings, ideal-simpleness and congruence-simpleness are not the same in the semiring setting. Furthermore, the yet uncharacterized simple infinite semirings may be better understood by characterization of simple Leavitt path algebras. Hence, studying Leavitt path algebras over semirings is significant on many accounts. Now Katsov showed that for a commutative semiring S, a necessary condition for $L_S(\Gamma)$ to be simple is that S should be a semifield. This acts as a motivation of studying Leavitt path algebras over semifields and its various generalizations. We earlier studied Leavitt path algebras with coefficients in Clifford semifields [13], where we looked at the full k-simplicity (the property of not having any full k-ideals). In this paper, we consider another class of commutative semirings, viz., the k-semifields.

A k-semifield is a generalization of a field in the sense that it is a commutative semiring free from a special kind of ideal called the k-ideals. It is known that kernels of semiring homomorphisms are ideals, but they are not ordinary ideals. If one considers the Bourne Congruence $\sigma_I = \{(x, y) \in S \times S \mid x + a = y + b$ for some $a, b \in I\}$ of a commutative semiring S with respect to an ideal I, then I is contained in the congruence class $\overline{I}$, i.e., the k-closure of I (defined in Section 2); and $\overline{I}$ is the kernel of the canonical homomorphism from S onto S/I $(= S/\overline{I})$. This led to the concept of k-ideals (defined later). Probably, the k stands for 'kernel'. For any ideal I of a semiring S, $\overline{I}$ is the smallest k-ideal containing I. Thus, k-ideals, characterized as kernels of homomorphisms, are often more important than ordinary ideals. The characteristic k-simplicity, i.e., absence of non-trivial k-ideals, makes k-semifields play a significant role. k-semifields not only generalize semifields (since every semifield is a k-semifield, although the converse is not true), but they also lead to many interesting characterizations and structures in the semiring theory (cf. [4, 11, 12]), often involving distributive lattices. Considering all this, it seemed worthwhile to look at the k-simplicity of $L_S(\Gamma)$ for any k-semifield S, from the perspectives of both semiring theory and Leavitt path algebra. We also note that (as shown in Section 4) simplicity implies k-simplicity, but not the other way around.

In this paper, we discuss some basic properties of k-semifields in Section 2. Some earlier results on Leavitt path algebras are given in Section 3, and then in Section 4 we consider k-semifields S with no non-zero zeroid elements, and find a set of necessary and sufficient conditions for $L_S(\Gamma)$ to be k-simple. Before we move on, we give some basic definitions and terminologies regarding graphs.

Definition 1.1. A directed graph $\Gamma = (V, E, r, s)$ consists of two sets, V $(\neq \emptyset)$ and E, and two maps $r, s : E \to V$. The elements of V are called *vertices* and the elements of E are called *edges*. For any $e \in E$, $s(e)$ and $r(e)$ are respectively called the *source* and *range* of e. If $s(e) = v$ and $r(e) = w$, then v emits e and w receives e. If $r(e_1) = s(e_2)$, then e_1 and e_2 are called *adjacent*.

We refer to directed graphs simply as graphs. For $v \in V$, $s^{-1}(v)$ and $r^{-1}(v)$ are respectively the set of all edges emitted by v and received by v. A vertex v is called a *sink* if $s^{-1}(v) = \emptyset$, and regular if $0 < |s^{-1}(v)| < \infty$. A graph is *row-finite* if $|s^{-1}(v)| < \infty, \forall v \in V$. A path $p = e_1e_2 \cdots e_n$ is a sequence of edges $e_1, e_2, \ldots, e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, 2, \ldots, n-1$. A path consisting of n edges is said to be of length n. The functions s, r are extended to paths by considering $s(e_1)$ as the source of $p = e_1e_2 \cdots e_n$, and (if p has finite length) $r(e_n)$ as the range of p. Every $v \in V$ is considered as a path of length 0, with $s(v) = v = r(v)$. $E^{(*)}$ denotes the set of all paths in Γ. A path p is called a *closed path based at* v if $s(p) = r(p) = v$. A closed path based at v is a *closed simple path based at* v if $s(e_i) \neq v \quad \forall i > 1$. $CP(v)$ denotes the set of all closed paths based at v, while $CSP(v)$ is the set of all closed simple paths based at v. A *cycle* based at v is a closed simple path based at v not visiting any vertex more than once, i.e., $s(e_i) \neq s(e_j)$ if $i \neq j$. An edge e is called an *exit* to the cycle $e_1e_2 \cdots e_n$ if $\exists$ some $i \in \{1, 2, \ldots, n\}$ such that $s(e) = s(e_i)$ but $e \neq e_i$.

2. Basic notions regarding k-semifields

In this section, we discuss some basic ideas regarding semirings and k-semifields. A semiring is an algebraic system $(S, +, \cdot)$ where S is a nonempty set and '$+$' and '$\cdot$' are two binary operations on S, called addition and multiplication respectively;

$(S, +)$ and $(S, \cdot)$ are semigroups connected to each other by ring-like distributivity. A semiring S is said to be *additively commutative* if $(S, +)$ is commutative; and simply *commutative* if it is both additively and multiplicatively commutative. A *zero* of a semiring S is an element $0 \in S$ satisfying $a + 0 = 0 + a = a$ and $a \cdot 0 = 0 \cdot a = 0$ for all $a \in S$. The set $S - \{0\}$ is denoted by S^*. An element $1 \in S$ is called the *identity* of a semiring S if $a \cdot 1 = 1 \cdot a = a$ for all $a \in S$. For more on semirings, one may see [7, 8]. Now we define a semifield.

Definition 2.1. A commutative semiring $(S, +, \cdot)$ with identity which satisfies $|S| \geq 2$ is called a *semifield* if $(S^*, \cdot)$ is a subgroup of $(S, \cdot)$.

We next consider the background of the concept of a k-semifield. In semiring theory, various generalizations of fields are found from semirings free of different types of ideals, e.g., k-ideals.

Definition 2.2. Let S be a semiring. A nonempty subset A of S is called

1. an *ideal* of S if $A + A \subseteq A$, and $SA \subseteq A, AS \subseteq A$
2. a *k-ideal* of S if A is an ideal such that for any $x, y \in S$, it happens that if $x \in A$ and either $x + y \in A$ or $y + x \in A$, then $y \in A$ (Golan, in [7], termed such an ideal as *subtractive*).

For a semiring S, the zero ideal (0) (if $0 \in S$) and S itself are the *trivial* ideals (as well as the trivial k-ideals) of S. S is called *simple* (respectively, *k-simple*) if S has no non-trivial ideals (respectively, k-ideals). For two semirings S and T, a homomorphism of S into T is a mapping $f : S \longrightarrow T$ such that $f(x + y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$ for all $x, y \in S$. If S, T have zero elements $0_S, 0_T$ respectively, then a homomorphism $f : S \to T$ is called a homomorphism of semirings with zero if $f(0_S) = 0_T$. We define the *kernel* of f by the set $ker f = \{x \in S \mid f(x) = 0_T\}$, which is easily seen to be a k-ideal of S. Next, we define the k-closure of a subset of a semiring. Let $A \subseteq S$, then the k-closure of A is defined by

$$\overline{A} = \{x \in S \mid x + y = z \text{ or } y + x = z \text{ for some } y, z \in A\}.$$

For any ideal I of S (where 0_S exists in S), the k-closure of I is the smallest k-ideal containing I. Thus, an ideal I of S (where $0_S \in S$) is a k-ideal of S if and only if $\overline{I} = I$.

Definition 2.3. The *zeroid* of a semiring $(S, +, \cdot)$ is the set $Z_S = \{a \in S \mid a + b = b \text{ or } b + a = b \text{ for some } b \in S\}$. The elements belonging to the zeroid of S are called zeroid elements of S.

Remark 2.4. The zeroid elements of a commutative semiring S have some interesting properties.

(i) If 1 is a zeroid element in S, then all elements of S are clearly zeroid elements of S.

(ii) The set Z_S forms a k-ideal of S. It is easily seen to be an ideal. Let $x, x+y \in Z_S$. Then $\exists\ c, d \in S$ such that $x + c = c$ and $x + y + d = d$. Thus, $c + d = x + c + y + d = y + c + d$. So $y \in Z_S$.

Clearly, it is reasonable to consider only those semirings in which 1 is not a zeroid element.

Definition 2.5. (i) Let S be a semiring with identity. An element $a \in S^*$ is said to be *semi-invertible* if there exist $r, s \in S$ such that $1 + ra = sa$ and $1 + ar = as$.

(ii) A commutative semiring S with identity is called a *k-semifield* (Golan [7] used the term *austere* semiring for such semirings) if every non-zero element of S is semi-invertible.

We now give a characterization of k-simple commutative semirings.

Theorem 2.6. *A commutative semiring S with identity is k-simple if and only if $\forall a \in S^*$, $\exists\ r, s \in S$ such that $1 + ra = sa$. Thus, a commutative semiring with identity is a k-semifield if and only if it has no non-trivial k-ideal; i.e., k-semifields are commutative k-simple semirings with identity.*

Remark 2.7. By definition, every semifield is a k-semifield. But not all k-semifields are semifields. For example, $S = \mathbf{Q_0^+}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbf{Q_0^+}\}$ with usual addition and multiplication is a k-semifield. However, S is not a semifield, as the element $1 + \sqrt{2}$ has no inverse in S.

Proposition 2.8. *If a k-semifield S has a non-zero zeroid element, then $Z_S = S$.*

Proof. Let $a \in Z_S - \{0\}$. Then $\exists\ b \in S$ such that $a + b = b$. As $a \neq 0$, $\exists\ r, s \in S$ such that $1 + ra = sa$. Now $a + b = b$ implies $sb = sa + sb = 1 + ra + sb$. Hence, $rb + sb = 1 + ra + rb + sb = 1 + rb + sb$ (since $ra + rb = rb$). Thus, $1 \in Z_S$, and hence, by Remark 2.4, all elements of S are zeroid elements. ∎

Remark 2.9. The above proposition shows that for a k-semifield S, either $Z_S = \{0_S\}$ (e.g., for $\mathbf{Q_0^+}(\sqrt{2})$) or $Z_S = S$ (e.g., for any distributive lattice). Thus, it seems reasonable to consider only those k-semifields in which 0 is the only zeroid element. We call them *non-zeroid k-semifields*. Hereafter, throughout the paper we shall assume that all the semirings we consider: (i) are additively commutative (ii) contain 1 and 0 with $1 \neq 0$ (iii) have 0 as their only zeroid element.

3. Definitions and some earlier results on Leavitt path algebras

In this section we discuss the basic notions regarding Leavitt path algebras defined over semirings.

Definition 3.1 [9]**.** Let $\Gamma = (V, E, s, r)$ be a graph, S be a commutative semiring with 1_S and 0_S and E^* be the set of formal symbols $\{e^* \mid e \in E\}$. The Leavitt path algebra $L_S(\Gamma)$ of the graph Γ with coefficients in S is defined to be the Universal S-algebra generated by the set of generators $V \cup E \cup E^*$ (where $e \to e^*$ is a bijection between E and E^* with $r(e) = s(e^*)$ and $r(e^*) = s(e)$, and V, E, E^* are pairwise disjoint), satisfying the following relations:

(A1) $vw = \delta_{v,w}v$ for all $v, w \in V$;

(A2) $s(e)e = e = er(e), r(e)e^* = e^* = e^*s(e)$ for all $e \in E$;

(CK1) $e^*f = \delta_{e,f}r(e)$ for all $e, f \in E$;

(CK2) $v = \sum_{e \in s^{-1}(v)} ee^*$ for any regular vertex v.

Note that elements of E and E^* are respectively, called *real edges* and *ghost edges*. Also, if S is additively commutative then so is $L_S(\Gamma)$.

Remark 3.2. Let $\Gamma = (V, E, s, r)$ be a graph, S be a commutative semiring and A be an S-algebra generated by the three subsets $\{a_v \mid v \in V\}, \{a_e \mid e \in E\}$, $\{a_{e^*} \mid e \in E\}$ of A such that:

(1) $a_v a_w = \delta_{v,w} a_v$ for all $v, w \in V$;

(2) $a_{s(e)} a_e = a_e = a_e a_{r(e)}, a_{r(e)} a_{e^*} = a_{e^*} = a_{e^*} a_{s(e)}$ for all $e \in E$;

(3) $a_{e^*} a_f = \delta_{e,f} a_{r(e)}$ for all $e, f \in E$;

(4) $a_v = \sum_{e \in s^{-1}(v)} a_e a_{e^*}$ for any regular vertex v.

Then there always exists a unique S-algebra homomorphism $\phi : L_S(\Gamma) \to A$ given by $\phi(v) = a_v, \phi(e) = a_e, \phi(e^*) = a_{e^*}$ for all $v \in V, e \in E$. This universal property ensures the uniqueness of $L_S(\Gamma)$ for a graph Γ and a semiring S.

Remark 3.3. From Definition 3.1, it is easy to deduce the following properties:

(i) $ef = er(e)s(f)f = \delta_{r(e),s(f)}ef$ for all $e, f \in E$;

(ii) $e^*f^* = \delta_{s(e),r(f)}e^*f^*$ for all $e^*, f^* \in E^*$;

(iii) $ve = \delta_{v,s(e)}e$ and $ev = \delta_{v,r(e)}e$ $\forall v \in V$, $e \in E$. (Thus, $ve \neq 0 \Rightarrow v = s(e)$ and $ev \neq 0 \Rightarrow v = r(e)$);

(iv) $ve^* = \delta_{v,r(e)}e^*$ and $e^*v = \delta_{v,s(e)}e^*$ for all $v \in V$, $e \in E^*$.

Remark 3.4 [9, Remark 2.7]**.** For a path $p = e_1e_2 \cdots e_n$, p^* is defined as $e_n^* e_{n-1}^* \cdots e_1^*$. Then

$$p^*q = \begin{cases} q' & \text{if } q = pq'; \\ r(p) & \text{if } p = q; \\ p'^* & \text{if } p = qp'; \\ 0 & \text{otherwise.} \end{cases}$$

A semiring R has a set of local units F, if F is a set of idempotents in R such that for each finite subset $\{r_1, r_2, \ldots, r_n\}$ in R, there exists an element $f \in F$ for which $fr_if = r_i$ for all $1 \leq i \leq n$. The following result shows that $L_S(\Gamma)$ either has an identity or it has a set of local units.

Proposition 3.5 [9, Proposition 2.5]**.** *Let $\Gamma = (V, E, s, r)$ be an arbitrary graph and let S be a commutative semiring. Then the S-algebra $L_S(\Gamma)$ has an identity $1(= \sum_{v \in V} v)$ if V is finite; and if V is infinite, the set of all finite sums of distinct elements of V is a set of local units of $L_S(\Gamma)$.*

Katsov *et al.* gave the general form of the monomials in $L_S(\Gamma)$, for a commutative semiring S.

Proposition 3.6 [9, Proposition 2.4]**.** *For a commutative semiring S and a graph $\Gamma = (V, E, s, r)$, the Leavitt path algebra $L_S(\Gamma)$ has the following properties:*

(i) *All elements of the set $V \cup E \cup E^*$ are non-zero.*

(ii) *If a, b are distinct elements in S, then $av \neq bv$ for all $v \in V$.*

(iii) *Every monomial in $L_S(\Gamma)$ is of the form λpq^*, where $\lambda \in S$ and p, q are paths in Γ such that $r(p) = r(q)$.*

The following result, which has been proved in our earlier paper [13], is interesting to note.

Proposition 3.7. *Let S be a commutative semiring and $\Gamma = (V, E, s, r)$ be a graph. Let c be a cycle in Γ which has no exit. If c is based at some vertex v then*

$$vL_S(\Gamma)v = \left\{ \sum_{i=-m}^{n} k_i c^i \mid m, n \in \mathbb{N}_0, k_i \in S \textit{ for } i = -m, \ldots, n \right\}$$

where $c^{-t} = (c^*)^t$ for all $t \in \mathbb{N}$, and $c^0 = v$.

A monomial is a *path in only real edges* (respectively, *path in only ghost edges*) if it contains no ghost edges (respectively, no real edges). A polynomial in only real edges (respectively, in only ghost edges) is a sum of paths in only real edges (respectively, paths in only ghost edges). The following result was used [9] for determining conditions for simplicity of $L_S(\Gamma)$ over a semifield S.

Theorem 3.8 [9, Lemma 3.2]**.** *Let $\Gamma = (V, E, s, r)$ be a graph such that every cycle in Γ has an exit. If S is a semifield and $\alpha \neq 0$ is a polynomial in $L_S(\Gamma)$ in only real edges, then $\exists\ a, b \in L_S(\Gamma)$ such that $a\alpha b \in V$.*

Definition 3.9. For a graph $\Gamma = (V, E, s, r)$, a subset $H \subseteq V$ is called a *hereditary* subset if $s(e) \in H \implies r(e) \in H$ for all $e \in E$; and $H \subseteq V$ is called *saturated* if for any regular vertex v, $r(s^{-1}(v)) \subseteq H \implies v \in H$. Clearly, $\emptyset$ and V are hereditary and saturated subsets of V.

The following necessary and sufficient conditions for the simplicity of $L_S(\Gamma)$ with coefficients in a semifield S were given by Katsov *et al.*, which involves hereditary and saturated vertex subsets.

Theorem 3.10 [9, Theorem 3.4]**.** *A Leavitt path algebra $L_S(\Gamma)$ of a graph $\Gamma = (V, E, s, r)$ with coefficients in a semifield S is simple if and only if both of the following conditions are satisfied:*

(i) *The only hereditary and saturated subsets of V are $\emptyset$ and V.*

(ii) *Every cycle in Γ has an exit.*

In fact, Katsov *et al.* provided the following result for any commutative semiring S.

Theorem 3.11 [9, Theorem 3.5]**.** *The Leavitt path algebra $L_S(\Gamma)$ of a graph $\Gamma = (V, E, s, r)$ with coefficients in a commutative semiring S is simple if and only if all the following conditions hold:*

(i) *S is a semifield.*

(ii) *The only hereditary and saturated subsets of V are $\emptyset$ and V.*

(iii) *Every cycle in Γ has an exit.*

4. k-SIMPLICITY OF $L_S(\Gamma)$ FOR A k-SEMIFIELD S

For a k-semifield S, Theorem 3.11 gives a set of necessary and sufficient conditions for $L_S(\Gamma)$ to be simple. However, not every ideal is a k-ideal, so it might happen that $L_S(\Gamma)$ is not simple but is k-simple (a semiring is k-simple if it has no non-trivial k-ideal), as seen in the following example.

Example 4.1. Consider the semiring $T = \mathbf{Q}_\mathbf{0}^+(\sqrt{2})$ with usual addition and multiplication. From [7, Proposition (6.45)], there is an inclusion preserving bijection f between the ideals of any semiring S and the ideals of $M_n(S)$; and an ideal of $M_n(S)$ is a k-ideal of $M_n(S)$ if and only if the corresponding ideal of S is a k-ideal of S. Now since T is a k-semifield, it has no non-trivial k-ideal (by Theorem 2.4), and thus, the same is true for $M_n(T)$. Hence, $M_n(T)$ is k-simple. However, T, being not a semifield, does contain a non-trivial ideal I, and consequently, $M_n(T)$ has a non-trivial ideal $f(I)$. Hence, $M_n(T)$ is not simple. Now let Γ_n denote the finite line graph.

It is known (cf. [1, Example 1.4(i)]) that $L_T(\Gamma_n) \cong M_n(T)$, via the maps $v_n \mapsto E_{nn}, v_i \mapsto E_{ii}, e_i \mapsto E_{i,i+1}, e_i^* \mapsto E_{i+1,i}$ for $i = 1, 2, \ldots, n-1$; E_{ij} being the matrix with 1 in its $(i,j)^{th}$ position and 0 elsewhere. Thus, $L_T(\Gamma_n)$ is k-simple but not simple. This shows that we can find a k-semifield S (which is not a semifield) and a graph Γ such that $L_S(\Gamma)$ is k-simple but not simple.

$$\bullet^{v_1} \xrightarrow{e_1} \bullet^{v_2} \xrightarrow{e_2} \bullet^{v_3} \cdots\cdots\cdots\cdots \bullet^{v_{n-1}} \xrightarrow{e_{n-1}} \bullet^{v_n}$$

Figure 1. Finite line graph Γ_n.

In view of the above case, we specifically consider the k-simplicity of $L_S(\Gamma)$ for a k-semifield S.

Proposition 4.2. *Let $\Gamma = (V, E, s, r)$ be a graph and S be a k-semifield. Then any non-zero k-ideal I of $L_S(\Gamma)$ contains a polynomial in only real edges.*

Proof. The result follows immediately by [9, Proposition 3.3] since S is a commutative semiring and I is an ideal of $L_S(\Gamma)$. ∎

Proposition 4.3. *Let $\Gamma = (V, E, s, r)$ be a graph with the property that every cycle in Γ has an exit. Let S be a k-semifield. If $\alpha \in L_S(\Gamma)$ is a polynomial in only real edges with $\alpha \neq 0$, then there exist $a, b \in L_S(\Gamma)$ such that $a\alpha b = \lambda v$ for some $\lambda\ (\neq 0_S) \in S$ and $v \in V$.*

Proof. We begin as in the proof of Theorem 3.8 given in [9]. The overall proof here (in particular, the initial steps) is exactly similar to our proof of the corresponding result for Clifford semifields [13]. However, we present a sketch of the proof for the sake of completeness. The polynomial α is written in the form $\alpha = \sum_i \lambda_i q_i$, where the q_i's are distinct paths in only real edges and $0_S \neq \lambda_i \in S$ for all i. We choose a path p from the set $\{q_i\}$ such that no proper initial subpath of p is contained in $\{q_i\}$. Let $v = r(p)$. Then, by Remark 3.4, we have that $p^*\alpha v = \lambda v + \sum_i \lambda_i p^* q_i$, where the sum runs over all q_i's having p as one of their proper initial subpaths and $r(q_i) = v$ (ensuring $p^* q_i \in CP(v)$). Denoting $p^*\alpha v$ by α_1, we get $\alpha_1 = \lambda v + \sum_{i=1}^n \lambda_i p_i$, where p_i is a closed path of positive length based at v and $0_S \neq \lambda \in S$. Fixing some $c \in CSP(v)$, any $p_i \in CP(v)$ can be written as $p_i = c^{n_i} p_i'$ with $n_i \in \mathbb{N}$ being maximal. Let $n = max\{n_i \mid i = 1, 2, \ldots, n\} + 1$. Then $(c^*)^n \alpha_1 c^n = \lambda v + \sum_j \lambda_j c^{n_j}$ with $n_j > 0$. Hence, $\alpha' = (c^*)^n \alpha_1 c^n = \lambda v + cP(c)$ for some polynomial P. Suppose $c = e_1 e_2 \cdots e_m$. By our hypothesis and [1, Lemma 2.5], c has an exit $g \in \Gamma$. Let $s(g) = s(e_j)$ with $g \neq e_j$. We consider the path $z = e_1 e_2 \cdots g$. Clearly, $s(z) = v$ and $z^* c = 0$. Now we have that $z^*\alpha' z = z^*\lambda v z + z^* cP(c) z = \lambda z^* v z = \lambda z^* s(z) z = \lambda z^* z = \lambda r(z)$. Writing $a = z^*(c^*)^n p^*$ and $b = v c^n z$, we get that $a\alpha b = \lambda r(z)$. ∎

Corollary 4.4. *Let $\Gamma = (V, E, s, r)$ be a graph where every cycle has an exit. If S is a k-semifield and $\alpha \neq 0$ is a polynomial in only real edges in a k-ideal J of $L_S(\Gamma)$, then J contains a vertex.*

Proof. By Proposition 4.3, there exist $a, b \in L_S(\Gamma)$ such that $a\alpha b = \lambda v$ for some $\lambda \in S - \{0_S\}$ and some $v \in V$. Now $\alpha \in J$ implies that $\lambda v = a\alpha b \in J$. As $\lambda \neq 0_S$ and S is a k-semifield, there exist $t_1, t_2 \in S$ such that $1 + \lambda t_1 = \lambda t_2$. Then, $t_2 a\alpha b = \lambda t_2 v = (1 + \lambda t_1)v = v + \lambda t_1 v$. Since $\lambda t_1 v \in J$ and $t_2 a\alpha b \in J$, by the property of k-ideal it follows that $v \in J$. Thus, J contains a vertex. ∎

Now, we give a set of sufficient conditions for k-simplicity of $L_S(\Gamma)$ over a k-semifield S.

Lemma 4.5. *For a graph $\Gamma = (V, E, s, r)$, the Leavitt path algebra $L_S(\Gamma)$ with coefficients in a k-semifield S is k-simple if both of the following conditions are satisfied:*

(i) *The only hereditary and saturated subsets of V are $\emptyset$ and V.*

(ii) *Every cycle in Γ has an exit.*

Proof. Let J be a non-zero k-ideal of $L_S(\Gamma)$. By Proposition 4.2, J contains a non-zero polynomial in only real edges. Then, by condition (ii) and Corollary 4.4, $J \cap V \neq \emptyset$. Considering the ideal J and the commutative semiring S, we have by [9, Lemma 2.6] that $J \cap V$ is hereditary and saturated in V. The condition (i) then gives that $J \cap V = V$, implying $V \subseteq J$. Hence, by Proposition 3.5, J contains an identity or a set of local units of $L_S(\Gamma)$. So $J = L_S(\Gamma)$. Thus, $L_S(\Gamma)$ is k-simple. ∎

As shown next, the condition (i) of Lemma 4.5 is a necessary condition for k-simplicity of $L_S(\Gamma)$.

Lemma 4.6. *For a graph $\Gamma = (V, E, s, r)$, if the Leavitt path algebra $L_S(\Gamma)$ with coefficients in a k-semifield S is k-simple, then the only hereditary and saturated subsets of V are $\emptyset$ and V.*

Proof. Let $L_S(\Gamma)$ be k-simple. We assume that V has a hereditary and saturated subset H which is non-trivial (i.e., $H \neq V, \emptyset$) and see if this leads to any contradiction. Consider the graph $F = (F^0, F^1, r_F, s_F)$, where $F^0 = V - H$, $F^1 = r^{-1}(V - H)$, $r_F = r|_{V-H}$ and $s_F = s|_{V-H}$. Clearly, $r_F(F^1) \cup s_F(F^1) \subseteq F^0$. Hence, F is well-defined. We then produce an S-algebra homomorphism $\Psi : L_S(\Gamma) \to L_S(F)$. To do so, define Φ on the generators of the free S-algebra $B = S[V \cup E \cup E^*]$ by setting $\Phi(v) = \chi_{F^0}(v)v, \Phi(e) = \chi_{F^1}(e)e$ and $\Phi(e^*) = \chi_{(F^1)^*}(e^*)$ for all $v \in V, e \in E, e^* \in E^*$, and extending it to B (note that χ_A is the characteristic function of a set A). Now we check if the relations (1)–(4) mentioned in Definition 3.1 are preserved under Ψ. Clearly, it suffices to check the same for Φ. We can proceed similarly to the converse part of the proof of [1, Theorem 3.11], to show that the aforementioned conditions are preserved under Ψ. Next, we consider the k-ideal $Ker(\Psi)$ of $L_S(\Gamma)$. Arguing as in the proof of [1, Theorem 3.11], we infer that $Ker(\Psi)$ is a proper non-trivial k-ideal of $L_S(\Gamma)$, contradicting that $L_S(\Gamma)$ is k-simple. Thus, we get a contradiction if such a subset H of V exists. So $\emptyset$ and V are the only hereditary and saturated subsets of V. ∎

As we shall see next, the condition (ii) of Lemma 4.5 also is a necessary condition for k-simplicity of $L_S(\Gamma)$ if S is a k-semifield. We now give our main result, which gives a set of necessary and sufficient conditions for $L_S(\Gamma)$ to be k-simple, where S is a (non-zeroid) k-semifield.

Theorem 4.7. *Let S be a (non-zeroid) k-semifield, and $\Gamma = (V, E, s, r)$ be a graph. Then $L_S(\Gamma)$ is k-simple if and only if both of the following conditions are satisfied:*

(i) *The only hereditary and saturated subsets of V are $\emptyset$ and V.*

(ii) *Every cycle in Γ has an exit.*

Proof. First, let the conditions (i) and (ii) hold. Then, by Lemma 4.5, $L_S(\Gamma)$ is k-simple. Conversely, let $L_S(\Gamma)$ be k-simple. Then, by Lemma 4.6, the condition (i) holds. Now we assume that there exists a cycle c in Γ with no exit, and see if this leads to any contradiction. Let c be based at v. As c has no exit, we have that $CSP(v) = \{c\}$. For the same reason, $cc^* = v$ (note that for any edge e of c, e is the only edge having $s(e)$ as its source, and thus $ee^* = s(e)$) and $c^*c = v$. We next consider the k-ideal $I = \overline{\langle v + c \rangle}$, which is the k-closure of the ideal $\langle v + c \rangle$. Clearly, I is non-zero.

Case I. Let $v \notin I$. Then $I \neq L_S(\Gamma)$. Thus, $L_S(\Gamma)$ has a non-trivial k-ideal I. This contradicts that $L_S(\Gamma)$ is k-simple. So a cycle without any exit in Γ leads to a contradiction, and we are through.

Case II. Let $v \in I$. Then there exist monic monomials $\alpha_i, \beta_i, \gamma_j, \delta_j$ such that

$$v + \sum_{i=1}^{n} k_i \alpha_i (v + c) \beta_i = \sum_{j=1}^{m} l_j \gamma_j (v + c) \delta_j \tag{4.1}$$

where $k_i, l_j \in S$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. Now noting that $v(v + c)v = v$, we can assume that (by multiplying both sides by v if necessary) $v\alpha_i v = \alpha_i, v\beta_i v = \beta_i$ for $i = 1, 2, \ldots, n$ and $v\gamma_j v = \gamma_j$ and $v\delta_j v = \delta_j$ for $j = 1, 2, \ldots, m$. This shows that the monomials $\alpha_i, \beta_i, \gamma_j, \delta_j$ are elements of $vL_S(\Gamma)v$. By Proposition 3.7, it then follows that (noting that $v = c^0$ and also that $(v + c)$ commutes with c and c^*) both $\sum_{i=1}^{n} k_i \alpha_i (v + c) \beta_i$ and $\sum_{j=1}^{m} l_j \gamma_j (v + c) \delta_j$ are products of $(v + c)$ with some polynomial in c, c^*. Thus, we have that

$$v + (v + c)P(c, c^*) = (v + c)Q(c, c^*).$$

Writing $(c^*)^n = c^{-n}$ for any $n \in \mathbb{N}$, let $Q(c,c^*) = b_{-m}c^{-m} + \cdots + b_{-1}c^{-1} + b_0 v + b_1 c^1 + \cdots + b_y c^y$ and $P(c,c^*) = a_{-n}c^{-n} + \cdots + a_{-1}c^{-1} + a_0 v + a_1 c^1 + \cdots + a_t c^t$, where $a_i \neq 0$ for $i = -n, \ldots, t$ and $b_i \neq 0$ for $i = -m, \ldots, y$. From (4.1), we then have that

(4.2)
$$a_{-n}c^{-n} + \left(\sum_{i=-n+1}^{-1} (a_{i-1} + a_i)c^i\right) + (1 + a_{-1} + a_0)v + \left(\sum_{i=1}^{t} (a_{i-1} + a_i)c^i\right) + a_t c^{t+1}$$
$$= b_{-m}c^{-m} + \left(\sum_{i=-m+1}^{-1} (b_{i-1} + b_i)c^i\right) + (b_{-1} + b_0)v + \left(\sum_{i=1}^{y} (b_{i-1} + b_i)c^i\right) + b_y c^{y+1}.$$

Comparing both sides, we have that $m = n$ and $t = y$. Now we can equate the coefficients of identical powers of c in both sides, since the equation (4.2) involves elements in a free S-semimodule with basis $\{c^i \mid i \in \mathbb{Z}\}$. Comparing coefficients, we obtain that $a_{-n} = b_{-n}, a_t = b_t, a_i + a_{i-1} = b_i + b_{i-1}$ for $-n+1 \leq i \leq -1$ and for $1 \leq i \leq t$. By invoking Proposition 3.6, we also get that

$$1 + a_{-1} + a_0 = b_{-1} + b_0. \tag{4.3}$$

Now let $a = a_{-n} + a_{-n+1} + \cdots + a_{-2}$, and $b = a_t + a_{t-1} + \cdots + a_1$. Then we have that
$$(1 + (b + a_0) + (a + a_{-1}))v = ((b + b_0) + (a + b_{-1}))\, v.$$

Putting $d = b + a_0 + a + a_{-1}$ (and noting that $a + a_{-1} = a + b_{-1}$ and $b + a_0 = b + b_0$), we get by invoking Proposition 3.6 that $1 + d = d$. This is a contradiction since 1 is not a zeroid element (cf. Remark 2.9). Thus, the existence of a cycle without an exit in Γ leads us to a contradiction. Hence, every cycle must have an exit in Γ, satisfying the condition (ii). This completes the proof. ∎

We conclude the paper by illustrating the utility of Theorem 4.7 by an example. We can use Theorem 4.7 to check the k-simplicity of $L_S(\Gamma_n)$ over a k-semifield S (where Γ_n is the finite line graph). As discussed in Example 4.1, $L_S(\Gamma_n)$ is k-simple. However, this can also be verified with the help of Theorem 4.7, since Γ_n clearly satisfies conditions (i) and (ii) mentioned in Theorem 4.7.

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