

INTERIOR GE-FILTERS OF GE-ALGEBRAS

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Abstract

The notions of an interior GE-filter, a weak interior GE-filter and a belligerent interior GE-filter are introduced, and their relations and properties are investigated. Example of a GE-filter which is neither an interior GE-filter nor a weak interior GE-filter is provided. Relations between a weak interior GE-filter and an interior GE-filter are discussed, and conditions under which every weak interior GE-filter is an interior GE-filter are investigated. Relations between a belligerent interior GE-filter and an interior GE-filter are displayed, and conditions for an interior GE-filter to be a belligerent interior

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GE-filter are considered. Given a subset and an element, an interior GE-filter is established, and conditions for a subset to be a belligerent interior GE-filter are discussed. The extensibility of the belligerent interior GE-filter is debated. Relationships between weak interior GE-filter and belligerent interior GE-filter of type 1, type 2 and type 3 are founded.

Keywords: (transitive, left exchangeable) GE-algebra, GE-filter, belligerent GE-filter, (weak) interior GE-filter, belligerent interior GE-filter (of type 1, type 2 and type 3).

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1. INTRODUCTION

Henkin and Skolem introduced Hilbert algebras in the fifties for investigations in intuitionistic and other non-classical logics. Diego [4] proved that Hilbert algebras form a variety which is locally finite. Bandaru *et al.* introduced the notion of GE-algebras which is a generalization of Hilbert algebras, and investigated several properties (see [1, 2]). The notion of interior operator is introduced by Vorster [8] in an arbitrary category, and it is used in [3] to study the notions of connectedness and disconnectedness in topology. Interior algebras are a certain type of algebraic structure that encodes the idea of the topological interior of a set, and are a generalization of topological spaces defined by means of topological interior operators. Rachůnek and Svoboda [6] studied interior operators on bounded residuated lattices, and Svrcek [7] studied multiplicative interior operators on GMV-algebras. Lee *et al.* [5] applied the interior operator theory to GE-algebras, and they introduced the concepts of (commutative, transitive, left exchangeable, belligerent, antisymmetric) interior GE-algebras and bordered interior GE-algebras, and investigated their relations and properties. They found and presented many examples to illustrate these concepts, and used the set of interior GE-algebras to make up a semigroup. They gave examples to show that the set of interior GE-algebras is not a GE-algebra. They established the internal GE-algebra using a bordered interior GE-algebra with certain conditions, and provided examples describing this.

The aim of this section is to introduce the notions of an interior GE-filter, a weak interior GE-filter and a belligerent interior GE-filter, and investigate their relations and properties. We provide example of a GE-filter which is neither an interior GE-filter nor a weak interior GE-filter. We discuss relations between a weak interior GE-filter and an interior GE-filter. We investigate the conditions under which every weak interior GE-filter is an interior GE-filter. We discuss relations between a belligerent interior GE-filter and an interior GE-filter. We consider conditions for an interior GE-filter to be a belligerent interior GE-filter. Given a subset and an element, we establish an interior GE-filter, and we con-

sider conditions for a subset to be a belligerent interior GE-filter. We discuss the extensibility of the belligerent interior GE-filter. We establish relationships between weak interior GE-filter and belligerent interior GE-filter of type 1, type 2 and type 3.

2. PRELIMINARIES

Definition 2.1 [1]. By a *GE-algebra* we mean a non-empty set X with a constant 1 and a binary operation $*$ satisfying the following axioms:

$$(GE1) \quad u * u = 1,$$

$$(GE2) \quad 1 * u = u,$$

$$(GE3) \quad u * (v * w) = u * (v * (u * w))$$

for all $u, v, w \in X$.

In a GE-algebra X , a binary relation “ \leq ” is defined by

$$(2.1) \quad (\forall x, y \in X) (x \leq y \Leftrightarrow x * y = 1).$$

Definition 2.2 [1, 2]. A GE-algebra X is said to be

- *transitive* if it satisfies:

$$(2.2) \quad (\forall x, y, z \in X) (x * y \leq (z * x) * (z * y)),$$

- *left exchangeable* if it satisfies:

$$(2.3) \quad (\forall x, y, z \in X) (x * (y * z) = y * (x * z)),$$

- *belligerent* if it satisfies:

$$(2.4) \quad (\forall x, y, z \in X) (x * (y * z) = (x * y) * (x * z)).$$

Proposition 2.3 [1]. *Every GE-algebra X satisfies the following items.*

$$(2.5) \quad (\forall u \in X) (u * 1 = 1).$$

$$(2.6) \quad (\forall u, v \in X) (u * (u * v) = u * v).$$

$$(2.7) \quad (\forall u, v \in X) (u \leq v * u).$$

$$(2.8) \quad (\forall u, v, w \in X) (u * (v * w) \leq v * (u * w)).$$

$$(2.9) \quad (\forall u \in X) (1 \leq u \Rightarrow u = 1).$$

$$(2.10) \quad (\forall u, v \in X) (u \leq (v * u) * u).$$

$$(2.11) \quad (\forall u, v \in X) (u \leq (u * v) * v).$$

$$(2.12) \quad (\forall u, v, w \in X) (u \leq v * w \Leftrightarrow v \leq u * w).$$

If X is transitive, then

$$(2.13) \quad (\forall u, v, w \in X) (u \leq v \Rightarrow w * u \leq w * v, v * w \leq u * w).$$

$$(2.14) \quad (\forall u, v, w \in X) (u * v \leq (v * w) * (u * w)).$$

Lemma 2.4 [1]. In a GE-algebra X , the following facts are equivalent each other.

$$(2.15) \quad (\forall x, y, z \in X) (x * y \leq (z * x) * (z * y)).$$

$$(2.16) \quad (\forall x, y, z \in X) (x * y \leq (y * z) * (x * z)).$$

Definition 2.5 [1]. A subset F of a GE-algebra X is called a *GE-filter* of X if it satisfies:

$$(2.17) \quad 1 \in F,$$

$$(2.18) \quad (\forall x, y \in X) (x * y \in F, x \in F \Rightarrow y \in F).$$

Lemma 2.6 [1]. In a GE-algebra X , every GE-filter F of X satisfies:

$$(2.19) \quad (\forall x, y \in X) (x \leq y, x \in F \Rightarrow y \in F).$$

Definition 2.7 [2]. A subset F of a GE-algebra X is called a *belligerent GE-filter* of X if it satisfies (2.17) and

$$(2.20) \quad (\forall x, y, z \in X) (x * (y * z) \in F, x * y \in F \Rightarrow x * z \in F).$$

3. INTERIOR GE-FILTERS

The aim of this section is to introduce the notions of an interior GE-filter and a weak interior GE-filter, and investigate their properties.

Definition 3.1 [5]. By an *interior GE-algebra* we mean a pair (X, f) in which X is a GE-algebra and $f : X \rightarrow X$ is a mapping such that

$$(3.1) \quad (\forall x \in X) (x \leq f(x)),$$

$$(3.2) \quad (\forall x \in X) ((f \circ f)(x) = f(x)),$$

$$(3.3) \quad (\forall x, y \in X) (x \leq y \Rightarrow f(x) \leq f(y)).$$

Definition 3.2. Let (X, f) be an interior GE-algebra. A GE-filter F of X is said to be *interior* if it satisfies:

$$(3.4) \quad (\forall x \in X) (f(x) \in F \Rightarrow x \in F).$$

Example 3.3. Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation $*$ which is given in the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	c	c
b	1	1	1	d	d
c	1	a	b	1	1
d	1	a	a	1	1

If we define a mapping f on X as follows:

$$f : X \rightarrow X, x \mapsto \begin{cases} 1 & \text{if } x \in \{1, c, d\}, \\ a & \text{if } x \in \{a, b\}, \end{cases}$$

then (X, f) is an interior GE-algebra and $F = \{1, c, d\}$ is an interior GE-filter in (X, f) .

Example 3.4. Let $X_1 := \mathbb{N}$ be the set of all natural numbers and $*$ be the binary operation on X_1 defined by

$$x * y = \begin{cases} y & \text{if } x = 1, \\ 1 & \text{otherwise.} \end{cases}$$

Then $(X_1, *, 1)$ is a GE-algebra. Let $X_2 := \{1, a, b, c, d\}$ be a set with the following Cayley table:

\circ	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	c	c
b	1	1	1	d	d
c	1	a	b	1	1
d	1	a	a	1	1

Then $(X_2, \circ, 1)$ is a GE-algebra. We can observe that $(X_1 \times X_2, \bullet, 1_{X_1 \times X_2})$, where $(x, y) \bullet (u, v) = (x * u, y \circ v)$ and $1_{X_1 \times X_2} = (1_{X_1}, 1_{X_2})$, is a GE-algebra. If we define a mapping f on $X_1 \times X_2$ as follows:

$$f : X_1 \times X_2 \rightarrow X_1 \times X_2, (x, y) \mapsto \begin{cases} (x, 1) & \text{if } (x, y) \in \{(x, 1), (x, c), (x, d)\}, \\ (x, a) & \text{if } (x, y) \in \{(x, a), (x, b)\}, \end{cases}$$

then $(X_1 \times X_2, f)$ is an interior GE-algebra.

It is clear that X is an interior GE-filter in every interior GE-algebra (X, f) . But the trivial GE-filter $\{1\}$ is not interior. In fact, in Example 3.3, the trivial GE-filter $\{1\}$ is not interior since $f(c) = 1 \in \{1\}$ but $c \notin \{1\}$.

Theorem 3.5. *In an interior GE-algebra, the intersection of interior GE-filters is also an interior GE-filter.*

Proof. Let $\{F_i \mid i \in \Lambda\}$ be a set of interior GE-filters in an interior GE-algebra (X, f) where Λ is an index set. It is clear that $\cap\{F_i \mid i \in \Lambda\}$ is a GE-filter of X . Let $x \in X$ be such that $f(x) \in \cap\{F_i \mid i \in \Lambda\}$. Then $f(x) \in F_i$ for all $i \in \Lambda$. Since F_i is an interior GE-filter in (X, f) for all $i \in \Lambda$, it follows from (3.4) that $x \in F_i$ for all $i \in \Lambda$. Thus $x \in \cap\{F_i \mid i \in \Lambda\}$, and therefore $\cap\{F_i \mid i \in \Lambda\}$ is an interior GE-filter in (X, f) . ■

The following example shows that the union of interior GE-filters may not be an interior GE-filter.

Example 3.6. Consider a GE-algebra $X = \{1, a, b, c, d, e\}$ with the binary operation $*$ which is given in the following table:

$*$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	1	1	1	e
b	1	c	1	c	1	1
c	1	d	b	1	d	1
d	1	c	b	c	1	e
e	1	a	b	c	d	1

Then it is routine to verify that (X, f) is an interior GE-algebra where

$$f : X \rightarrow X, x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ b & \text{if } x \in \{a, b\}, \\ d & \text{if } x = d, \\ e & \text{if } x \in \{c, e\}. \end{cases}$$

Let $F = \{1, d\}$ and $G = \{1, c, e\}$. Then F and G are interior GE-filters in (X, f) . But $F \cup G = \{1, c, d, e\}$ is not a GE-filter of X since $d * a = c \in F \cup G, d \in F \cup G$ but $a \notin F \cup G$ and hence not an interior GE-filter in (X, f) .

Definition 3.7. Let (X, f) be an interior GE-algebra. A subset F of X is called a *weak interior GE-filter* in (X, f) if it satisfies (2.17) and

$$(3.5) \quad (\forall x, y \in X)(x * f(y) \in F, f(x) \in F \Rightarrow y \in F).$$

Example 3.8. Let $X = \{1, a, b, c, d\}$ be a set with the binary operation $*$ given in the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	c	c
b	1	1	1	d	d
c	1	1	b	1	1
d	1	1	1	1	1

Then X is a GE-algebra. Define a mapping f as follows:

$$f : X \rightarrow X, x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ a & \text{if } x \in \{a, b\}, \\ d & \text{if } x \in \{c, d\}, \end{cases}$$

Then (X, f) is an interior GE-algebra and $F := \{1, a, b\}$ is a weak interior GE-filter in (X, f) .

Example 3.9. Consider the GE-algebra $X_1 := \mathbb{N}$ which is given in Example 3.4 and let $X_2 := \{1, a, b, c, d\}$ be a GE-algebra with the following Cayley table:

\circ	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	1	d
b	1	1	1	c	1
c	1	b	b	1	1
d	1	a	a	1	1

Then $(X_1 \times X_2, \bullet, 1_{X_1 \times X_2})$ is a GE-algebra, where $(x, y) \bullet (u, v) = (x * u, y \circ v)$ for every $(x, y), (u, v) \in X_1 \times X_2$ and $1_{X_1 \times X_2} = (1_{X_1}, 1_{X_2})$. If we define a mapping f on $X_1 \times X_2$ as follows:

$$f : X_1 \times X_2 \rightarrow X_1 \times X_2, (x, y) \mapsto \begin{cases} (x, 1) & \text{if } (x, y) = (x, 1), \\ (x, c) & \text{if } (x, y) \in \{(x, a), (x, c)\}, \\ (x, d) & \text{if } (x, y) \in \{(x, b), (x, d)\}, \end{cases}$$

then $(X_1 \times X_2, f)$ is an interior GE-algebra. It is routine to verify that the set $F := \{(x, 1), (x, b) \mid x \in X_1\}$ is a weak interior GE-filter in $(X_1 \times X_2, \bullet, 1_{X_1 \times X_2})$.

The following example shows that there is a GE-filter which is neither an interior GE-filter nor a weak interior GE-filter.

Example 3.10. Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation $*$ which is given in the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	c	c
b	1	1	1	d	d
c	1	a	b	1	1
d	1	a	a	1	1

Then it is routine to verify that (X, f) is an interior GE-algebra where

$$f : X \rightarrow X, x \mapsto \begin{cases} 1 & \text{if } x \in \{1, c, d\}, \\ a & \text{if } x \in \{a, b\}, \end{cases}$$

and the set $F := \{1, a, b\}$ is a GE-filter of X . But it is not an interior GE-filter in (X, f) since $f(c) = 1 \in F$ but $c \notin F$. Also it is not a weak interior GE-filter in (X, f) since $b * f(c) = b * 1 = 1 \in F$ and $f(b) = a \in F$ but $c \notin F$.

Theorem 3.11. *In an interior GE-algebra, the intersection of weak interior GE-filters is also a weak interior GE-filter.*

Proof. Let $\{F_i \mid i \in \Lambda\}$ be a set of weak interior GE-filters in an interior GE-algebra (X, f) where Λ is an index set. It is clear that $1 \in \cap\{F_i \mid i \in \Lambda\}$. Let $x, y \in X$ be such that $x * f(y) \in \cap\{F_i \mid i \in \Lambda\}$ and $f(x) \in \cap\{F_i \mid i \in \Lambda\}$. Then $x * f(y) \in F_i$ and $f(x) \in F_i$ for all $i \in \Lambda$. Since F_i is a weak interior GE-filter in (X, f) , it follows from (3.5) that $y \in F_i$ for all $i \in \Lambda$. Hence $y \in \cap\{F_i \mid i \in \Lambda\}$, and therefore $\cap\{F_i \mid i \in \Lambda\}$ is a weak interior GE-filter in (X, f) . ■

We discuss relations between a weak interior GE-filter and an interior GE-filter.

Theorem 3.12. *In an interior GE-algebra, every interior GE-filter is a weak interior GE-filter.*

Proof. Let F be an interior GE-filter in an interior GE-algebra (X, f) . Let $x, y \in X$ be such that $x * f(y) \in F$ and $f(x) \in F$. Then $x \in F$ by (3.4). Since F is a GE-filter of X , it follows from (2.18) that $f(y) \in F$. Hence $y \in F$ by (3.4), and therefore F is a weak interior GE-filter in (X, f) . ■

The following example shows that any weak interior GE-filter may not be an interior GE-filter in an interior GE-algebra. Hence the converse of Theorem 3.12 is not true in general.

Example 3.13. Let $X = \{1, a, b, c, d\}$ be a set with the binary operation $*$ given in the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	1	d
b	1	1	1	c	1
c	1	b	b	1	1
d	1	a	a	1	1

Then X is a GE-algebra. Define a mapping f as follows:

$$f : X \rightarrow X, x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ c & \text{if } x \in \{a, c\}, \\ d & \text{if } x \in \{b, d\}, \end{cases}$$

Then (X, f) is an interior GE-algebra and $F := \{1, b\}$ is a weak interior GE-filter in (X, f) . But it is not a GE-filter of X since $b * d = 1 \in F$ and $b \in F$ but $d \notin F$. Hence F is not an interior GE-filter in (X, f) .

In the following theorem, we investigate the conditions under which every weak interior GE-filter is an interior GE-filter.

Theorem 3.14. *If F is a weak interior GE-filter in an interior GE-algebra (X, f) which is also a GE-filter of X , then F is an interior GE-filter in (X, f) .*

Proof. Let F be a weak interior GE-filter in an interior GE-algebra (X, f) which is also a GE-filter of X . Let $x \in X$ be such that $f(x) \in F$. Then $1 * f(x) = f(x) \in F$ and $f(1) = 1 \in F$. It follows from (3.5) that $x \in F$. Therefore F is an interior GE-filter in (X, f) . ■

Let (X, f) be an interior GE-algebra. Given a non-empty subset F of X , the interior GE-filter in (X, f) generated by F is defined to be the smallest interior GE-filter in (X, f) containing F , and it is denoted by $\langle F \rangle_f$.

Example 3.15. Consider the interior GE-algebra (X, f) in Example 3.10. For a subset $F = \{1, c\}$ of X , we have $\langle F \rangle_f = \{1, c, d\}$.

We present a question about the interior GE-filter generated by a set as follows:

Question 3.16. *How is the interior GE-filter generated by a subset of an interior GE-algebra described?*

4. BELLIGERENT INTERIOR GE-FILTERS

In this section, we introduce the concept of a belligerent interior GE-filter and discuss relations between an interior GE-filter and a belligerent interior GE-filter.

Definition 4.1. Let (X, f) be an interior GE-algebra. Then a subset F of X is called a *belligerent interior GE-filter* in (X, f) if F is a belligerent GE-filter of X which satisfies the condition (3.4).

Example 4.2. Consider the interior GE-algebra (X, f) in Example 3.6. It is routine to verify that $F := \{1, d\}$ is a belligerent interior GE-filter in (X, f) .

Theorem 4.3. *In an interior GE-algebra (X, f) , every belligerent interior GE-filter is an interior GE-filter.*

Proof. It is straightforward because every belligerent GE-filter is a GE-filter in a GE-algebra. ■

The following example shows that any interior GE-filter may not be a belligerent interior GE-filter.

Example 4.4. Consider a GE-algebra $X = \{1, a, b, c\}$ with the binary operation $*$ which is given in the following table:

$*$	1	a	b	c
a	1	1	1	1
1	1	a	b	c
b	1	1	1	c
c	1	b	b	1

Then it is routine to verify that (X, f) is an interior GE-algebra where

$$f : X \rightarrow X, x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ a & \text{if } x \in \{a, b\}, \\ c & \text{if } x = c, \end{cases}$$

and the set $F = \{1\}$ is an interior GE-filter in (X, f) . But it is not a belligerent interior GE-filter since $b*(a*c) = b*1 = 1 \in F$ and $b*a = 1 \in F$ but $b*c = c \notin F$.

Given a GE-filter F and an element w in an interior GE-algebra (X, f) , consider the set

$$(4.1) \quad F_w := \{x \in X \mid w * x \in F\}.$$

It is clear that $1 \in F_w$. The following example shows that F_w is neither an interior GE-filter nor a weak interior GE-filter.

Example 4.5. Consider the interior GE-algebra (X, f) in Example 3.13. Then $F := \{1, a, b\}$ is a GE-filter of X and $F_b = \{1, a, b\} = F$. But F_b is not an interior GE-filter in (X, f) since $f(c) = 1 \in F_b$ but $c \notin F_b$. Also F_b is not a weak interior GE-filter in (X, f) since $b * f(c) = b * 1 = 1 \in F_b$ and $f(b) = a \in F_b$ but $c \notin F_b$. Therefore F_b is neither an interior GE-filter nor a weak interior GE-filter in (X, f) .

Question 4.6. Under what conditions can the set F_w be a (weak) interior GE-filter in an interior GE-algebra (X, f) ?

We consider conditions for an interior GE-filter to be a belligerent interior GE-filter in an interior GE-algebra.

Theorem 4.7. Let F be an interior GE-filter in an interior GE-algebra (X, f) and suppose that F_w is a GE-filter of X for all $w \in X$. Then F is a belligerent interior GE-filter in (X, f) .

Proof. It is sufficient to show that F is a belligerent GE-filter of X . Let $x, y, z \in X$ be such that $x * (y * z) \in F$ and $x * y \in F$. Then $y * z \in F_x$ and $y \in F_x$. Since F_x is a GE-filter of X , we have $z \in F_x$ and so $x * z \in F$. This shows that F is a belligerent GE-filter of X , and hence F is a belligerent interior GE-filter in (X, f) . ■

Given a subset F and an element z in an interior GE-algebra (X, f) , we present conditions for the set F_z to be an interior GE-filter.

Lemma 4.8 [5]. Every transitive interior GE-algebra (X, f) satisfies:

$$(4.2) \quad (\forall x, y \in X)(f(x) * y \leq x * f(y)),$$

$$(4.3) \quad (\forall x, y \in X)(f(x) * y \leq f(x * y)).$$

Theorem 4.9. Let F be a belligerent interior GE-filter in a transitive interior GE-algebra (X, f) such that

$$(4.4) \quad (\forall x \in X)(f(x) \leq x),$$

$$(4.5) \quad (\forall x, y \in X)(x \leq y, y \in F \Rightarrow x \in F).$$

Then F_z is an interior GE-filter in (X, f) for all $z \in X$.

Proof. Note that $1 \in F_z$ for all $z \in X$. Let $x, y, z \in X$ be such that $x * y \in F_z$ and $x \in F_z$. Then $z * (x * y) \in F$ and $z * x \in F$. Since F is a belligerent GE-filter of X , it follows that $z * y \in F$, i.e., $y \in F_z$. Hence F_z is a GE-filter of X . Let $x \in X$ be such that $f(x) \in F_z$. Then $z * f(x) \in F$, which implies from (3.2), (4.4)

and (4.5) that $f(z * f(x)) \in F$. Using (2.13), Lemma 4.8 and (4.4), the following is induced:

$$z * x \leq f(z) * x = f(f(z)) * x \leq f(z) * f(x) \leq f(z * f(x)).$$

It follows from (4.5) that $z * x \in F$, that is, $x \in F_z$. Therefore F_z is an interior GE-filter in (X, f) for all $z \in X$. ■

We consider conditions for a subset to be a belligerent interior GE-filter.

Theorem 4.10. *Let (X, f) be an interior GE-algebra in which X is a left exchangeable transitive GE-algebra. If a subset F of X in (X, f) satisfies (2.17), (3.4) and*

$$(4.6) \quad (\forall x, y \in X)(\forall a \in F)(x * (x * y) \in F_a \Rightarrow x * y \in F),$$

then F is a belligerent interior GE-filter in (X, f) .

Proof. Let $x, y \in X$ be such that $x * y \in F$ and $x \in F$. Then $1 * (1 * x) = x \in F$, and thus $x = 1 * x \in F$ by (GE2) and (4.6). Hence F is an interior GE-filter in (X, f) . Let $x, y, z \in X$ be such that $x * (y * z) \in F$ and $x * y \in X$. Since $x * (y * z) = y * (x * z) \leq (x * y) * (x * (x * z))$, it follows from (GE2), (2.18) and (2.19) that $1 * (x * (x * z)) = x * (x * z) \in F$, that is, $x * (x * z) \in F_1$. Hence $x * z \in F$ by (4.6), and therefore F is a belligerent interior GE-filter in (X, f) . ■

In the next theorem, we discuss the extensibility of the belligerent interior GE-filter.

Theorem 4.11. *Let (X, f) be an interior GE-algebra in which X is a left exchangeable transitive GE-algebra. For any interior GE-filters F and G in (X, f) with $F \subseteq G$, if F is a belligerent interior GE-filter in (X, f) , then so is G .*

Proof. If F is a belligerent interior GE-filter in (X, f) , then

$$(4.7) \quad (\forall x, y \in X)(x * (x * y) \in F \Rightarrow x * y \in F).$$

Now, we will show that

$$(4.8) \quad (\forall x, y, z \in X)(x * (y * z) \in F \Rightarrow (x * y) * (x * z) \in F).$$

Assume that $x * (y * z) \in F$ for all $x, y, z \in X$. The conditions (2.2), (2.3) and (2.13) induce:

$$x * (y * z) \leq x * ((x * y) * (x * z)) = x * (x * ((x * y) * z)).$$

It follows from Lemma 2.6 that $x * (x * ((x * y) * z)) \in F$. Thus $(x * y) * (x * z) = x * ((x * y) * z) \in F$ by (2.3) and (4.7), which shows (4.8) is valid. Let $x, y, z \in X$ be such that $x * (y * z) \in G$. Using the left exchangeability of X induces:

$$(4.9) \quad x * (y * ((x * (y * z)) * z)) = (x * (y * z)) * (x * (y * z)) = 1 \in F.$$

Hence $(x * (y * z)) * ((x * y) * (x * z)) = (x * y) * (x * ((x * (y * z)) * z)) \in F \subseteq G$ by (2.3), (4.8) and (4.9), which implies from (2.18) that $(x * y) * (x * z) \in G$, that is, we shown that G satisfies:

$$(4.10) \quad (\forall x, y, z \in X) (x * (y * z) \in G \Rightarrow (x * y) * (x * z) \in G).$$

Let $x, y, z \in X$ be such that $x * (y * (y * z)) \in G$ and $x \in G$. Then $y * (y * (x * z)) \in G$ by (2.3), and so $x * (y * z) = (y * y) * (y * (x * z)) \in G$ by (GE1), (GE2) and (4.10). Since $x \in G$ and G is a GE-filter of X , we get $y * z \in G$. This shows that G satisfies:

$$(4.11) \quad (\forall x, y, z \in X) (x * (y * (y * z)) \in G, x \in G \Rightarrow y * z \in G).$$

Let $x, y, z \in X$ be such that $x * (y * z) \in G$ and $x * y \in G$. Since $x * (y * z) = y * (x * z) \leq (x * y) * (x * (x * z))$ by (2.3) and (2.15), we obtain $(x * y) * (x * (x * z)) \in G$ by Lemma 2.6. Thus $x * z \in G$ by (4.11). Therefore G is a belligerent interior GE-filter in (X, f) . ■

Definition 4.12. Let (X, f) be an interior GE-algebra and let F be a subset of X which satisfies (2.17). Then F is called

- a *belligerent interior GE-filter of type 1* in (X, f) if it satisfies:

$$(4.12) \quad (\forall x, y, z \in X) (x * (y * f(z)) \in F, f(x * y) \in F \Rightarrow x * z \in F).$$

- a *belligerent interior GE-filter of type 2* in (X, f) if it satisfies:

$$(4.13) \quad (\forall x, y, z \in X) (x * (y * f(z)) \in F, f(x * y) \in F \Rightarrow f(x) * z \in F).$$

- a *belligerent interior GE-filter of type 3* in (X, f) if it satisfies:

$$(4.14) \quad (\forall x, y, z \in X) (x * (y * f(z)) \in F, f(x * y) \in F \Rightarrow x * f(z) \in F).$$

Example 4.13. 1. Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation $*$ which is given in the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	c	c
b	1	1	1	d	d
c	1	a	b	1	1
d	1	a	b	1	1

If we define a mapping f as follows:

$$f : X \rightarrow X, x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ b & \text{if } x \in \{a, b\}, \\ c & \text{if } x \in \{c, d\}, \end{cases}$$

then (X, f) is an interior GE-algebra. It is routine to verify that $F := \{1, a\}$ is a belligerent interior GE-filter of type 1.

2. Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation $*$ which is given in the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	c	c
b	1	1	1	d	d
c	1	a	b	1	1
d	1	a	b	1	1

Then it is routine to verify that (X, f) is an interior GE-algebra where

$$f : X \rightarrow X, x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ b & \text{if } x \in \{a, b\}, \\ c & \text{if } x = c, \\ d & \text{if } x = d. \end{cases}$$

and the set $F := \{1, a\}$ is a belligerent interior GE-filter of type 2.

3. Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation $*$ which is given in the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	c	c
b	1	1	1	d	d
c	1	1	1	1	1
d	1	1	1	1	1

Then it is routine to verify that (X, f) is an interior GE-algebra where

$$f : X \rightarrow X, x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ a & \text{if } x \in \{a, b, c, d\} \end{cases}$$

and the set $F := \{1, a\}$ is a belligerent interior GE-filter of type 3.

The belligerent interior GE-filter of type 3 may not be an interior GE-filter as seen in the following example.

Example 4.14. Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation $*$ which is given in the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	d	d
b	1	a	1	d	d
c	1	1	1	1	1
d	1	a	b	1	1

Then it is routine to verify that (X, f) is an interior GE-algebra where

$$f : X \rightarrow X, x \mapsto \begin{cases} 1 & \text{if } x \in \{1, c, d\}, \\ a & \text{if } x \in \{a, b\}, \end{cases}$$

and the set $F := \{1\}$ is a belligerent interior GE-filter of type 3. But F is not an interior GE-filter in (X, f) since $f(c) = 1 \in F$ but $c \notin F$.

Question 4.15. *Is a belligerent interior GE-filter of type 1 or type 2 an interior GE-filter?*

We establish relationships between weak interior GE-filter and belligerent interior GE-filter of type 1, type 2 and type 3.

Theorem 4.16. *In an interior GE-algebra, every belligerent interior GE-filter of type 1 is a weak interior GE-filter.*

Proof. Let F be a belligerent interior GE-filter of type 1 in an interior GE-algebra (X, f) . Let $x, y \in F$ be such that $x * f(y) \in F$ and $f(x) \in F$. Then $1 * (x * f(y)) = x * f(y) \in F$ and $f(1 * x) = f(x) \in F$ which imply from (GE2) and (4.12) that $y = 1 * y \in F$. Therefore F is a weak interior GE-filter in (X, f) . ■

The following example shows that the converse of Theorem 4.16 is not true in general.

Example 4.17. Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation $*$ which is given in the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	c	c
b	1	a	1	d	d
c	1	a	1	1	1
d	1	a	1	1	1

Then it is routine to verify that (X, f) is an interior GE-algebra where

$$f : X \rightarrow X, \quad x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ b & \text{if } x \in \{a, b, c, d\}, \end{cases}$$

and the set $F := \{1, a\}$ is a weak interior GE-filter (X, f) . But it is not a belligerent interior GE-filter of type 1, since $a*(b*f(c)) = a*(b*b) = a*1 = 1 \in F$ and $f(a*b) = f(1) = 1 \in F$ but $a*c = c \notin F$.

Theorem 4.18. *In an interior GE-algebra, every belligerent interior GE-filter of type 2 is a weak interior GE-filter.*

Proof. Let F be a belligerent interior GE-filter of type 2 in an interior GE-algebra (X, f) . Assume that $x*f(y) \in F$ and $f(x) \in F$ for all $x, y \in X$. Then $1*(x*f(y)) = x*f(y) \in F$ and $f(1*x) = f(x) \in F$. Since $f(1) = 1$ in an interior GE-algebra (X, f) , it follows from (GE2) and (4.13) that $y = 1*y = f(1)*y \in F$. Therefore F is a weak interior GE-filter in (X, f) . ■

The following example shows that the converse of Theorem 4.18 is not true in general.

Example 4.19. Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation $*$ which is given in the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	1	1
b	1	a	1	c	d
c	1	a	1	1	1
d	1	a	1	c	1

Then it is routine to verify that (X, f) is an interior GE-algebra where

$$f : X \rightarrow X, \quad x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ b & \text{if } x \in \{a, b, c, d\}, \end{cases}$$

and the set $F := \{1, a\}$ is a weak interior GE-filter in (X, f) . But it is not a belligerent interior GE-filter of type 2 since $a*(1*f(c)) = a*(1*b) = a*b = 1 \in F$ and $f(a*b) = f(1) = 1 \in F$ but $f(a)*c = b*c = c \notin F$.

In the following example, we know that any belligerent interior GE-filter of type 3 is not a weak interior GE-filter.

Example 4.20. Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation $*$ which is given in the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	d	d
b	1	a	1	d	d
c	1	1	1	1	1
d	1	a	b	1	1

Then it is routine to verify that (X, f) is an interior GE-algebra where

$$f : X \rightarrow X, x \mapsto \begin{cases} 1 & \text{if } x \in \{1, a, b\}, \\ d & \text{if } x \in \{c, d\}, \end{cases}$$

and the set $F := \{1\}$ is a belligerent interior GE-filter of type 3. But it is not a weak interior GE-filter in (X, f) since $a * f(a) = a * 1 = 1 \in F$ and $f(a) = 1 \in F$ but $a \notin F$.

Theorem 4.21. *In an interior GE-algebra, every belligerent interior GE-filter is a belligerent interior GE-filter of type 3.*

Proof. Let F be a belligerent interior GE-filter in an interior GE-algebra (X, f) . Let $x, y, z \in X$ be such that $x * (y * f(z)) \in F$ and $f(x * y) \in F$. Then $x * y \in F$ by (3.4). Since F is a belligerent GE-filter of X , we have $x * f(z) \in F$ by (2.20). Therefore F is a belligerent interior GE-filter of type 3. ■

The converse of Theorem 4.21 may not be true as seen in the following example.

Example 4.22. Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation $*$ which is given in the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	d	d
b	1	a	1	d	d
c	1	1	1	1	1
d	1	a	b	1	1

Then it is routine to verify that (X, f) is an interior GE-algebra where

$$f : X \rightarrow X, x \mapsto \begin{cases} 1 & \text{if } x \in \{1, c, d\}, \\ a & \text{if } x \in \{a, b\}, \end{cases}$$

and the set $F := \{1\}$ is a belligerent interior GE-filter of type 3. But it is not a belligerent interior GE-filter in (X, f) since $f(c) = 1 \in F$ but $c \notin F$.

The combination of Theorem 4.7 and Theorem 4.21 produces the following corollary:

Corollary 4.23. *Let F be an interior GE-filter in an interior GE-algebra (X, f) and suppose that F_w is a GE-filter of X for all $w \in X$. Then F is a belligerent interior GE-filter of type 3.*

The combination of Theorem 4.10 and Theorem 4.21 produces the following corollary.

Corollary 4.24. *Let (X, f) be an interior GE-algebra in which X is a left exchangeable transitive GE-algebra. If a subset F of X in (X, f) satisfies (2.17), (3.4) and (4.6), then F is a belligerent interior GE-filter of type 3.*

The following example shows that any belligerent interior GE-filter may not be a belligerent interior GE-filter of type 1 or type 2.

Example 4.25. 1. Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation $*$ which is given in the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	d	d
b	1	a	1	d	d
c	1	1	1	1	1
d	1	a	b	1	1

Then it is routine to verify that (X, f) is an interior GE-algebra where

$$f : X \rightarrow X, x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ a & \text{if } x \in \{a, b, c, d\}, \end{cases}$$

and the set $F := \{1\}$ is a belligerent interior GE-filter in (X, f) . But it is not a belligerent interior GE-filter of type 1 since $a*(a*(f(b))) = a*(a*a) = a*1 = 1 \in F$ and $f(a*a) = f(1) = 1 \in F$ but $a*b = b \notin F$.

2. Consider a GE-algebra $X = \{1, a, b, c, d\}$ with the binary operation $*$ which is given in the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	1	1
b	1	1	1	1	1
c	1	1	b	1	1
d	1	1	b	1	1

Then it is routine to verify that (X, f) is an interior GE-algebra where

$$f : X \rightarrow X, x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ a & \text{if } x \in \{a, b, c, d\}, \end{cases}$$

and the set $F := \{1\}$ is a belligerent interior GE-filter in (X, f) . But it is not a belligerent interior GE-filter of type 2 since $a*(a*(f(b))) = a*(a*a) = a*1 = 1 \in F$ and $f(a*a) = f(1) = 1 \in F$ but $f(a)*b = a*b = b \notin F$.

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