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GENERALIZED CENTROID OF HYPERRINGS

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Abstract

In this paper, the notion of generalized centroid is applied to hyperrings. We show that the generalized centroid C of a semiprime hyperring R is a regular hyperring. Also, we show that if C is a hyperfield, then R is a prime hyperring.

Keywords: hyperring, semiprime hyperring, generalized centroid, regular hyperring.

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1. Introduction

Algebraic hyperstructures are a generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements of a set is again an element of the same set, while in an algebraic hyperstructure, the composition of two elements is a non-empty subset of the same set. The theory of hyperstructures was introduced by Marty in 1934 at the 8th Congress of the Scandinavian Mathematicians [11]. Marty defined hypergroups, began to analyze their properties and applied them to groups. Hyperstructures have many applications to several sectors of both pure and applied mathematics in [3] and [4]. The hyperrings were introduced and studied by Krasner, Nakasis, Mirvakili, and especially studied by Davvaz and Leoreanu-Fotea in [5]. Also, we refer the reader to see [6–8, 12–18]. A well-known type of a hyperring is called the Krasner hyperring [9]. Krasner hyperring is essentially ring with approximately modified axioms in which addition is hyperoperation, while the multiplication is an operation. This type of hyperrings has been studied by a variety of authors. In [1], the idempotent elements of Krasner hyperrings were studied by Asokkumar. The notion of Martindale rings of quotients, jointly with the notion of the extended centroid, play an important role in the study of prime rings satisfying a generalized polynomial identity [10]. After this study, many authors have investigated the properties of extended centroid in various rings. In [19], Öztürk and Jun proved that generalized centroid of a semiprime Γ-ring is a regular Γ-ring. Our paper is concerned about the centroid of semiprime Krasner hyperrings and it is constructed as follows. After an Introduction, Section 2 consists the Preliminaries that are needed in the investigations. In Section 3, we show that the generalized centroid C of a semiprime hyperring R is regular hyperring. Also, we show that if C is a hyperfield, then R is prime hyperring.

2. Preliminaries

In this section we give some definitions and results of hyperstructures which we need to develop our paper. A mapping $\circ : H \times H \to P^*(H)$ is called a hyperoperation, where $P^*(H)$ is the set of non-empty subsets of H. An algebraic system $(H, \circ)$ is called a hypergroupoid.

Let A and B be any two non-empty subsets of H and $x \in H$, we define

$$A \circ B = \bigcup_{a\in A,\ b\in B} a \circ b,\ A \circ x = A \circ \{x\}\,,\ \ x \circ B = \{x\} \circ B.$$

A hyperoperation "$\circ$" is called associative if for all $a, b, c \in H$, $a\circ(b\circ c) = (a\circ b)\circ c$, which means that

$$\bigcup_{u\in b\circ c} a \circ u = \bigcup_{v\in a\circ b} v \circ c.$$

A hypergroupoid with the associative hyperoperation is called a semihypergroup.

A hypergroupoid $(H, \circ)$ is a quasihypergroup, whenever $a \circ H = H = H \circ a$ for all $a \in H$. If $(H, \circ)$ is semihypergroup and quasihypergroup, then $(H, \circ)$ is called a hypergroup.

Definition 1 [5]**.** A Krasner hyperring is an algebraic structure $(R, +, \cdot)$ which satisfies the following axioms:

(1) $(R, +)$ is a canonical hypergroup, i.e.,

(i) $(x + y) + z = x + (y + z)$, for every $x, y, z \in R$,

(ii) $x + y = y + x$, for every $x, y \in R$,

(iii) For all $x \in R$, there exists $0 \in R$ such that $0 + x = \{x\}$

(iv) For every $x \in R$ there exists an unique element denoted by $-x \in R$ such that $0 \in x + (-x)$,

(v) For every $x, y, z \in R$, $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$;

(2) $(R, \cdot)$ is a semigroup having zero as a bilaterally absorbing element, i.e.,

(i) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$, for every $x, y, z \in R$,

(ii) For all $x \in R$, $x \cdot 0 = 0 \cdot x = 0$;

(3) The multiplication is distributive with respect to the hyperoperation $+$, i.e., for every $x, y, z \in R$, $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x + y) \cdot z = x \cdot z + y \cdot z$.

From definition of $-A = \{-a \mid a \in A\}$, we have $-(-x) = x$ and $-(x + y) = -x - y$. In definition, for simplicity of notations we write sometimes xy instead of $x \cdot y$ and in (iii), $0 + x = x$ instead of $0 + x = \{x\}$.

In a hyperring R, if there exists an element $1 \in R$ such that $1a = a1 = a$ for every $a \in A$, then the element 1 is called the identity element of the hyperring R. If $ab = ba$ for every $a, b \in R$ then the hyperring R is called a commutative hyperring.

An element a of a hyperring R is called idempotent if $a = a^2$. A hyperring R is called a Boolean hyperring if every element of R is an idempotent.

An element a of a hyperring R is regular if and only if there exists an element a' of R satisfying $aa'a = a$. A hyperring R is called regular if and only if each element of R is regular.

A hyperring R is called a hyperdomain if R does not have zero divisors. In other words, for $x, y \in R$ if $xy = 0$ then either $x = 0$ or $y = 0$.

A Krasner hyperring is called a Krasner hyperfield, if $(R \backslash \{0\}, \cdot)$ is a group.

Throughout this paper, by a hyperring we mean that Krasner hyperring.

Let R be a hyperring. A non-empty subset S of R is called a subhyperring of R, if $x - y \subseteq S$ and $xy \in S$ for all $x, y \in S$.

A subhyperring I of a hyperring R is a left (resp. right) hyperideal of R if $ra \in I$ (resp. $ar \in I$) for all $r \in R$, $a \in I$. A hyperideal of R is both a left and a right hyperideal.

Lemma 1 [5]**.** *A non-empty subset A of a hyperrring R is a left* (*right*) *hyperideal if and only if*

(1) $a,\ b \in A$ *implies* $a - b \subseteq A$,

(2) $a \in A,\ r \in R$ *imply* $ra \in A$ $(ar \in A)$.

Let A and B be non-empty subsets of a hyperring R

$$A + B = \{x \mid x \in a + b \ \text{ for some } \ a \in A,\ b \in B\}$$

and

$$AB = \left\{ x \mid x \in \sum_{i=1}^{n} a_i b_i,\ \ a_i \in A,\ \ b_i \in B,\ \ n \in \mathbb{Z}^+ \right\}.$$

If A and B are hyperideals of R, then $A + B$ and AB are also hyperideals of R.

A hyperideal P of a hyperring R is called prime hyperideal, if for hyperideals I and J of R, satisfying $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$. A hyperideal I of a hyperring R is called semiprime hyperideal if for any hyperideal H of R, satisfying $H^2 \subseteq I$ implies $H \subseteq I$. Prime hyperideals are surely semiprime hyperideals. A hyperring R is called a prime hyperring if $aRb = 0$ for all $a, b \in R$ implies $a = 0$ or $b = 0$. Also, R is called semiprime if $aRa = 0$ implies that $a = 0$. Clearly, every prime hyperring is a semiprime hyperring but the converse is not always true.

Remark 1. Note that every prime ring is a prime hyperring and every semiprime ring is a semiprime hyperring.

Example 1.

(1) Any domain is a prime ring.

(2) Any simple ring is a prime ring, and more generally: every left or right primitive ring is a prime ring.

(3) Any matrix ring over an integral domain is a prime ring. In particular, the ring of 2×2 integer matrices is a prime ring.

Example 2. Let $R = \{0, x\}$ with the hyperoperation "+" and the multiplication "$\cdot$" given in the following tables:

$+$	0	x
0	0	x
x	x	$\{0, x\}$

$\cdot$	0	x
0	0	0
x	0	x

Then, R is a semiprime hyperring.

Example 3. Let $R = \{0, x, y\}$ with the hyperoperation and the multiplication given in the following tables:

$+$	0	x	y
0	0	x	y
x	x	x	R
y	y	R	y

$\cdot$	0	x	y
0	0	0	0
x	0	x	y
y	0	y	x

Then, R is a semiprime hyperring.

Definition 2 [17]**.** Let R be a hyperring. For a non-empty subset A of R,

$$Ann_l(A) = \{r \in R \mid ra = 0 \text{ for all } a \in A\}$$

is called the left annihilator hyperideal of A. A right annihilator hyperideal $Ann_r(A)$ can be defined similarly.

Definition 3 [5]**.** Let R_1 and R_2 be hyperrings. A mapping φ from R_1 into R_2 is said to be a good homomorphism if for all $a, b \in R_1$,

$$\varphi(a+b) = \varphi(a) + \varphi(b), \ \ \varphi(ab) = \varphi(a)\varphi(b) \ \text{ and } \ \varphi(0) = 0.$$

A good homomorphism φ is an isomorphism if φ is one to one and onto. If there exists isomorphism between hyperrings R_1 and R_2, we write $R_1 \cong R_2$.

Since R_1 is a hyperring, $0 \in a - a$ for all $a \in R_1$, then we have $\varphi(0) \in \varphi(a) + \varphi(-a)$ or $0 \in \varphi(a) + \varphi(-a)$ which implies that $\varphi(-a) \in -\varphi(a) + 0$, therefore $\varphi(-a) = -\varphi(a)$ for all $a \in R_1$. Moreover, if φ is a good homomorphism from R_1 into R_2, then the kernel of φ is the set $\{x \in R_1 \mid \varphi(x) = 0\}$. It is trivial that $\ker\varphi$ is a hyperideal of R_1 and $\mathrm{Im}(\varphi) = \{\varphi(r) \mid r \in R\}$ is a subhyperring of R_2.

Corollary 1. *Let φ be a good homomorphism from R_1 into R_2. Then φ is one to one if and only if* $\ker\varphi = \{0\}$*.*

Let R be a hyperring. A canonical hypergroup $(M, +)$ together with the map $\cdot : R \times M \to M$ is called a left hypermodule over R if for all $r_1, r_2 \in R$, $m_1, m_2 \in M$ the following axioms hold:

(1) $r_1(m_1 + m_2) = r_1m_1 + r_1m_2$,

(2) $(r_1 + r_2)m_1 = r_1m_1 + r_2m_1$,

(3) $(r_1r_2)m_1 = r_1(r_2m_1)$,

(4) $0_Rm_1 = 0_M$.

A subhypermodule N of M is a subhypergroup of M which is closed under multiplication by elements of R.

Definition 4 [5]**.** Let M and N be two R-hypermodules. A function $f : M \to N$ that satisfies the conditions:

(1) $f(x + y) \subseteq f(x) + f(y)$,

(2) $f(xr) = f(x)r$, for all $r \in R$ and all $x, y \in M$ is called be a right R-homomorphism from M into N.

In definition, if the equality holds, then f is called a good right R-homomorphism.

Definition 5 [20]**.** Let R be a prime hyperring and Q_r the quotient hyperring of R. The set

$$C := \{ g \in Q_r \mid gf = fg \text{ for all } f \in Q_r\}$$

is called the extended centroid of a hyperring R.

Theorem 1 [20]**.** *The extended centroid C of R is a hyperfield.*

3. Main results

Let R be a semiprime hyperring. Let us denote set of all non-zero hyperideal of R which have zero annihilator in R by M. That is,

$$M = \{U \mid U \neq \{0_R\} \text{ is a hyperideal of } R \text{ such that } AnnU = \{0_R\}\}.$$

In this case, the set M is closed under multiplication. Let U, $V \in M$. Then, $UVx = \{0_R\}$, for all $x \in R$ yields $Vx \subseteq AnnU = \{0_R\}$, that is $Vx = \{0_R\}$. Hence $x \in AnnV = \{0_R\}$ which implies $x = 0_R$. Therefore we have $UV \in M$.

Define a relation "$\approx$" on

$$\Gamma = \{f_U \mid f : U \to R \text{ is a good right } R\text{-homomorphism and } U \in M\}$$

as follows

$$f_U \approx g_V :\Leftrightarrow \text{ there exists } K \in M \text{ and } K \subseteq U \cap V \text{ such that } f = g \text{ on } K.$$

Since the set M is closed under multiplication, it is possible to find such a hyperideal $K \in M$ and $\approx$ is an equivalence relation on Γ. This gives a chance for us to get a partition of Γ. We denote the equivalence class by $\overline{f}_U$, where $\overline{f}_U := \{ g : V \to R | \, f_U \approx g_V\}$ and denote by Q_r set of all equivalence classes. We define a hyperaddition "$+$" on Q_r as follows

$$\overline{f}_U + \overline{g}_V := \overline{(f+g)}_{U\cap V}$$

for all $\overline{f}_U, \overline{g}_V \in Q_r$,where $f + g : U \cap V \to R$ is a good right R homomorphism. Assume that $f_{1U_1} \approx f_{2U_2}$ and $g_{1V_1} \approx g_{2V_2}$. Then $\exists K_1(\in M) \subseteq U_1 \cap U_2$ such that $f_1 = f_2$ on K_1 and $\exists K_2(\in M) \subseteq V_1 \cap V_2$ such that $g_1 = g_2$ on K_2. Taking $K = K_1 \cap K_2$ and so $K \in M$. For any $x \in K$, we have

$$\begin{aligned}(f_1 + g_1)(x) &= f_1(x) + g_1(x) = \bigcup \{t(x) \mid t(x) \in f_1(x) + g_1(x)\} \\ &= \bigcup \{t(x) \in f_2(x) + g_2(x)\} = f_2(x) + g_2(x) = (f_2 + g_2)(x),\end{aligned}$$

and so $f_1 + g_1 = f_2 + g_2$ on K. Therefore $f_1 + g_{1_{U_1 \cap V_1}} \approx f_2 + g_{2_{U_2 \cap V_2}}$, which means that the addition in Q_r is well-defined.

Now we will prove that Q_r is a canonical hypergroup. Let $\overline{f}_U$, $\overline{g_V}$, $\overline{h}_W$ be elements of Q_r. Since $U \cap (V \cap W) = (U \cap V) \cap W$, for all $x \in U \cap (V \cap W)$

$$\begin{aligned}
[(f+g)+h]\,(x) = (f+g)(x)+h(x) &= \bigcup_{t(x)\in(f+g)(x)} t(x)+h(x) \\
&= \bigcup_{t(x)\in(f+g)(x)} \{k(x) \mid k(x) \in t(x)+h(x)\} \\
&= \bigcup \{k(x) \mid k(x) \in (f(x)+g(x))+h(x)\} \\
&= \bigcup \{k(x) \mid k(x) \in f(x)+(g(x)+h(x))\} \\
&= \bigcup_{p(x)\in g(x)+h(x)} \{k(x) \mid k(x) \in f(x)+p(x)\} \\
&= \bigcup_{p(x)\in(g+h)(x)} f(x)+p(x) = f(x)+(g+h)(x) \\
&= [f+(g+h)]\,(x).
\end{aligned}$$

Hence $(f+g)+h = f+(g+h)$ on $U \cap (V \cap W)$. That is $\left(\overline{f}_U + \overline{g}_V\right) + \overline{h}_W = \overline{f}_U + \left(\overline{g}_V + \overline{h}_W\right)$. One can easily check that $\overline{f}_U + \overline{g}_V = \overline{g}_V + \overline{f}_U$. Taking $\overline{\theta_R} \in Q_r$ where $\theta : R \to R$, $x \mapsto 0$ for all $x \in R$. Since $U \subseteq U \cap R$, $(\theta+f)(x) = \theta(x)+f(x) = 0+f(x) = f(x)$ for all $x \in U$. Then we have $\overline{\theta_R}+\overline{f}_U = \overline{f}_U$ and similarly $\overline{f}_U+\overline{\theta}_R = \overline{f}_U$ for all $\overline{f}_U \in Q_r$. Hence $\overline{\theta_R}$ is the additive identity in Q_r. Let $-\overline{f}_U \in Q_r$, where $-f : U \to R$, $x \mapsto -f(x) = (-f)(x)$ for all $x \in U$. Since $-f(x)$ is the unique inverse of $f(x)$ in R, we have $\theta(x) \in f(x) - f(x) = f(x) + (-f)(x)$ for all $x \in U$. So $\overline{\theta_R} \in \overline{f}_U + (-\overline{f}_U)$. Finally, let $\overline{f}_U, \overline{g}_V, \overline{h}_W$ be elements of Q_r and $\overline{h}_W \in \overline{f}_U + \overline{g}_V$. Then, there exists $f_1 \in \overline{f}_U$ and $g_1 \in \overline{g}_V$ such that $h = f_1 + g_1$. For any $x \in K (\in M) \subseteq U \cap V$, we get $h(x) = (f_1+g_1)(x) = f_1(x)+g_1(x) \subseteq f(x)+g(x)$. Since R is a hyperring, $h(x) \in f(x)+g(x)$ implies $g(x) \in -f(x)+h(x)$ and $f(x) \in h(x) - g(x)$. Thus we have $g(x) \in (-f+h)(x)$ and $f(x) \in (h-g)(x)$. That is, $\overline{g}_V \in -\overline{f}_U + \overline{h}_W$ and $\overline{f}_U \in \overline{h}_W - \overline{g}_V$. Therefore $(Q_r, +)$ is a canonical hypergroup.

Now we define a multiplication "." on Q_r as follows: for all $\overline{f}_U, \overline{g}_V \in Q_r$

$$\overline{f}_U \overline{g}_V := \overline{fg}_{VU}$$

where $fg : VU \to R$ is a good right R-homomorphism. Assume that $f_{1U_1} \approx f_{2U_2}$ and $g_{1V_1} \approx g_{2V_2}$. Then $\exists K_1 (\in M) \subseteq U_1 \cap U_2$ such that $f_1 = f_2$ on K_1 and $\exists K_2 (\in M) \subseteq V_1 \cap V_2$ such that $g_1 = g_2$ on K_2. Also $V_1U_1 \cap V_2U_2 \subseteq (U_1 \cap U_2) \cap (V_1 \cap V_2) = (U_1 \cap V_1) \cap (U_2 \cap V_2)$ and there exists $\{0_R\} \neq K \in M$ such that $K \subseteq V_1U_1 \cap V_2U_2$. For any $x \in K$, $x \in V_1U_1 \cap V_2U_2$. Thus $x \in V_1U_1$ and

$x \in V_2U_2$. Then $x = \sum_{finite} a_ib_i$, $a_i \in V_1 \cap V_2$, $b_i \in U_1 \cap U_2$. Therefore,

$$\begin{aligned}(f_1g_1)(x) &= f_1(g_1(x)) = f_1\left(g_1\left(\sum_{finite} a_ib_i\right)\right)\\ &= f_1\left(\left(\sum_{finite} g_1(a_i)b_i\right)\right) = f_1\left(\left(\sum_{finite} g_2(a_i)b_i\right)\right)\\ &= f_2\left(\left(\sum_{finite} g_2(a_i)b_i\right)\right) = f_2\left(g_2\left(\sum_{finite} a_ib_i\right)\right)\\ &= f_2(g_2(x)) = (f_2g_2)(x).\end{aligned}$$

Thus $f_1g_1 = f_2g_2$ on K. Hence "$\cdot$" is well-defined. Now we will prove that $(Q_r, \cdot)$ is a semigroup having zero as a bilaterally element. Let $\overline{f}_U, \overline{g}_V, \overline{h}_W \in Q_r$. Since $W(VU) = (WV)U$, we get for all $x \in W(VU)$,

$$\begin{aligned}[(fg)h](x) &= (fg)(h(x)) = f(g(h(x)))\\ &= f((gh)(x)) = [f(gh)](x).\end{aligned}$$

Hence $(fg)h = f(gh)$ on $W(VU)$. That is, $\left(\overline{f}_U\overline{g}_V\right)\overline{h}_W = \overline{f}_U\left(\overline{g}_V\overline{h}_W\right)$. Now we prove that $\overline{f}_U\ \overline{\theta}_R = \overline{\theta}_R = \overline{\theta}_R\overline{f}_U$ for all $\overline{f}_U \in Q_r$. Since $RU \subseteq RU \cap R$ and $f\theta = \theta$ on RU, we get $\overline{f}_U\ \overline{\theta}_R = \overline{\theta}_R$. Similarly $\overline{\theta}_R\overline{f}_U = \overline{\theta}_R$.

Let $\overline{f}_U, \overline{g}_V, \overline{h}_W$ be elements of Q_r. Since $(V \cap W)U \subseteq VU \cap WU$, we get for all $x \in (V \cap W)U$,

$$\begin{aligned}[f(g+h)](x) &= f((g+h)(x)) = f(g(x)+h(x))\\ &= f(g(x)) + f(h(x)) = (fg+fh)(x).\end{aligned}$$

Then $f(g+h) = fg+fh$ on $(V \cap W)U$. That is, $\overline{f}_U\left(\overline{g}_V + \overline{h}_W\right) = \overline{f}_U\overline{g}_V + \overline{f}_U\overline{h}_W$. Similarly, $\left(\overline{f}_U + \overline{g}_V\right)\overline{h}_W = \overline{f}_U\overline{h}_W + \overline{g}_V\overline{h}_W$.

Therefore, $(Q_r, +, \cdot)$ is a Krasner hyperring.

Taking $\overline{1}_R \in Q_r$ where $1 : R \to R$, $x \mapsto x$ for all $x \in R$. Let $\overline{f}_U \in Q_r$. Since $RU \subseteq U$, we get for all $x \in RU$, $(f1)(x) = f(1(x)) = f(x)$ and $(1f)(x) = 1(f(x)) = f(x)$. Thus, we have $\overline{f}_U\overline{1}_R = \overline{1}_R\overline{f}_U = \overline{f}_U$, that is $\overline{1}_R$ is the multiplicative identity in Q_r.

Hence, $(Q_r, +, \cdot)$ is a hyperring with multiplicative identity $\overline{1}_R$.

Let R be semiprime hyperring. For a fixed element a in R, consider a mapping $\lambda_a : R \to R$ by $\lambda_a(r) = ar$ for all $r \in R$. It is easy to prove that the mapping λ_a is a good right R-homomorphism. Define a mapping $\Psi : R \to Q_r$ by $\Psi(a) = \overline{\lambda}_{a_R}$ for $a \in R$. Clearly the mapping Ψ is injective good homomorphism and so R is a subring of Q_r, and in this case, we call Q_r the right quotient hyperring of R. The left quotient hyperring Q_l can characterize in similar manner. For purpose of convenience, we use q instead of $\overline{q}_U \in Q_r$.

Definition 6. Let $(R, +, \cdot)$ be a semiprime hyperring. Then

$$C := \{g \in Q_r \mid gf = fg \text{ for all } f \in Q_r\}$$

is called the generalized centroid of a hyperring R.

Remark 2. Assume that $q = f_U \in C$. For all $r \in R$, $\lambda_{r_R} f_U = f_U \lambda_{r_R}$ and so there exists $K(\in M) \subseteq RU$ such that $\lambda_r f = f\lambda_r$ on K. From here, $(\lambda_r f)(x) = (f\lambda_r)(x)$ for all $x \in K$, i.e., $rf(x) = f(rx)$. Hence f acts as an R-homomorphism on K.

The following theorem characterizes the quotient hyperring Q_r of R. The proof is similar to the proof of the corresponding theorem in ring theory, so we omit it.

Theorem 2. *Let R be a semiprime hyperring and Q_r be the quotient hyperring of R. Then the hyperring Q_r satisfies the following properties:*

(i) *Q_r is a semiprime hyperring.*

(ii) *For any element q of Q_r, there exists a hyperideal of $U_q \in M$ which has zero annihilator with a good right R-homomorphism $q : U \to R$, such that $q(U_q) \subseteq R$ (or $qU_q \subseteq R$).*

(iii) *If $q \in Q_r$ and $q(U_q) = \{0_R\}$ for a certain $U_q \in M$ ($qU_q = \{0_R\}$ for a certain $U_q \in M$), then $q = 0$.*

(iv) *If $U \in M$ and $\Psi : U \to R$ is a good right R-homomorphism, then there exists an element $q \in Q_r$ such that $\Psi(u) = q(u)$ for all $u \in U$ (or $\Psi(u) = qu$ for all $u \in U$).*

(v) *Let W be a subhypermodule in Q_r and $\Psi : W \to Q_r$ a good right R-homomorphism. If W contains the hyperideal U of R such that $\Psi(U) \subseteq R$ and $AnnU = Ann_r W$, then there is an element $q \in Q_r$ such that $\Psi(b) = q(b)$ for any $b \in W$ (or $\Psi(b) = qb$ for any $b \in W$) and $q(a) = 0$ for any $a \in Ann_r W$ (or $qa = 0_R$ for any $a \in Ann_r W$).*

Theorem 3. *Let R be a semiprime hyperring and C the generalized centroid of R. Then C is a regular hyperring.*

Proof. Let a be an element of C. Then $a, a^2 \in Q_r$ and so we get that U_a and U_{a^2} are non-zero hyperideals which have zero annihilators in R. Hence $J = U_a \cap U_{a^2} \in M$ we consider the mapping $\Psi : J \to R$ defined by $\Psi(a^2x) = ax$ where x runs through the set J. Let $a^2x = 0$. Then $(ax)R(ax) = 0$ implies $ax = 0$. Therefore, Ψ is a good right R-homomorphism. There exists $a_1 \in Q_r$ such that $a_1a^2x = ax$ for all $x \in J$. Hence, we have $0 \in a_1a^2x - ax = (a_1a^2 - a)x$ and so $a_1a^2 = a$. Let us prove that the element a_1 in C. Let q be an arbitrary element of Q_r. Then, we get $(a_1a^2)^2q = q(a_1a^2)^2$ and so $a^4a_1^2q = a^4qa_1^2$. Multiplying this equality from left by aa_1^3, we get $aa_1q = aqa_1$. Thus, we get $0 \in aa_1q - aqa_1 = a(a_1q - qa_1)$. So, $0 \in (a_1q - qa_1)a(a_1q - qa_1)$. This implies that $0 \in a_1q - qa_1$ by Theorem 2. That is, $a_1q = qa_1$. This completes the proof. ■

Remark 3. We show that all elements of C are regular. If $a \in C$, then there exists an element a_1 in C such that $a_1a^2 = a$. Hence we get $(a_1a)^2 = (a_1a)(a_1a) = a_1(aa_1a) = a_1a$. Thus we have $e = a_1a$ is an idempotent and so $ea = a$. Therefore, C has a sufficient number of idempotents the relation $\leq$ defined by

$$e_1 \leq e_2 \Leftrightarrow e_2e_1 = e_1$$

is a partial order.

Definition 7. Let R be a semiprime hyperring, Q_r the quotient hyperring of R and $S \subseteq Q_r$. The least of idempotent elements $e(S) = e \in C$ such that $es = s$ for all $s \in S$ is called the support of the set S.

Lemma 2. *Let R be a semiprime hyperring, Q_r the quotient hyperring of R and $S \subseteq Q_r$. If S has a support $e(S) = e \in C$, then the equality $qRS = 0$ for an element $q \in Q_r$ is equivalent to $qe(S) = 0$.*

Proof. Let V be an R-subhypermodule in Q_r which is generated by S. Then, $U = V \cap R$ is a hyperideal of the hyperring R. We prove that its annihilator in the R coincides with the annihilator of V in R. Let $qU = 0$. If $v \in V$, then $vU_v \in U$ and so $qvU_v = 0$. In this case, $qv = 0$, by Theorem 2(iv). From Theorem 2(v), we get for the identical mapping $\Psi : V \to V$ there exists an element $e \in Q_r$ such that $ev = v$ for all $v \in V$ and e annihilates the annihilator L of the set V in the hyperring Q_r. Hence, for any $1 \in L$, $v \in V$ and $q \in Q_r$

$$0 \in (eq - qe)(1 + v)\ ,\ 0 \in (e^2 - e)(1 + v).$$

Since the annihilator of $L + V$ has a zero multiplication, $e \in C$ and e is an idempotent. If e_1 is a central idempotent such that $e_1s = s$ for all $s \in S$, then $e_1v = v$ for $v \in V$ and so $1 - e_1 \in L$, i.e.,

$$0 \in e(1 - e_1) = e - ee_1 \Rightarrow ee_1 = e$$
$$e_1e = e\ (e_1 \in C) \Rightarrow e \leq e_1 \text{ by Remark 3.}$$

Finally, let $qRS = 0$. Then qR is in the annihilator of V and so $qRe = 0$ which implies $Rqe = 0$ and $qe = 0$. This completes the proof. ■

Lemma 3. *Let R be a semiprime hyperring, Q_r the quotient hyperring of R and $S \subseteq Q_r$ that has a support $e(S) = e \in C$. If $0 \neq e_1 \leq e(S)$, then $e_1S \neq 0$.*

Proof. If $e_1 S = 0$, then the idempotent $f \in 1 - e_1$ such that $fs \in (1 - e_1)s = s - e_1 s$. Hence $fs = s$ for all $s \in S$. Therefore $f \geq e(S) \geq e_1$ i.e., $fe_1 = e_1 = 0$. This is a contradiction. ■

Proposition 1. *Let R be a semiprime hyperring and C the extended centroid of R. If C is a hyperfield, then the hyperring R is a prime hyperring.*

Proof. Suppose that $xRy = 0$. Hence, we get $e(x)y = 0$ by Lemma 3. Therefore, $e(x)Ry = 0$. Since C is hyperfield, $Ry = 0$ which implies $y = 0$. The proof is completed. ■

4. CONCLUSIONS

In this study, we deal with a special type of hyperrring; Krasner hyperrings. This paper has shown that the generalized centroid C of a semiprime hyperring R is a regular hyperring. Furthermore, it was found that if C is a hyperfield, then R is a prime hyperring.

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