Discussiones Mathematicae
General Algebra and Applications 42 (*2022*) *217–235*
https://doi.org/10.7151/dmgaa.1385

INTERIOR GE-FILTERS OF GE-ALGEBRAS

Seok-Zun Song

Department of Mathematics
Jeju National University, Jeju 63243, Korea

e-mail: szsong@jejunu.ac.kr

Ravikumar Bandaru[1]

Department of Mathematics
GITAM (*Deemed to be University*)
Hyderabad Campus, Telangana-502329, India

e-mail: ravimaths83@gmail.com

Daniel A. Romano

International Mathematical Virtual Institute
6, Kordunaška Street, 78000 Banja Luka, Bosnia and Herzegovina

e-mail: bato49@hotmail.com

and

Young Bae Jun

Department of Mathematics Education
Gyeongsang National University, Jinju 52828, Korea

e-mail: skywine@gmail.com

Abstract

The notions of an interior GE-filter, a weak interior GE-filter and a belligerent interior GE-filter are introduced, and their relations and properties are investigated. Example of a GE-filter which is neither an interior GE-filter nor a weak interior GE-filter is provided. Relations between a weak interior GE-filter and an interior GE-filter are discussed, and conditions under which every weak interior GE-filter is an interior GE-filter are investigated. Relations between a belligerent interior GE-filter and an interior GE-filter are displayed, and conditions for an interior GE-filter to be a belligerent interior

[1]Correponding author.

GE-filter are considered. Given a subset and an element, an interior GE-filter is established, and conditions for a subset to be a belligerent interior GE-filter are discussed. The extensibility of the beligerant interior GE-filter is debated. Relationships between weak interior GE-filter and belligerent interior GE-filter of type 1, type 2 and type 3 are founded.

Keywords: (transitive, left exchangeable) GE-algebra, GE-filter, belligerent GE-filter, (weak) interior GE-filter, belligerent interior GE-filter (of type 1, type 2 and type 3).

2010 Mathematics Subject Classification: 03G25, 06F35.

1. Introduction

Henkin and Skolem introduced Hilbert algebras in the fifties for investigations in intuitionistic and other non-classical logics. Diego [4] proved that Hilbert algebras form a variety which is locally finite. Bandaru *et al.* introduced the notion of GE-algebras which is a generalization of Hilbert algebras, and investigated several properties (see [1, 2]). The notion of interior operator is introduced by Vorster [8] in an arbitrary category, and it is used in [3] to study the notions of connectedness and disconnectedness in topology. Interior algebras are a certain type of algebraic structure that encodes the idea of the topological interior of a set, and are a generalization of topological spaces defined by means of topological interior operators. Rachůnek and Svoboda [6] studied interior operators on bounded residuated lattices, and Svrcek [7] studied multiplicative interior operators on GMV-algebras. Lee *et al.* [5] applied the interior operator theory to GE-algebras, and they introduced the concepts of (commutative, transitive, left exchangeable, belligerent, antisymmetric) interior GE-algebras and bordered interior GE-algebras, and investigated their relations and properties. They found and presented many examples to illustrate these concepts, and used the set of interior GE-algebras to make up a semigroup. They gave examples to show that the set of interior GE-algebras is not a GE-algebra. They established the internal GE-algebra using a bordered interior GE-algebra with certain conditions, and provided examples describing this.

The aim of this section is to introduce the notions of an interior GE-filter, a weak interior GE-filter and a belligerent interior GE-filter, and investigate their relations and properties. We provide example of a GE-filter which is neither an interior GE-filter nor a weak interior GE-filter. We discuss relations between a weak interior GE-filter and an interior GE-filter. We investigate the conditions under which every weak interior GE-filter is an interior GE-filter. We discuss relations between a belligerent interior GE-filter and an interior GE-filter. We consider conditions for an interior GE-filter to be a belligerent interior GE-filter. Given a subset and an element, we establish an interior GE-filter, and we consider conditions for a subset to be a belligerent interior GE-filter. We discuss the extensibility of the beligerent interior GE-filter. We establish relationships between weak interior GE-filter and belligerent interior GE-filter of type 1, type 2 and type 3.

2. Preliminaries

Definition 2.1 [1]**.** By a *GE-algebra* we mean a non-empty set X with a constant 1 and a binary operation $*$ satisfying the following axioms:
(GE1) $u * u = 1$,
(GE2) $1 * u = u$,
(GE3) $u * (v * w) = u * (v * (u * w))$
for all $u, v, w \in X$.

In a GE-algebra X, a binary relation "$\leq$" is defined by

$$(\forall x, y \in X)\,(x \leq y \;\Leftrightarrow\; x * y = 1)\,. \tag{2.1}$$

Definition 2.2 [1, 2]**.** A GE-algebra X is said to be

- *transitive* if it satisfies:

$$(\forall x, y, z \in X)\,(x * y \leq (z * x) * (z * y))\,, \tag{2.2}$$

- *left exchangeable* if it satisfies:

$$(\forall x, y, z \in X)\,(x * (y * z) = y * (x * z))\,, \tag{2.3}$$

- *belligerent* if it satisfies:

$$(\forall x, y, z \in X)\,(x * (y * z) = (x * y) * (x * z))\,. \tag{2.4}$$

Proposition 2.3 [1]**.** *Every GE-algebra X satisfies the following items.*

$$(\forall u \in X)\,(u * 1 = 1)\,. \tag{2.5}$$
$$(\forall u, v \in X)\,(u * (u * v) = u * v)\,. \tag{2.6}$$
$$(\forall u, v \in X)\,(u \leq v * u)\,. \tag{2.7}$$
$$(\forall u, v, w \in X)\,(u * (v * w) \leq v * (u * w))\,. \tag{2.8}$$
$$(\forall u \in X)\,(1 \leq u \Rightarrow u = 1)\,. \tag{2.9}$$
$$(\forall u, v \in X)\,(u \leq (v * u) * u)\,. \tag{2.10}$$
$$(\forall u, v \in X)\,(u \leq (u * v) * v)\,. \tag{2.11}$$
$$(\forall u, v, w \in X)\,(u \leq v * w \Leftrightarrow v \leq u * w)\,. \tag{2.12}$$

If X is transitive, then

(2.13) $$(\forall u, v, w \in X)\,(u \leq v \;\Rightarrow\; w * u \leq w * v,\; v * w \leq u * w)\,.$$

(2.14) $$(\forall u, v, w \in X)\,(u * v \leq (v * w) * (u * w))\,.$$

Lemma 2.4 [1]**.** *In a GE-algebra X, the following facts are equivalent each other.*

(2.15) $$(\forall x, y, z \in X)\,(x * y \leq (z * x) * (z * y))\,.$$

(2.16) $$(\forall x, y, z \in X)\,(x * y \leq (y * z) * (x * z))\,.$$

Definition 2.5 [1]**.** A subset F of a GE-algebra X is called a *GE-filter* of X if it satisfies:

(2.17) $$1 \in F,$$

(2.18) $$(\forall x, y \in X)(x * y \in F,\; x \in F \Rightarrow y \in F).$$

Lemma 2.6 [1]**.** *In a GE-algebra X, every GE-filter F of X satisfies:*

(2.19) $$(\forall x, y \in X)\,(x \leq y,\; x \in F \Rightarrow y \in F)\,.$$

Definition 2.7 [2]**.** A subset F of a GE-algebra X is called a *belligerent GE-filter* of X if it satisfies (2.17) and

(2.20) $$(\forall x, y, z \in X)(x * (y * z) \in F,\; x * y \in F \Rightarrow x * z \in F).$$

3. Interior GE-filters

The aim of this section is to introduce the notions of an interior GE-filter and a weak interior GE-filter, and investigate their properties.

Definition 3.1 [5]**.** By an *interior GE-algebra* we mean a pair (X, f) in which X is a GE-algebra and $f : X \to X$ is a mapping such that

(3.1) $$(\forall x \in X)(x \leq f(x)),$$

(3.2) $$(\forall x \in X)((f \circ f)(x) = f(x)),$$

(3.3) $$(\forall x, y \in X)(x \leq y \Rightarrow f(x) \leq f(y)).$$

Definition 3.2. Let (X, f) be an interior GE-algebra. A GE-filter F of X is said to be *interior* if it satisfies:

(3.4) $$(\forall x \in X)(f(x) \in F \Rightarrow x \in F).$$

Example 3.3. Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation $*$ which is given in the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	c	c
b	1	1	1	d	d
c	1	a	b	1	1
d	1	a	a	1	1

If we define a mapping f on X as follows:

$$f : X \to X, \ x \mapsto \begin{cases} 1 & \text{if } x \in \{1, c, d\}, \\ a & \text{if } x \in \{a, b\}, \end{cases}$$

then (X, f) is an interior GE-algebra and $F = \{1, c, d\}$ is an interior GE-filter in (X, f).

Example 3.4. Let $X_1 := \mathbb{N}$ be the set of all natural numbers and $*$ be the binary operation on X_1 defined by

$$x * y = \begin{cases} y & \text{if } x = 1, \\ 1 & \text{otherwise.} \end{cases}$$

Then $(X_1, *, 1)$ is a GE-algebra. Let $X_2 := \{1, a, b, c, d\}$ be a set with the following Cayley table:

$\circ$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	c	c
b	1	1	1	d	d
c	1	a	b	1	1
d	1	a	a	1	1

Then $(X_2, \circ, 1)$ is a GE-algebra. We can observe that $(X_1 \times X_2, \bullet, 1_{X_1 \times X_2})$, where $(x, y) \bullet (u, v) = (x * u, y \circ v)$ and $1_{X_1 \times X_2} = (1_{X_1}, 1_{X_2})$, is a GE-algebra. If we define a mapping f on $X_1 \times X_2$ as follows:

$$f : X_1 \times X_2 \to X_1 \times X_2, \ (x, y) \mapsto \begin{cases} (x, 1) & \text{if } (x, y) \in \{(x, 1), (x, c), (x, d)\}, \\ (x, a) & \text{if } (x, y) \in \{(x, a), (x, b)\}, \end{cases}$$

then $(X_1 \times X_2, f)$ is an interior GE-algebra.

It is clear that X is an interior GE-filter in every interior GE-algebra (X, f). But the trivial GE-filter $\{1\}$ is not interior. In fact, in Example 3.3, the trivial GE-filter $\{1\}$ is not interior since $f(c) = 1 \in \{1\}$ but $c \notin \{1\}$.

Theorem 3.5. *In an interior GE-algebra, the intersection of interior GE-filters is also an interior GE-filter.*

Proof. Let $\{F_i \mid i \in \Lambda\}$ be a set of interior GE-filters in an interior GE-algebra (X, f) where Λ is an index set. It is clear that $\cap\{F_i \mid i \in \Lambda\}$ is a GE-filter of X. Let $x \in X$ be such that $f(x) \in \cap\{F_i \mid i \in \Lambda\}$. Then $f(x) \in F_i$ for all $i \in \Lambda$. Since F_i is an interior GE-filter in (X, f) for all $i \in \Lambda$, it follows from (3.4) that $x \in F_i$ for all $i \in \Lambda$. Thus $x \in \cap\{F_i \mid i \in \Lambda\}$, and therefore $\cap\{F_i \mid i \in \Lambda\}$ is an interior GE-filter in (X, f). ∎

The following example shows that the union of interior GE-filters may not be an interior GE-filter.

Example 3.6. Consider a GE-algebra $X = \{1, a, b, c, d, e\}$ with the binary operation $*$ which is given in the following table:

$*$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	1	1	1	e
b	1	c	1	c	1	1
c	1	d	b	1	d	1
d	1	c	b	c	1	e
e	1	a	b	c	d	1

Then it is routine to verify that (X, f) is an interior GE-algebra where

$$f : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ b & \text{if } x \in \{a, b\}, \\ d & \text{if } x = d, \\ e & \text{if } x \in \{c, e\}. \end{cases}$$

Let $F = \{1, d\}$ and $G = \{1, c, e\}$. Then F and G are interior GE-filters in (X, f). But $F \cup G = \{1, c, d, e\}$ is not a GE-filter of X since $d * a = c \in F \cup G, d \in F \cup G$ but $a \notin F \cup G$ and hence not an interior GE-filter in (X, f).

Definition 3.7. Let (X, f) be an interior GE-algebra. A subset F of X is called a *weak interior GE-filter* in (X, f) if it satisfies (2.17) and

$$(\forall x, y \in X)(x * f(y) \in F,\ f(x) \in F \Rightarrow y \in F). \tag{3.5}$$

Example 3.8. Let $X = \{1, a, b, c, d\}$ be a set with the binary operation $*$ given in the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	c	c
b	1	1	1	d	d
c	1	1	b	1	1
d	1	1	1	1	1

Then X is a GE-algebra. Define a mapping f as follows:

$$f : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ a & \text{if } x \in \{a, b\}, \\ d & \text{if } x \in \{c, d\}, \end{cases}$$

Then (X, f) is an interior GE-algebra and $F := \{1, a, b\}$ is a weak interior GE-filter in (X, f).

Example 3.9. Consider the GE-algebra $X_1 := \mathbb{N}$ whcih is given in Example 3.4 and let $X_2 := \{1, a, b, c, d\}$ be a GE-algebra with the following Cayley table:

$\circ$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	1	d
b	1	1	1	c	1
c	1	b	b	1	1
d	1	a	a	1	1

Then $(X_1 \times X_2, \bullet, 1_{X_1 \times X_2})$ is a GE-algebra, where $(x, y) \bullet (u, v) = (x * u, y \circ v)$ for every $(x, y), (u, v) \in X_1 \times X_2$ and $1_{X_1 \times X_2} = (1_{X_1}, 1_{X_2})$. If we define a mapping f on $X_1 \times X_2$ as follows:

$$f : X_1 \times X_2 \to X_1 \times X_2,\ (x, y) \mapsto \begin{cases} (x, 1) & \text{if } (x, y) = (x, 1), \\ (x, c) & \text{if } (x, y) \in \{(x, a), (x, c)\}, \\ (x, d) & \text{if } (x, y) \in \{(x, b), (x, d)\}, \end{cases}$$

then $(X_1 \times X_2, f)$ is an interior GE-algebra. It is routine to verify that the set $F := \{(x, 1), (x, b) \mid x \in X_1\}$ is a weak interior GE-filter in $(X_1 \times X_2, \bullet, 1_{X_1 \times X_2})$.

The following example shows that there is a GE-filter which is neither an interior GE-filter nor a weak interior GE-filter.

Example 3.10. Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation $*$ which is given in the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	c	c
b	1	1	1	d	d
c	1	a	b	1	1
d	1	a	a	1	1

Then it is routine to verify that (X, f) is an interior GE-algebra where

$$f : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x \in \{1, c, d\}, \\ a & \text{if } x \in \{a, b\}, \end{cases}$$

and the set $F := \{1, a, b\}$ is a GE-filter of X. But it is not an interior GE-filter in (X, f) since $f(c) = 1 \in F$ but $c \notin F$. Also it is not a weak interior GE-filter in (X, f) since $b * f(c) = b * 1 = 1 \in F$ and $f(b) = a \in F$ but $c \notin F$.

Theorem 3.11. *In an interior GE-algebra, the intersection of weak interior GE-filters is also a weak interior GE-filter.*

Proof. Let $\{F_i \mid i \in \Lambda\}$ be a set of weak interior GE-filters in an interior GE-algebra (X, f) where Λ is an index set. It is clear that $1 \in \cap\{F_i \mid i \in \Lambda\}$. Let $x, y \in X$ be such that $x * f(y) \in \cap\{F_i \mid i \in \Lambda\}$ and $f(x) \in \cap\{F_i \mid i \in \Lambda\}$. Then $x * f(y) \in F_i$ and $f(x) \in F_i$ for all $i \in \Lambda$. Since F_i is a weak interior GE-filter in (X, f), it follows from (3.5) that $y \in F_i$ for all $i \in \Lambda$. Hence $y \in \cap\{F_i \mid i \in \Lambda\}$, and therefore $\cap\{F_i \mid i \in \Lambda\}$ is a weak interior GE-filter in (X, f). ∎

We discuss relations between a weak interior GE-filter and an interior GE-filter.

Theorem 3.12. *In an interior GE-algebra, every interior GE-filter is a weak interior GE-filter.*

Proof. Let F be an interior GE-filter in an interior GE-algebra (X, f). Let $x, y \in X$ be such that $x * f(y) \in F$ and $f(x) \in F$. Then $x \in F$ by (3.4). Since F is a GE-filter of X, it follows from (2.18) that $f(y) \in F$. Hence $y \in F$ by (3.4), and therefore F is a weak interior GE-filter in (X, f). ∎

The following example shows that any weak interior GE-filter may not be an interior GE-filter in an interior GE-algebra. Hene the converse of Theorem 3.12 is not true in general.

Example 3.13. Let $X = \{1, a, b, c, d\}$ be a set with the binary operation $*$ given in the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	1	d
b	1	1	1	c	1
c	1	b	b	1	1
d	1	a	a	1	1

Then X is a GE-algebra. Define a mapping f as follows:

$$f : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ c & \text{if } x \in \{a, c\}, \\ d & \text{if } x \in \{b, d\}, \end{cases}$$

Then (X, f) is an interior GE-algebra and $F := \{1, b\}$ is a weak interior GE-filter in (X, f). But it is not a GE-filter of X since $b * d = 1 \in F$ and $b \in F$ but $d \notin F$. Hence F is not an interior GE-filter in (X, f).

In the following theorem, we investigate the conditions under which every weak interior GE-filter is an interior GE-filter.

Theorem 3.14. *If F is a weak interior GE-filter in an interior GE-algebra (X, f) which is also a GE-filter of X, then F is an interior GE-filter in (X, f).*

Proof. Let F be a weak interior GE-filter in an interior GE-algebra (X, f) which is also a GE-filter of X. Let $x \in X$ be such that $f(x) \in F$. Then $1 * f(x) = f(x) \in F$ and $f(1) = 1 \in F$. It follows from (3.5) that $x \in F$. Therefore F is an interior GE-filter in (X, f). ■

Let (X, f) be an interior GE-algebra. Given a non-empty subset F of X, the interior GE-filter in (X, f) generated by F is defined to be the smallest interior GE-filter in (X, f) containing F, and it is denoted by $\langle F \rangle_f$.

Example 3.15. Consider the interior GE-algebra (X, f) in Example 3.10. For a subset $F = \{1, c\}$ of X, we have $\langle F \rangle_f = \{1, c, d\}$.

We present a question about the interior GE-filter generated by a set as follows:

Question 3.16. *How is the interior GE-filter generated by a subset of an interior GE-algebra described?*

4. Belligerent interior GE-filters

In this section, we introduce the concept of a belligerent interior GE-filter and discuss relations between an interior GE-filter and a belligerent interior GE-filter.

Definition 4.1. Let (X, f) be an interior GE-algebra. Then a subset F of X is called a *belligerent interior GE-filter* in (X, f) if F is a belligerent GE-filter of X which satisfies the condition (3.4).

Example 4.2. Consider the interior GE-algebra (X, f) in Example 3.6. It is routine to verify that $F := \{1, d\}$ is a belligerent interior GE-filter in (X, f).

Theorem 4.3. *In an interior GE-algebra (X, f), every belligerent interior GE-filter is an interior GE-filter.*

Proof. It is straightforward because every belligerent GE-filter is a GE-filter in a GE-algebra. ■

The following example shows that any interior GE-filter may not be a belligerent interior GE-filter.

Example 4.4. Consider a GE-algebra $X = \{1, a, b, c\}$ with the binary operation $*$ which is given in the following table:

$*$	1	a	b	c
a	1	1	1	1
1	1	a	b	c
b	1	1	1	c
c	1	b	b	1

Then it is routine to verify that (X, f) is an interior GE-algebra where

$$f : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ a & \text{if } x \in \{a, b\}, \\ c & \text{if } x = c, \end{cases}$$

and the set $F = \{1\}$ is an interior GE-filter in (X, f). But it is not a belligerent interior GE-filter since $b*(a*c) = b*1 = 1 \in F$ and $b*a = 1 \in F$ but $b*c = c \notin F$.

Given a GE-filter F and an element w in an interior GE-algebra (X, f), consider the set

$$F_w := \{x \in X \mid w * x \in F\}. \tag{4.1}$$

It is clear that $1 \in F_w$. The following example shows that F_w is neither an interior GE-filter nor a weak interior GE-filter.

Example 4.5. Consider the interior GE-algebra (X, f) in Example 3.13. Then $F := \{1, a, b\}$ is a GE-filter of X and $F_b = \{1, a, b\} = F$. But F_b is not an interior GE-filter in (X, f) since $f(c) = 1 \in F_b$ but $c \notin F_b$. Also F_b is not a weak interior GE-filter in (X, f) since $b * f(c) = b * 1 = 1 \in F_b$ and $f(b) = a \in F_b$ but $c \notin F_b$. Therefore F_b is neither an interior GE-filter nor a weak interior GE-filter in (X, f).

Question 4.6. *Under what conditions can the set F_w be a (weak) interior GE-filter in an interior GE-algebra (X, f)?*

We consider conditions for an interior GE-filter to be a belligerent interior GE-filter in an interior GE-algebra.

Theorem 4.7. *Let F be an interior GE-filter in an interior GE-algebra (X, f) and suppose that F_w is a GE-filter of X for all $w \in X$. Then F is a belligerent interior GE-filter in (X, f).*

Proof. It is sufficient to show that F is a belligerent GE-filter of X. Let $x, y, z \in X$ be such that $x * (y * z) \in F$ and $x * y \in F$. Then $y * z \in F_x$ and $y \in F_x$. Since F_x is a GE-filter of X, we have $z \in F_x$ and so $x * z \in F$. This shows that F is a belligerent GE-filter of X, and hence F is a belligerent interior GE-filter in (X, f). ■

Given a subset F and an element z in an interior GE-algebra (X, f), we present conditions for the set F_z to be an interior GE-filter.

Lemma 4.8 [5]**.** *Every transitive interior GE-algebra (X, f) satisfies:*

$$(\forall x, y \in X)(f(x) * y \leq x * f(y)), \tag{4.2}$$

$$(\forall x, y \in X)(f(x) * y \leq f(x * y)). \tag{4.3}$$

Theorem 4.9. *Let F be a belligerent interior GE-filter in a transitive interior GE-algebra (X, f) such that*

$$(\forall x \in X)(f(x) \leq x), \tag{4.4}$$

$$(\forall x, y \in X)(x \leq y,\ y \in F \Rightarrow x \in F). \tag{4.5}$$

Then F_z is an interior GE-filter in (X, f) for all $z \in X$.

Proof. Note that $1 \in F_z$ for all $z \in X$. Let $x, y, z \in X$ be such that $x * y \in F_z$ and $x \in F_z$. Then $z * (x * y) \in F$ and $z * x \in F$. Since F is a belligerent GE-filter of X, it follows that $z * y \in F$, i.e., $y \in F_z$. Hence F_z is a GE-filter of X. Let $x \in X$ be such that $f(x) \in F_z$. Then $z * f(x) \in F$, which implies from (3.2), (4.4) and (4.5) that $f(z * f(x)) \in F$. Using (2.13), Lemma 4.8 and (4.4), the following is induced:

$$z * x \leq f(z) * x = f(f(z)) * x \leq f(z) * f(x) \leq f(z * f(x)).$$

It follows from (4.5) that $z * x \in F$, that is, $x \in F_z$. Therefore F_z is an interior GE-filter in (X, f) for all $z \in X$. ■

We consider conditions for a subset to be a belligerent interior GE-filter.

Theorem 4.10. *Let (X, f) be an interior GE-algebra in which X is a left exchangeable transitive GE-algebra. If a subset F of X in (X, f) satisfies* (2.17), (3.4) *and*

$$(\forall x, y \in X)(\forall a \in F)\,(x * (x * y) \in F_a \Rightarrow x * y \in F)\,, \tag{4.6}$$

then F is a belligerent interior GE-filter in (X, f).

Proof. Let $x, y \in X$ be such that $x * y \in F$ and $x \in F$. Then $1 * (1 * x) = x \in F$, and thus $x = 1 * x \in F$ by (GE2) and (4.6). Hence F is an interior GE-filter in (X, f). Let $x, y, z \in X$ be such that $x * (y * z) \in F$ and $x * y \in X$. Since $x * (y * z) = y * (x * z) \leq (x * y) * (x * (x * z))$, it follows from (GE2), (2.18) and (2.19) that $1 * (x * (x * z)) = x * (x * z) \in F$, that is, $x * (x * z) \in F_1$. Hence $x * z \in F$ by (4.6), and therefore F is a belligerent interior GE-filter in (X, f). ■

In the next theorem, we discuss the extensibility of the beligerent interior GE-filter.

Theorem 4.11. *Let (X, f) be an interior GE-algebra in which X is a left exchangeable transitive GE-algebra. For any interior GE-filters F and G in (X, f) with $F \subseteq G$, if F is a belligerent interior GE-filter in (X, f), then so is G.*

Proof. If F is a belligerent interior GE-filter in (X, f), then

$$(\forall x, y \in X)\,(x * (x * y) \in F \Rightarrow x * y \in F)\,. \tag{4.7}$$

Now, we will show that

$$(\forall x, y, z \in X)\,(x * (y * z) \in F \Rightarrow (x * y) * (x * z) \in F)\,. \tag{4.8}$$

Assume that $x * (y * z) \in F$ for all $x, y, z \in X$. The conditions (2.2), (2.3) and (2.13) induce:

$$x * (y * z) \leq x * ((x * y) * (x * z)) = x * (x * ((x * y) * z)).$$

It follows from Lemma 2.6 that $x*(x*((x*y)*z)) \in F$. Thus $(x*y)*(x*z) = x*((x*y)*z) \in F$ by (2.3) and (4.7), which shows (4.8) is valid. Let $x, y, z \in X$ be such that $x*(y*z) \in G$. Using the left exchangeablity of X induces:

$$x*(y*((x*(y*z))*z)) = (x*(y*z))*(x*(y*z)) = 1 \in F. \tag{4.9}$$

Hence $(x*(y*z))*((x*y)*(x*z)) = (x*y)*(x*((x*(y*z))*z)) \in F \subseteq G$ by (2.3), (4.8) and (4.9), which implies from (2.18) that $(x*y)*(x*z) \in G$, that is, we shown that G satisfies:

$$(\forall x, y, z \in X)\,(x*(y*z) \in G \Rightarrow (x*y)*(x*z) \in G)\,. \tag{4.10}$$

Let $x, y, z \in X$ be such that $x*(y*(y*z)) \in G$ and $x \in G$. Then $y*(y*(x*z)) \in G$ by (2.3), and so $x*(y*z) = (y*y)*(y*(x*z)) \in G$ by (GE1), (GE2) and (4.10). Since $x \in G$ and G is a GE-filter of X, we get $y*z \in G$. This shows that G satisfies:

$$(\forall x, y, z \in X)\,(x*(y*(y*z)) \in G,\ x \in G \Rightarrow y*z \in G)\,. \tag{4.11}$$

Let $x, y, z \in X$ be such that $x*(y*z) \in G$ and $x*y \in G$. Since $x*(y*z) = y*(x*z) \le (x*y)*(x*(x*z))$ by (2.3) and (2.15), we obtain $(x*y)*(x*(x*z)) \in G$ by Lemma 2.6. Thus $x*z \in G$ by (4.11). Therefore G is a belligerent interior GE-filter in (X, f). ■

Definition 4.12. Let (X, f) be an interior GE-algebra and let F be a subset of X which satisfies (2.17). Then F is called

- a *belligerent interior GE-filter of type* 1 in (X, f) if it satisfies:

$$(\forall x, y, z \in X)\,(x*(y*f(z)) \in F,\ f(x*y) \in F \Rightarrow x*z \in F)\,. \tag{4.12}$$

- a *belligerent interior GE-filter of type* 2 in (X, f) if it satisfies:

$$(\forall x, y, z \in X)\,(x*(y*f(z)) \in F,\ f(x*y) \in F \Rightarrow f(x)*z \in F)\,. \tag{4.13}$$

- a *belligerent interior GE-filter of type* 3 in (X, f) if it satisfies:

$$(\forall x, y, z \in X)\,(x*(y*f(z)) \in F,\ f(x*y) \in F \Rightarrow x*f(z) \in F)\,. \tag{4.14}$$

Example 4.13. 1. Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation $*$ which is given in the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	c	c
b	1	1	1	d	d
c	1	a	b	1	1
d	1	a	b	1	1

If we define a mapping f as follows:

$$f : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ b & \text{if } x \in \{a, b\}, \\ c & \text{if } x \in \{c, d\}, \end{cases}$$

then (X, f) is an interior GE-algebra. It is routine to verify that $F := \{1, a\}$ is a belligerent interior GE-filter of type 1.

2. Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation $*$ which is given in the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	c	c
b	1	1	1	d	d
c	1	a	b	1	1
d	1	a	b	1	1

Then it is routine to verify that (X, f) is an interior GE-algebra where

$$f : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ b & \text{if } x \in \{a, b\}, \\ c & \text{if } x = c, \\ d & \text{if } x = d. \end{cases}$$

and the set $F := \{1, a\}$ is a belligerent interior GE-filter of type 2.

3. Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation $*$ which is given in the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	c	c
b	1	1	1	d	d
c	1	1	1	1	1
d	1	1	1	1	1

Then it is routine to verify that (X, f) is an interior GE-algebra where

$$f : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ a & \text{if } x \in \{a, b, c, d\} \end{cases}$$

and the set $F := \{1, a\}$ is a belligerent interior GE-filter of type 3.

The belligerent interior GE-filter of type 3 may not be an interior GE-filter as seen in the following example.

Example 4.14. Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation $*$ which is given in the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	d	d
b	1	a	1	d	d
c	1	1	1	1	1
d	1	a	b	1	1

Then it is routine to verify that (X, f) is an interior GE-algebra where

$$f : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x \in \{1, c, d\}, \\ a & \text{if } x \in \{a, b\}, \end{cases}$$

and the set $F := \{1\}$ is a belligerent interior GE-filter of type 3. But F is not an interior GE-filter in (X, f) since $f(c) = 1 \in F$ but $c \notin F$.

Question 4.15. *Is a belligerent interior GE-filter of type* 1 *or type* 2 *an interior GE-filter?*

We establish relationships between weak interior GE-filter and belligerent interior GE-filter of type 1, type 2 and type 3.

Theorem 4.16. *In an interior GE-algebra, every belligerent interior GE-filter of type* 1 *is a weak interior GE-filter.*

Proof. Let F be a belligerent interior GE-filter of type 1 in an interior GE-algebra (X, f). Let $x, y \in F$ be such that $x * f(y) \in F$ and $f(x) \in F$. Then $1 * (x * f(y)) = x * f(y) \in F$ and $f(1 * x) = f(x) \in F$ which imply from (GE2) and (4.12) that $y = 1 * y \in F$. Therefore F is a weak interior GE-filter in (X, f). ■

The following example shows that the converse of Theorem 4.16 is not true in general.

Example 4.17. Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation $*$ which is given in the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	c	c
b	1	a	1	d	d
c	1	a	1	1	1
d	1	a	1	1	1

Then it is routine to verify that (X, f) is an interior GE-algebra where

$$f : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ b & \text{if } x \in \{a, b, c, d\}, \end{cases}$$

and the set $F := \{1, a\}$ is a weak interior GE-filter (X, f). But it is not a belligerent interior GE-filter of type 1, since $a*(b*f(c)) = a*(b*b) = a*1 = 1 \in F$ and $f(a*b) = f(1) = 1 \in F$ but $a*c = c \notin F$.

Theorem 4.18. *In an interior GE-algebra, every belligerent interior GE-filter of type* 2 *is a weak interior GE-filter.*

Proof. Let F be a belligerent interior GE-filter of type 2 in an interior GE-algebra (X, f). Assume that $x * f(y) \in F$ and $f(x) \in F$ for all $x, y \in X$. Then $1*(x*f(y)) = x*f(y) \in F$ and $f(1*x) = f(x) \in F$. Since $f(1) = 1$ in an interior GE-algebra (X, f), it follows from (GE2) and (4.13) that $y = 1*y = f(1)*y \in F$. Therefore F is a weak interior GE-filter in (X, f). ■

The following example shows that the converse of Theorem 4.18 is not true in general.

Example 4.19. Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation $*$ which is given in the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	1	1
b	1	a	1	c	d
c	1	a	1	1	1
d	1	a	1	c	1

Then it is routine to verify that (X, f) is an interior GE-algebra where

$$f : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ b & \text{if } x \in \{a, b, c, d\}, \end{cases}$$

and the set $F := \{1, a\}$ is a weak interior GE-filter in (X, f). But it is not a belligerent interior GE-filter of type 2 since $a*(1*f(c)) = a*(1*b) = a*b = 1 \in F$ and $f(a*b) = f(1) = 1 \in F$ but $f(a)*c = b*c = c \notin F$.

In the following example, we know that any belligerent interior GE-filter of type 3 is not a weak interior GE-filter.

Example 4.20. Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation $*$ which is given in the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	d	d
b	1	a	1	d	d
c	1	1	1	1	1
d	1	a	b	1	1

Then it is routine to verify that (X, f) is an interior GE-algebra where

$$f : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x \in \{1, a, b\}, \\ d & \text{if } x \in \{c, d\}, \end{cases}$$

and the set $F := \{1\}$ is a belligerent interior GE-filter of type 3. But it is not a weak interior GE-filter in (X, f) since $a * f(a) = a * 1 = 1 \in F$ and $f(a) = 1 \in F$ but $a \notin F$.

Theorem 4.21. *In an interior GE-algebra, every belligerent interior GE-filter is a belligerent interior GE-filter of type* 3.

Proof. Let F be a belligerent interior GE-filter in an interior GE-algebra (X, f). Let $x, y, z \in X$ be such that $x * (y * f(z)) \in F$ and $f(x * y) \in F$. Then $x * y \in F$ by (3.4). Since F is a belligerent GE-filter of X, we have $x * f(z) \in F$ by (2.20). Therefore F is a belligerent interior GE-filter of type 3. ■

The convese of Theorem 4.21 may not be true as seen in the following example.

Example 4.22. Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation $*$ which is given in the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	d	d
b	1	a	1	d	d
c	1	1	1	1	1
d	1	a	b	1	1

Then it is routine to verify that (X, f) is an interior GE-algebra where

$$f : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x \in \{1, c, d\}, \\ a & \text{if } x \in \{a, b\}, \end{cases}$$

and the set $F := \{1\}$ is a belligerent interior GE-filter of type 3. But it is not a belligerent interior GE-filter in (X, f) since $f(c) = 1 \in F$ but $c \notin F$.

The combination of Theorem 4.7 and Theorem 4.21 produces the following corollary:

Corollary 4.23. *Let F be an interior GE-filter in an interior GE-algebra (X, f) and suppose that F_w is a GE-filter of X for all $w \in X$. Then F is a belligerent interior GE-filter of type* 3.

The combination of Theorem 4.10 and Theorem 4.21 produces the following corollary.

Corollary 4.24. *Let (X, f) be an interior GE-algebra in which X is a left exchangeable transitive GE-algebra. If a subset F of X in (X, f) satisfies* (2.17), (3.4) *and* (4.6), *then F is a belligerent interior GE-filter of type* 3.

The following example shows that any belligerent interior GE-filter may not be a belligerent interior GE-filter of type 1 or type 2.

Example 4.25. 1. Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation $*$ which is given in the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	d	d
b	1	a	1	d	d
c	1	1	1	1	1
d	1	a	b	1	1

Then it is routine to verify that (X, f) is an interior GE-algebra where

$$f : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ a & \text{if } x \in \{a, b, c, d\}, \end{cases}$$

and the set $F := \{1\}$ is a belligerent interior GE-filter in (X, f). But it is not a belligerent interior GE-filter of type 1 since $a*(a*(f(b)) = a*(a*a) = a*1 = 1 \in F$ and $f(a * a) = f(1) = 1 \in F$ but $a * b = b \notin F$.

2. Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation $*$ which is given in the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	1	1
b	1	1	1	1	1
c	1	1	b	1	1
d	1	1	b	1	1

Then it is routine to verify that (X, f) is an interior GE-algebra where

$$f : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ a & \text{if } x \in \{a, b, c, d\}, \end{cases}$$

and the set $F := \{1\}$ is a belligerent interior GE-filter in (X, f). But it is not a belligerent interior GE-filter of type 2 since $a{*}(a{*}(f(b)) = a{*}(a{*}a) = a{*}1 = 1 \in F$ and $f(a * a) = f(1) = 1 \in F$ but $f(a) * b = a * b = b \notin F$.

Acknowledgments

This research was supported by Basic Science Research Program to Research Institute for Basic Sciences (RIBS) of Jeju National University through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (2019R1A6A1A10072987).

References

[1] R.K. Bandaru, A. Borumand Saeid and Y.B. Jun, *On GE-algebras*, Bulletin of the Section of Logic **50** (2021) 81–90.
https://doi.org/10.18778/0138-0680.2020.20

[2] R.K. Bandaru, A. Borumand Saeid and Y.B. Jun, *Belligerent GE-filters in GE-algebras*, T. Indones. Math. Soc. (submitted).

[3] G. Castellini and J. Ramos, *Interior operators and topological connectedness*, Quaest. Math. **33** (2010) 290–304.
https://doi.org/10.2989/16073606.2010.507322

[4] A. Diego, *Sur algébres de Hilbert*, Collect. Logique Math. Ser. A **21** (1967) 177–198.

[5] J.G. Lee, R.K. Bandaru, K. Hur and Y.B. Jun, *Interior GE-algebras*, J. Math. **2021** (2021) 1–10.
https://doi.org/10.1155/2021/6646091

[6] J. Rachůnek and Z. Svoboda, *Interior and closure operators on bounded residuated lattices*, Cent. Eur. J. Math. **12** (3) (2014) 534–544.
https://doi.org/10.2478/s11533-013-0349-y

[7] F. Svrcek, *Operators on GMV-algebras*, Math. Bohem. **129** (2004) 337–347.
https://doi.org/10.21136/MB.2004.134044

[8] S.J.R. Vorster, *Interior operators in general categories*, Quaest. Math. **23** (2000) 405–416.
https://doi.org/10.2989/16073600009485987

Received 7 December 2020
Revised 6 January 2020
Accepted 6 January 2020