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PRIME IDEALS OF TRANSITIVE *BE*-ALGEBRAS

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Abstract

The notion of prime ideals is introduced in transitive BE-algebras. Prime ideals are characterized with the help of principal ideals. Prime ideal theorem is stated and derived for BE-algebras. The concept of minimal prime ideals is introduced in transitive BE-algebras. A decomposition theorem of proper ideals into minimal prime ideals is derived.

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Introduction

The concept of BE-algebras was introduced and extensively studied in [7]. The class of BE-algebras was introduced as a generalization of the class of BCK-algebras of Iseki and Tanaka [5]. Some properties of filters of BE-algebras were studied by Ahn and Kim in [1] and by Meng in [9]. The notion of dual ideals in BCK-algebras was introduced by Deeba [3] in 1979. Later 2000, Sun [11] investigated the homomorphism theorems via dual ideals in bounded BCK-algebras. In [8], Meng introduced the notion of BCK-filters in BCK-algebras and presented a description of the BCK-filter generated by a set. In this paper, he discussed prime decompositions and irreducible decompositions. In [6], Jun, Hong and Meng, considered the fuzzification of the concept of BCK-filters, and investigate their properties.

In this work, we initially study some properties of ideals and the ideals generated by an arbitrary set. The notions of maximal ideals and prime ideals are introduced in transitive BE-algebras and they are characterized with the help of principal ideals. Properties of a prime ideal containing an arbitrary ideal are investigated. A necessary and sufficient condition is obtained for every proper ideal of a transitive BE-algebra to become a prime ideal. The famous prime ideal theorem of many algebraic structures is generalized to the case of prime ideals of transitive BE-algebras. Finally, some properties of prime ideals of transitive BE-algebras are derived with respect to inverse homomorphic images and cartesian products.

The concept of minimal prime ideals is introduced in transitive BE-algebras. Some properties of minimal prime ideals belonging to a proper ideal are investigated. Decomposition of a proper ideal as the intersection of all minimal prime ideals belonging to that proper ideal is derived. Another version of prime ideal theorem is derived with respect to a finite $\cap$-structure. Minimal prime ideals are characterized with the help of finite $\cap$-structures.

1. Preliminaries

In this section, we present certain definitions and results which are taken mostly from the papers [1, 2, 7, 9] and [10] for the ready reference of the reader.

Definition 1.1 [7]**.** An algebra $(X, *, 1)$ of type $(2, 0)$ is called a BE-algebra if it satisfies the following properties:

(1) $x * x = 1$,

(2) $x * 1 = 1$,

(3) $1 * x = x$,

(4) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$.

A BE-algebra X is called self-distributive if $x * (y * z) = (x * y) * (x * z)$ for all $x, y, z \in X$. A BE-algebra X is called transitive if $y * z \leq (x * y) * (x * z)$ for all $x, y, z \in X$. Every self-distributive BE-algebra is transitive. A BE-algebra X is called commutative if $(x * y) * y = (y * x) * x$ for all $x, y \in X$. We introduce a relation $\leq$ on a BE-algebra X by $x \leq y$ if and only if $x * y = 1$ for all $x, y \in X$. Clearly $\leq$ is reflexive. If X is commutative, then $\leq$ is both anti-symmetric, transitive and so it is a partial order on X.

Theorem 1.2 [9]**.** *Let X be a transitive BE-algebra and $x, y, z \in X$. Then*

(1) $1 \leq x$ *implies* $x = 1$,

(2) $y \leq z$ *implies* $x * y \leq x * z$ *and* $z * x \leq y * x$.

Definition 1.3 [1]**.** A non-empty subset F of a BE-algebra X is called a filter of X if, for all $x, y \in X$, it satisfies the following properties:

(1) $1 \in F$,

(2) $x \in F$ and $x * y \in F$ imply that $y \in F$.

For any $a \in X, \langle a \rangle = \{x \in X \mid a^n * x = 1 \text{ for some } n \in \mathbb{N}\}$ is called the principal filter generated a. If X is self-distributive, then $\langle a \rangle = \{x \in X \mid a * x = 1\}$. For a commutative BE-algebra, define $x \vee y = (y * x) * x$ for any $x, y \in X$. Then $x \vee y = y \vee x$ and the suprimum of x and y is $x \vee y$ for all $x, y \in X$. Hence $(X, \vee)$ will become a semilattice which is called a BE-semilattice.

A BE-algebra X is called bounded [2], if there exists an element 0 satisfying $0 \leq x$ (or $0 * x = 1$) for all $x \in X$. Define an unary operation N on a bounded BE-algebra X by $xN = x * 0$ for all $x \in X$.

Theorem 1.4 [2]**.** *Let X be a transitive BE-algebra and $x, y, z \in X$. Then*

(1) $1N = 0$ *and* $0N = 1$,

(2) $x \leq xNN$,

(3) $x * yN = y * xN$.

An element x of a bounded BE-algebra X is called *dense* [10] if $xN = 0$.We denote the set of all dense elements of a BE-algebra X by $\mathcal{D}(X)$.A BE-algebra X is called a *dense BE-algebra* if every non-zero element of X is dense (i.e., $xN = 0$ for all $0 \neq x \in X$). Let X and Y be two bounded BE-algebras, then a homomorphism $f : X \to Y$ is called bounded if $f(0) = 0$. If f is a bounded homomorphism, then it is easily observed that $f(xN) = f(x)N$ for all $x \in X$. For any bounded homomorphism $f : X \to Y$, define the *dual kernel* of the homomorphism f as $Dker(f) = \{x \in X \mid f(x) = 0\}$. It is easy to check that $Dker(f) = \{0\}$ whenever f is an injective homomorphism.

2. IDEALS OF TRANSITIVE BE-ALGEBRAS

In this section, some properties of ideals of a transitive BE-algebra are studied and the notion of maximal ideals is introduced in transitive BE-algebras. Some properties of maximal ideals are studied.

Definition 2.1. A non-empty subset I of a BE-algebra X is called an *ideal* of X if it satisfies the following conditions for all $x, y \in X$:

(I1) $0 \in I$,

(I2) $x \in I$ and $(xN * yN)N \in I$ imply that $y \in I$.

Obviously the single-ton set $\{0\}$ is an ideal of a BE-algebra X. For, suppose $x \in \{0\}$ and $(xN * yN)N \in \{0\}$ for $x, y \in X$. Then $x = 0$ and $yNN = (0N * yN)N \in \{0\}$. Hence $y \leq yNN = 0 \in \{0\}$. Thus $\{0\}$ is an ideal of X. In the following example, we observe non-trivial ideals of a BE-algebra.

Example 2.2. Let $X = \{1, a, b, c, d, 0\}$. Define an operation $*$ on X as follows:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Clearly $(X, *, 0, 1)$ is a bounded BE-algebra. It can be easily verified that the set $I = \{0, c, d\}$ is an ideal of X. However, the set $J = \{0, a, b, d\}$ is not an ideal of X, because of $a \in J$ and $(aN * cN)N = (d * b)N = aN = d \in J$ but $c \notin J$.

Lemma 2.3. *Let X be a transitive BE-algebra X. For any $x, y, z \in X$, we have*

(1) $xNNN \leq xN$,

(2) $x * y \leq yN * xN$,

(3) $x * yN \leq xNN * yN$,

(4) $(x * yNN)NN \leq x * yNN$,

(5) $(xN * yN)NN \leq xN * yN$,

(6) $x \leq y$ *implies* $yN \leq xN$,

(7) $x \leq y$ *implies* $y * zN \leq x * zN$.

Proof. (1) Let $x \in X$. Then $1 = (x * 0) * (x * 0) = x * ((x * 0) * 0) = x * xNN \leq x * xNNNN = xNNN * xN$. Hence $xNNN * xN = 1$, which gives $xNNN \leq xN$.

(2) Let $x, y \in X$. Since X is transitive, we get $yN = y*0 \le (x*y)*(x*0) = (x*y)*xN$. Hence $1 = yN*yN \le yN*((x*y)*xN) = (x*y)*(yN*xN)$. Thus, we get $(x*y)*(yN*xN) = 1$. Therefore $x*y \le yN*xN$.

(3) Let $x, y \in X$. Then, we get $x*yN = y*xN \le y*xNNN = xNN*yN$.

(4) Let $x, y \in X$. Clearly $(x*yNN)N \le (x*yNN)NNN$. Since X is transitive, we get $yN*(x*yNN)N \le yN*(x*yNN)NNN$ and so $x*(yN*(x*yNN)N) \le x*(yN*(x*yNN)NNN)$. Hence, we get

$$\begin{aligned} 1 &= (x*yNN)*(x*yNN) \\ &= x*((x*yNN)*yNN) \\ &= x*(yN*(x*yNN)N) \\ &\le x*(yN*(x*yNN)NNN) \\ &= x*((x*yNN)NN*yNN \\ &= (x*yNN)NN*(x*yNN). \end{aligned}$$

Thus $(x*yNN)NN*(x*yNN) = 1$. Therefore $(x*yNN)NN \le (x*yNN)$.

(5) Form (4), it can be easily verified.

(6) Let $x, y \in X$ be such that $x \le y$. Then by (2), we get $1 = x*y \le yN*xN$. Hence $yN*xN = 1$. Therefore $yN \le xN$.

(7) Let $x, y \in X$ be such that $x \le y$. Then by (6), we get $yN \le xN$. Since X is transitive, we get $z*yN \le z*xN$. Therefore $y*zN \le x*zN$. ■

Proposition 2.4. *Let I be an ideal of a transitive BE-algebra X. Then we have:*

(1) *For any $x, y \in X, x \in I$ and $y \le x$ imply $y \in I$,*

(2) *For any $x, y \in X, xN = yN, x \in I$ imply $y \in I$,*

(3) *For any $x \in X, x \in I$ if and only if $xNN \in I$.*

Proof. (1) Let $x, y \in X$. Suppose $x \in I$ and $y \le x$. Then $xN \le yN$, which implies $xN*yN = 1$. Hence $(xN*yN)N = 0 \in I$. Since $x \in I$, we get $y \in I$.

(2) Let $x, y \in X$. Assume that $xN = yN$. Suppose $x \in I$. Then we get $(xN*yN)N = 1N = 0 \in I$. Since I is an ideal of X, we get $y \in I$.

(3) Let $x \in X$. Suppose $x \in I$. Then we get $(xN*xNNN)N = (xNN*xNN)N = 1N = 0 \in I$. Since $x \in I$, it yields $xNN \in I$. Conversely, let $xNN \in I$ for any $x \in X$. Since $x \le xNN$, by property (1) we get that $x \in I$. ■

We denote by $\mathcal{I}(X)$ the set of all ideals of a BE-algebra X and $\mathcal{F}(X)$ the set of all filters of X. Let A be a non-empty subset of X, then the set

$$[A] = \bigcap \{I \in \mathcal{I}(X) \mid A \subseteq I\}$$

is called the ideal generated by A, written $[A]$. In the following proposition, we characterize the elements of a principal ideal generated by a set.

Theorem 2.5. *Let X be a transitive BE-algebra and $\emptyset \neq A \subseteq X$. Then*

$$[A] = \{x \in X \mid a_1N * (a_2N * (\cdots (a_nN * xN)\cdots)) = 1 \textit{ for some } a_1, a_2, \ldots, a_n \in A \textit{ and } n \in \mathbb{N}\}.$$

Proof. It is enough to show that $[A]$ is the smallest ideal of X containing the set A. Clearly $0 \in [A]$. Let $x \in [A]$ and $(xN * yN)N \in [A]$. Then there exist $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m \in A$ such that $a_1N * (a_2N * (\cdots (a_nN * xN)\cdots)) = 1$ and $b_1N * (b_2N * (\cdots (b_mN * (xN * yN)NN)\cdots)) = 1$. Hence we get

$$\begin{aligned} 1 &= b_mN * (\cdots * (b_1N * (xN * yN)NN)\cdots) \\ &\leq b_mN * (\cdots * (b_1N * (xN * yN))\cdots) \\ &= b_mN * (\cdots * (xN * (b_1N * yN))\cdots) \\ &\quad \cdots \\ &\quad \cdots \\ &= xN * (b_mN * (\cdots * (b_1N * yN))\cdots). \end{aligned}$$

Hence $xN \leq b_mN * (\cdots * (b_1N * yN)\cdots)$. Since X is transitive, we get $1 = a_nN*(\cdots*(a_1N*xN)\cdots) \leq a_nN*(\cdots*(a_1N*(b_mN*(\cdots*(b_1N*yN)\cdots)))\cdots)$. Hence

$$a_nN * (\cdots * (a_1N * (b_mN * (\cdots * (b_1N * yN)\cdots)))\cdots) = 1$$

where $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m \in A$. From the structure of $[A]$, it yields that $y \in [A]$. Therefore $[A]$ is an ideal of X. For any $x \in A$, we get $xN * (\cdots * (xN * xN)\cdots) = 1$. Hence $x \in [A]$. Therefore $A \subseteq [A]$.

Let I be an ideal of X containing A. Let $x \in [A]$. Then there exists $a_1, a_2, \ldots, a_n \in A \subseteq I$ such that $a_nN * (\cdots * (a_1N * xN)\cdots) = 1$. Hence $(a_nN * (\cdots * (a_1N * xN)\cdots)NN)N \leq (a_nN * (\cdots * (a_1N * xN)\cdots))N = 0 \in I$. Thus by Proposition 2.4(1), we get $(a_nN*(\cdots*(a_1N*xN)\cdots)NN)N \in I$. Since $a_n \in I$ and I is an ideal, we get $(a_{n-1}N*(\cdots*(a_1N*xN)\cdots))N \in I$. Continuing in this way, we finally get $x \in I$. Hence $[A] \subseteq I$. Therefore $[A]$ is the smallest ideal containing A. ∎

For $A = \{a\}$, we then denote $[\{a\}]$, briefly by $[a]$. We call this ideal by *principal ideal generated by a* and is represented by $[a] = \{x \in X \mid (aN)^n * xN = 1 \text{ for some } n \in \mathbb{N}\}$. We can easily observe, if X is self-distributive and $a \in X$, then $[a] = \{x \in X \mid aN * xN = 1\}$.

The following is a direct consequence of the above theorem.

Corollary 2.6. *Let X be a transitive BE-algebra. For any $a, b \in X$, and $A, B \subseteq X$, we have*

(1) $[0] = \{0\}$,

(2) $[X] = X$ *and* $[1] = X$,

(3) $A \subseteq B$ *implies* $[A] \subseteq [B]$,

(4) $a \leq b$ *implies* $[a] \subseteq [b]$,

(5) *if* A *is an ideal, then* $[A] = A$,

(6) *if* A *is an ideal and* $a \in A$, *then* $[a] \subseteq A$.

Proof. (1) Let $x \in [0]$. Then $(0N)^n * xN = 1$ for some $n \in \mathbb{N}$. Hence $xN = 1$. Thus $x \leq xNN = 1N = 0$. Therefore $x = 0$, which means $[0] = \{0\}$.

(2) For all $x \in X$, we get $1N * xN = 1 = 0 * xN = 1$. Hence $[1] = X$.

(3) Suppose $A \subseteq B$ and let $x \in [A]$ then $a_1N * (a_2N * (\cdots (a_nN * xN) \cdots)) = 1$ for some $a_1, a_2, \ldots, a_n \in A$ and $n \in \mathbb{N}$. Since $A \subseteq B$ implies $a_1N * (a_2N * (\cdots (a_nN * xN) \cdots)) = 1$ for some $a_1, a_2, \ldots, a_n \in B$ and $n \in \mathbb{N}$,we get $x \in [B]$ and hence $[A] \subseteq [B]$.

(4) Suppose $a \leq b$. By Lemma 2.3(6), we get $bN \leq aN$. Again by Lemma 2.3(7), we get $aN * xN \leq bN * xN$ for any $x \in X$. Since X is transitive, we get $(bN)^{n-1} * (aN * xN) \leq (bN)^{n-1} * (bN * xN) = (bN)^n * xN$. Thus $1 = (bN)^{n-1} * 1 \leq (bN)^n * xN$, which gives $x \in [b]$. Therefore $[a] \subseteq [b]$.

(5) From the construction of $[A]$, it is obvious.

(6) Let A be an ideal and $a \in A$. Suppose $x \in [a]$. Then there exists $n \in \mathbb{N}$ such that $(aN)^n * xN = 1$. Hence $1 = aN * ((aN)^{n-1} * xN) \leq aN * ((aN)^{n-1} * xN)NN$. Hence $aN * ((aN)^{n-1} * xN)NN = 1$, which gives $(aN * ((aN)^{n-1} * xN)NN)N = 0 \in A$. Since $a \in A$ and A is an ideal, we get $((aN)^{n-1} * xN)N \in A$. Now

$$\begin{aligned} (aN * ((aN)^{n-2} * xN)NN)N &\leq (aN * ((aN)^{n-2} * xN))N \\ &= ((aN)^{n-1} * xN)N \in A. \end{aligned}$$

which yields $(aN * ((aN)^{n-2} * xN)NN)N \in A$. Since $a \in A$, we get $(aN)^{n-2} * xN)N \in A$. Continuing in this way, we finally get $x \in A$. Therefore $[a] \subseteq A$. ∎

Corollary 2.7. *Let X be a transitive BE-algebra and $a \in X$. For any $A \subseteq X$, the set $[A \cup \{a\}]$ is the smallest ideal of X that contains both A and a.*

Proposition 2.8. *Let I an ideal of a transitive BE-algebra X and $A \subseteq X$. Then,*

$$\begin{aligned} [I \cup A] = \{x \in X \mid (a_1N * (a_2N * (\cdots (a_nN * xN))))N \in I \\ \textit{for some } a_1, a_2, \ldots, a_n \in A \textit{ and } n \in \mathbb{N}\}. \end{aligned}$$

Proof. Let us consider $B = \{x \in X \mid (a_1N * (a_2N * (\cdots (a_nN * xN))))N \in I$ for some $a_1, a_2, \ldots, a_n \in A$ and $n \in \mathbb{N}\}$. It is enough to show that B is the smallest ideal of X containing both I and A. Clearly $0 \in B$. Let $x, y \in X$ be such that $x \in B$ and $(xN * yN)N \in B$. Then there exists $m, n \in \mathbb{N}$ such that $(a_1N * (a_2N * (\cdots (a_nN * xN))))N \in I$ and $(b_1N * (b_2N * (\cdots (b_mN * (xN * yN)NN))))N \in I$. By Lemma 2.3(5), we have

$$\begin{aligned} &(b_1N * (b_2N * (\cdots (b_mN * (xN * yN)NN)))) \\ &\leq (b_1N * (b_2N * (\cdots (b_mN * (xN * yN))))) \\ &= (xN * (b_1N * (b_2N(\cdots * (b_mN * yN))))). \end{aligned}$$

By Lemma 2.3(6), we get $(xN * (b_1N * (b_2N(\cdots * (b_mN * yN)))))N \leq (b_1N * (b_2N * (\cdots (b_mN * (xN * yN)NN))))N \in I$. Therefore $(xN * (b_1N * (b_2N(\cdots * (b_mN * yN)))))N \in I$.

By applying the transitivity of X and Lemma 2.3(2), we get

$(xN * (b_1N * (b_2N(\cdots * (b_mN * yN))))) \leq (a_1N * (a_2N * (\cdots (a_nN * xN)))) * (a_1N * (a_2N * (\cdots (a_nN * (b_1N * (b_2N(\cdots * (b_mN * yN)))))))) \leq (a_1N * (a_2N * (\cdots (a_nN * xN))))NN * (a_1N * (a_2N * (\cdots (a_nN * (b_1N * (b_2N(\cdots * (b_mN * yN))))))))NN$.

By Lemma 2.3(6), $((a_1N * (a_2N * (\cdots (a_nN * xN))))NN * (a_1N * (a_2N * (\cdots (a_nN * (b_1N * (b_2N(\cdots * (b_mN * yN))))))))NN)N \leq ((xN * (b_1N * (b_2N(\cdots * (b_mN * yN))))))N \in I$. Therefore $((a_1N * (a_2N * (\cdots (a_nN * xN))))NN * (a_1N * (a_2N * (\cdots (a_nN * (b_1N * (b_2N(\cdots * (b_mN * yN))))))))NN)N \in I$. Since $(a_1N * (a_2N * (\cdots (a_nN * xN))))N \in I$ and I is an ideal, we get $(a_1N * (a_2N * (\cdots (a_nN * (b_1N * (b_2N(\cdots * (b_mN * yN))))))))N \in I$. Hence $y \in B$. Therefore B is an ideal of X. Let $x \in I$. Clearly $(a_1N * (a_2N * (\cdots (a_nN * xN)))) \leq (a_1N * (a_2N * (\cdots (a_nN * xN))))NN$. Then by Lemma 2.3(6), we get

$$\begin{aligned} &(xN * ((a_1N * (a_2N * (\cdots (a_nN * xN)))))NN)N \\ &\leq (xN * (a_1N * (a_2N * (\cdots (a_nN * xN)))))N \\ &= (a_1N * (a_2N * (\cdots (a_nN * (xN * xN)))))N \\ &= (a_1N * (a_2N * (\cdots (a_nN * (1)))))N \\ &= 0. \end{aligned}$$

Hence $(xN * ((a_1N * (a_2N * (\cdots (a_nN * xN)))))NN)N = 0 \in I$. Since $x \in I$ and I is an ideal, we get $((a_1N * (a_2N * (\cdots (a_nN * xN)))))N \in I$. Therefore $x \in B$ and hence $I \subseteq B$. Since for any $a \in A, (aN * aN)N = 0 \in I$, we get $a \in B$. Therefore $A \subseteq B$. Thus B is an ideal of X containing both I and A.

Suppose K is an ideal of X such that $I \subseteq K$ and $A \subseteq K$. Let $x \in B$. Then $(a_1N * (a_2N * (\cdots (a_nN * xN))))N \in I \subseteq K$ and $a_1, a_2, \ldots, a_n \in A \subseteq K$ for some $n \in \mathbb{N}$. Since $(a_1N * (a_2N * (\cdots (a_nN * xN)))NN)N \leq (a_1N * (a_2N * (\cdots (a_nN * xN))))N \in K$. Therefore $(a_1N * (a_2N * (\cdots (a_nN * xN)))NN)N$. Since $a_1 \in K$

and K is an ideal, we get $(a_2N * (\cdots(a_nN * xN)))N \in K$. Continuing in this way, we get $(a_nN * xN))N \in K$ and hence $x \in K$. Hence $B \subseteq K$. Thus B is the smallest ideal of X containing both I and A. ∎

The following corollaries are direct consequence of the above proposition.

Corollary 2.9. *Let X be a transitive BE-algebra and I an ideal of X. For any $a \in X$,*

$$[I \cup \{a\}] = \{x \in X \mid ((aN)^n * xN)N \in I \text{ for some } n \in \mathbb{N}\}.$$

Corollary 2.10. *Let X be a self-distributive BE-algebra and I an ideal of X. Then, for any $a \in X$, $[I \cup \{a\}] = \{x \in X \mid (aN * xN)N \in I\}$.*

Definition 2.11. An ideal I of a BE-algebra X is said to be *proper* if $I \neq X$.

Definition 2.12. A proper ideal M of a BE-algebra X is said to be *maximal* if M is not properly contained in any other proper ideal of X (i.e., $M \subseteq I \subseteq X$ implies $M = I$ or $I = X$ for any ideal I of X).

Example 2.13. Let $X = \{0, a, b, c, d, 1\}$. Define an operation $*$ on X as follows:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	1	1	d	d
b	1	c	1	c	d	c
c	1	b	b	1	d	b
d	1	a	b	c	1	a
0	1	1	1	1	1	1

Clearly $(X, *, 0, 1)$ is a bounded BE-algebra. It is easy to check that $I_1 = \{0\}$, $I_2 = \{0, a\}$,$I_3 = \{0, b\}$,$I_4 = \{0, c\}$, $I_5 = \{0, a, b\}$ and $I_6 = \{0, a, c\}$ are ideals of X in which I_2, I_3, I_4, I_5 and I_6 are proper ideals. Also here we can easily observe that I_5 and I_6 are only maximal ideals of X.

Theorem 2.14. *A proper ideal M of a transitive BE-algebra X is maximal if and only if $[M \cup \{x\}] = X$ for any $x \in X - M$.*

Proof. Let M be a proper ideal of X. Assume that M is maximal. Let $x \in X - M$. Suppose $[M \cup \{x\}] \neq X$. Hence $M \subseteq [M \cup \{x\}] \subset X$. Since M is maximal, we get $M = [M \cup \{x\}]$. Then $x \in M$, which is a contradiction. Therefore $[M \cup \{x\}] = X$.

Conversely, assume the condition. Suppose there exists an ideal I of X such that $M \subseteq I \subseteq X$. Let $M \neq I$. Then $M \subset I$. Choose $x \in I$ such that $x \notin M$. By the assumed condition, we get $[M \cup \{x\}] = X$. If $a \in X$, then $a \in [M \cup \{x\}]$. Hence $((xN)^n * aN)N \in M \subseteq I$ for some $n \in \mathbb{N}$. Then

$$(xN)^n * aN = xN * ((xN)^{n-1} * aN) \leq xN * ((xN)^{n-1} * aN)NN.$$

By Lemma 2.3(6) and Proposition 2.4(1) we get $(xN * ((xN)^{n-1} * aN)NN)N \leq ((xN)^n * aN)N \in I$. Since $x \in I$, implies $((xN)^{n-1} * aN)N \in I$. Continuing in this way, finally we get $a \in I$. Hence $I = X$. Therefore M is a maximal ideal of X. ■

3. Prime ideals of BE-algebras

In this section, the notion of prime ideals is introduced in transitive BE-algebras. A necessary and sufficient condition is derived for every proper ideal of a BE-algebra to become a prime ideal. Prime ideal theorem is stated and derived analogous to that in a distributive lattice.

Definition 3.1. A proper ideal P of a BE-algebra X is said to be *prime* if for any two ideals I and J of X, $I \cap J \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$.

Example 3.2. Let $X = \{0, a, b, c, d, 1\}$. Define an operation $*$ on X as follows:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	1	1	d	d
b	1	c	1	c	d	c
c	1	b	b	1	d	b
d	1	a	b	c	1	a
0	1	1	1	1	1	1

Clearly $(X, *, 0, 1)$ is a bounded BE-algebra. It is easy to check that $I_1 = \{0\}$, $I_2 = \{0, a\}$,$I_3 = \{0, b\}$,$I_4 = \{0, c\}$, $I_5 = \{0, a, b\}$ and $I_6 = \{0, a, c\}$ are ideals of X in which I_2, I_3, I_4, I_5 and I_6 are proper ideals. Also here we can easily observe that I_5 and I_6 are prime ideals of X.

Theorem 3.3. *A proper ideal P of a transitive BE-algebra X is prime if and only if for any $x, y \in X$, $[x] \cap [y] \subseteq P$ implies $x \in P$ or $y \in P$*

Proof. Let P be a proper ideal of X. Assume that P is prime. Let $x, y \in X$. Suppose $[x] \cap [y] \subseteq P$. By the definition of prime ideal, we get $[x] \subseteq P$ or $[y] \subseteq P$. Hence $x \in [x] \subseteq P$ or $\in [y] \subseteq P$. Therefore $x \in P$ or $y \in P$.

Conversely, assume the condition. Suppose I and J are two ideals of X such that $I \cap J \subseteq P$. Let $x \in I, y \in J$. Then $[x] \subseteq I$ and $[y] \subseteq J$. Hence $[x] \cap [y] \subseteq I \cap J \subseteq P$. By the assumed condition, we get $x \in P$ or $y \in P$. Thus $I \subseteq P$ or $J \subseteq P$. Therefore P is a prime ideal of X. ■

Theorem 3.4. *Let I be an ideal of a transitive BE-algebra X. For any $A, B \subseteq X$,*

$$[A] \cap [B] \subseteq I \textit{ if and only if } [I \cup A] \cap [I \cup B] = I.$$

Proof. Let I be an ideal of X. Suppose $[I \cup A] \cap [I \cup B] = I$ for $A, B \subseteq X$. Since $A \subseteq [I \cup A]$ and $B \subseteq [I \cup B]$. Hence $[A] \cap [B] \subseteq [I \cup A] \cap [I \cup B] = I$. Therefore $[A] \cap [B] \subseteq I$.

Conversely, assume that $[A] \cap [B] \subseteq I$ for any $A, B \subseteq X$. Clearly $I \subseteq [I \cup A] \cap [I \cup B]$. Let $x \in [I \cup A] \cap [I \cup B]$. Then $(a_1N * (a_2N * (\cdots (a_mN * xN))))N \in I$ and $(b_1N * (b_2N * (\cdots (b_nN * xN))))N \in I$ where $a_1, a_2, \ldots, a_m \in A$; $b_1, b_2, \ldots, b_n \in B$ for some $m, n \in \mathbb{N}$. Then there exist $m_1, m_2 \in I$ such that $m_1 = (a_1N * (a_2N * (\cdots (a_mN * xN))))N \in I$ and $m_2 = (b_1N * (b_2N * (\cdots (b_nN * xN))))N \in I$. Now, Lemma 2.3(5) gives

$$\begin{aligned}
1 &= m_1N * m_1N \\
&= m_1N * ((a_1N * (a_2N * (\cdots (a_mN * xN)))))NN \\
&\leq m_1N * ((a_1N * (a_2N * (\cdots (a_mN * xN))))) \\
&= a_1N * (a_2N * (\cdots (a_mN * (m_1N * xN)))) \\
&\leq a_1N * (a_2N * (\cdots (a_mN * (m_1N * xN)NN))).
\end{aligned}$$

Therefore $a_1N * (a_2N * (\cdots (a_mN * (m_1N * xN)NN))) = 1$. Then $(m_1N * xN)N \in [A]$. Similarly, we get $(m_2N * xN)N \in [B]$. Observe

$$(m_1N * xN) \leq m_2N * (m_1N * xN) = m_1N * (m_2N * xN)$$

$$\text{and} \quad (m_2N * xN) \leq m_1N * (m_2N * xN).$$

Then by Lemma 2.3(6), we obtain the following:

$$(m_1N * (m_2N * xN))N \leq (m_1N * xN)N$$

$$\text{and} \quad (m_1N * (m_2N * xN))N \leq (m_2N * xN)N.$$

Since $(m_1N * xN)N \in [A]$, $(m_2N * xN)N \in [B]$ and $[A], [B]$ are ideals, we get

$$(m_1N * (m_2N * xN))N \in [A] \quad \text{and} \quad (m_1N * (m_2N * xN))N \in [B].$$

Hence $(m_1N * (m_2N * xN))N \in [A] \cap [B] \subseteq I$. Since $(m_1N * (m_2N * xN)NN)N \leq (m_1N * (m_2N * xN))N$, we get $(m_1N * (m_2N * xN)NN)N \in I$. Since $m_1 \in I$, we get $(m_2N * xN)N \in I$. Since $m_2 \in I$, we get $x \in I$. Hence $[I \cup A] \cap [I \cup B] \subseteq I$. ■

Corollary 3.5. *Let I be an ideal of a transitive BE-algebra X. For any $a, b \in X$,*

$$[a] \cap [b] \subseteq I \textit{ if and only if } [I \cup \{a\}] \cap [I \cup \{b\}] = I.$$

Theorem 3.6. *Every maximal ideal of a transitive BE-algebra is prime.*

Proof. Let M be a maximal ideal of a transitive BE-algebra X. Let $x, y \in X$. Suppose $[x] \cap [y] \subseteq M$. If $x \notin M$ and $y \notin M$, then by Theorem 2.14, we have $[M \cup \{x\}] = X$ and $[M \cup \{y\}] = X$. Hence $[M \cup \{x\}] \cap [M \cup \{y\}] = X \neq M$. Thus $[x] \cap [y] \nsubseteq M$, which is a contradiction. So $x \in M$ or $y \in M$. Therefore M is prime. ∎

Theorem 3.7. *Let X be a transitive BE-algebra and $a \in X$. If I is an ideal of X such that $a \notin I$, then there exist a prime ideal P such that $a \notin P$ and $I \subseteq P$.*

Proof. Suppose I is an ideal of X such that $a \notin I$. Let $\mathcal{T} = \{G \in \mathcal{I}(X) \mid a \notin G, I \subseteq G\}$. Clearly $I \in \mathcal{T}$. Then $\mathcal{T} \neq \emptyset$. By Zorn's lemma, $\mathcal{T}$ has a maximal element say M. Clearly $a \notin M$. Now we prove that M is a prime ideal. Let $x, y \in X$ such that $[x] \cap [y] \subseteq M$. By Corollary 3.5, we get $[M \cup \{x\}] \cap [M \cup \{y\}] = M$. Since $a \notin M$, we get either $a \notin [M \cup \{x\}]$ or $a \notin [M \cup \{y\}]$. Since M is maximal, we get $[M \cup \{x\}] = M$ or $[M \cup \{y\}] = M$. Hence $x \in M$ or $y \in M$. Therefore M is a prime ideal such that $a \notin M$ and $I \subseteq M$. ∎

Corollary 3.8. *Let I be a proper ideal of a transitive BE-algebra X. Then*

$$I = \cap \{P \ / \ P \textit{ is a prime ideal of } X \textit{ such that } I \subseteq P\}.$$

Proof. Clearly $I \subseteq \cap \{P \ / \ P$ is a prime ideal of X such that $I \subseteq P\}$. Conversely, let $x \notin I$. Then by the above theorem 3.7, there exist a prime ideal P_x such that $x \notin P_x$ and $I \subseteq P_x$. Hence $x \notin \cap \{P \ / \ P$ is a prime ideal of X such that $I \subseteq P\}$. Therefore $\cap \{P \ / \ P$ is a prime ideal of X such that $I \subseteq P\} \subseteq I$.

Hence $I = \cap \{P \ / \ P$ is a prime ideal of X such that $I \subseteq P\}$. ∎

Corollary 3.9. *Let X be a transitive BE-algebra and $0 \neq x \in X$. Then there exist a prime ideal P such that $x \notin P$.*

Proof. Let $0 \neq x \in X$ and $I = \{0\}$. Then I is an ideal and $x \notin I$. By the above theorem 3.7, there exist a prime ideal P such that $x \notin P$. ∎

Corollary 3.10. *The intersection of all prime ideals of a transitive BE algebra is equal to $\{0\}$.*

Theorem 3.11. *Let X, Y be two transitive BE algebras and $f : X \to Y$ is homomorphism such that $f(X)$ is an ideal of Y. If I is a prime ideal of Y and $f^{-1}(I) \neq X$, then $f^{-1}(I)$ is a prime ideal of X.*

Proof. Let $f : X \to Y$ is homomorphism such that $f(X)$ is an ideal of Y. Suppose I is an ideal of Y. Let $x \in f^{-1}(I)$ and $(xN * yN)N \in f^{-1}(I)$. Then $f(x) \in I$ and $f(xN*yN)N \in I$. Hence $(f(x)N*f(y))N = (f(xN)*f(yN))N \in I$. Since $f(x) \in I$ and I is an ideal, we get $f(y) \in I$. Hence $y \in f^{-1}(I)$. Therefore $f^{-1}(I)$ is an ideal of X.

Let $x, y \in X$ such that $[x] \cap [y] \subseteq f^{-1}(I)$. Let $u \in [f(x)] \cap [f(y)]$ where $u \in Y$. Then there exist $m, n \in \mathbb{N}$ such that $(f(x)N)^m * uN = 1$ and $(f(y)N)^n * uN = 1$. Hence $((f(x)N)^m * uN)N = 1N = 0 \in I$ and $((f(y)N)^n * uN) = 1N = 0 \in I$. Since $f(x) \in f(X)$ and $f(X)$ is an ideal, we get $u \in f(X)$. Then $u = f(a)$ for some $a \in X$. Now, we have

$$\begin{aligned} (f(x)N)^m * f(a)N = 1 &\Rightarrow f((xN)^m * aN) = 1 \\ &\Rightarrow f((xN)^m * aN)N = 0 \in I \\ &\Rightarrow ((xN)^m * aN)N \in f^{-1}(I) \\ &\Rightarrow a \in [f^{-1}(I) \cup \{x\}]. \end{aligned}$$

Similarly, we get $a \in [f^{-1}(I) \cup \{y\}]$. Hence $a \in [f^{-1}(I) \cup \{x\}] \cap [f^{-1}(I) \cup \{y\}]$. Since $[x] \cap [y] \subseteq f^{-1}(I)$, we get $[f^{-1}(I) \cup \{x\}] \cap [f^{-1}(I) \cup \{y\}] = f^{-1}(I)$. Hence $a \in f^{-1}(I)$, which means $u = f(a) \in I$. Therefore $[f(x)] \cap [f(y)] \subseteq I$. Since I is a prime ideal of Y, we get $f(x) \in [f(x)] \subseteq I$ or $f(y) \in [f(y)] \subseteq I$. Hence $x \in f^{-1}(I)$ or $y \in f^{-1}(I)$. Therefore $f^{-1}(I)$ is a prime ideal of X. ■

Theorem 3.12. *Let X be a transitive BE-algebra. Then $\mathcal{I}(X)$ is a totally ordered set or a chain if and only if every proper ideal of X is a prime ideal.*

Proof. Assume that $\mathcal{I}(X)$ is a totally totally ordered set. Suppose I is a proper ideal of X. Choose $a, b \in X$ such that $[a] \cap [b] \subseteq I$. Since $[a]$ and $[b]$ are ideals of X, we get $[a] \subseteq [b]$ or $[b] \subseteq [a]$. Hence $a \in I$ or $b \in I$, which implies I is prime.

Conversely, assume that every proper ideal of X is a prime ideal. Let I and J be two proper ideals of X. Then $I \cap J$ is a proper ideal of X. Hence $I \cap J$ is a prime ideal of X. Thus $I \subseteq I \cap J$ or $J \subseteq I \cap J$, which implies $I \subseteq J$ or $J \subseteq I$. Therefore $\mathcal{I}(X)$ is a totally ordered set. ■

Theorem 3.13. *For any two ideals I and J of a transitive BE-algebra, $I \vee J = \{x \in X \mid aN * (bN * xN) = 1$ for some $a \in I, b \in J\}$ is the smallest ideal that is containing both I and J. Hence the set $(\mathcal{I}(X), \cap, \vee)$ is a complete lattice with smallest element $\{0\}$ and the greatest element X.*

Theorem 3.14. *Let $\mathcal{I}(X)$ be the set of all ideals of a transitive BE-algebra X. Then the algebraic structure $(\mathcal{I}(X), \cap, \vee)$ forms a distributive lattice.*

Proof. Let I, J, K be three ideals of X. Clearly $(I \cap J) \vee (I \cap K) \subseteq I \cap (J \vee K)$. Conversely, let $x \in I \cap (J \vee K)$. Then $x \in I$ and $x \in J \vee K$. Hence $aN * (bN * xN) = 1$ for some $a \in J$ and $b \in K$. Now, let $d_1 = (bN * xN)N$ and $d_2 = (d_1 N * xN)N$. Since $a \in J$, we get $d_1 = (bN * xN)N \in I$ and $d_2 = (d_1 N * xN)N \in I$. Then

$$\begin{aligned} (aN * d_1 N)N &= (aN * (bN * xN)NN)N \\ &\leq (aN * (bN * xN))N \\ &= 1N \\ &= 0 \in J. \end{aligned}$$

Since $a \in J$, we get $(bN * xN)N = d_1 \in J$. Hence $d_1 \in I \cap J$. Again

$$\begin{aligned}(bN * d_2N)N &= (bN * (d_1N * xN)NN)N \\ &\leq (bN * (d_1N * xN))N \\ &= (bN * ((bN * xN)NN * xN))N \\ &= ((bN * xN)NN * (bN * xN))N \\ &= 1N \\ &= 0 \in K.\end{aligned}$$

Therefore $(bN * d_2N)N = 0 \in K$. Since $b \in K$ implies $d_2 \in K$. Hence $d_2 \in I \cap K$. Now

$$\begin{aligned}d_1N * (d_2N * xN) &= (bN * xN)NN * ((d_1N * xN)NN * xN) \\ &\geq (bN * xN)NN * ((d_1N * xN) * xN) \\ &= (bN * xN)NN * (((bN * xN)NN * xN) * xN) \\ &= ((bN * xN)NN * xN) * ((bN * xN)NN * xN) \\ &= 1.\end{aligned}$$

Hence $d_1N * (d_2N * xN) = 1$. Since $d_1 \in I \cap J$ and $d_2 \in I \cap K$, we get $x \in (I \cap J) \vee (I \cap K)$. Hence $I \cap (J \vee K) \subseteq (I \cap J) \vee (I \cap K)$. Therefore $I \cap (J \vee K) = (I \cap J) \vee (I \cap K)$. Thus $(\mathcal{I}(X), \cap, \vee)$ is a distributive lattice. ∎

We now generalise the famous *prime ideal theorem* of various algebraic structures in transitive BE-algebras. Let us define a $\cap$-closed subset of a BE-algebra as the subset S of X in which $[a] \cap [b] \subseteq S$ for all $a, b \in S$.

Proposition 3.15. *Let P be a prime ideal of a transitive BE-algebra X and $a \in X$. Then the set $S = \{x \in X \mid [x] \subseteq [a] \vee J$ for some ideal J with $J \nsubseteq P\}$ is a $\cap$-closed subset of X.*

Proof. Let P be a prime ideal of X and $x, y \in X$. Suppose $x, y \in S$. Then there exist ideals J_1 and J_2 of X with $J_1 \nsubseteq P, J_2 \nsubseteq P$ such that $[x] \subseteq [a] \vee J_1$ and $[y] \subseteq [a] \vee J_2$. Hence

$$[x] \cap [y] \subseteq ([a] \vee J_1) \cap ([a] \vee J_2) = [a] \vee (J_1 \cap J_2).$$

Since P is prime, we get $J_1 \cap J_2 \nsubseteq P$. Let $t \in [x] \cap [y]$. Then $[t] \subseteq [x] \cap [y] \subseteq [a] \vee (J_1 \cap J_2)$. Hence $t \in S$, which gives $[x] \cap [y] \subseteq S$. Therefore S is $\cap$-closed. ∎

Theorem 3.16 (Prime ideal theorem)**.** *Let I be an ideal and S be a $\cap$-closed subset of a transitive BE-algebra X such that $I \cap S = \emptyset$. Then there exists a prime ideal P of X such that $I \subseteq P$ and $P \cap S = \emptyset$.*

Proof. Let I be an ideal and S be a $\cap$-closed subset of a transitive BE-algebra X such that $I \cap S = \emptyset$. Consider $\mathcal{F} = \{J \in \mathcal{I}(X) \mid I \subseteq J \text{ and } J \cap S = \emptyset\}$. Clearly $I \in \mathcal{F}$ and so $\mathcal{F} \neq \emptyset$. Let $\{J_\alpha\}_{\alpha \in \Delta}$ be a chain of elements of $\mathcal{F}$. Then clearly $\bigcup_{\alpha \in \Delta} J_\alpha$ is an upper bound of $\{J_\alpha\}_{\alpha \in \Delta}$. Hence the hypothesis of Zorn's lemma is satisfied. Thus $\mathcal{F}$ has a maximal element, say M. Clearly M is an ideal such that $I \subseteq M$ and $M \cap S = \emptyset$. We now prove that M is prime. Let $x, y \in X$ be such that $x \notin M$ and $y \notin M$. Then $M \subset M \vee [x]$ and $M \subset M \vee [y]$. By the maximality of M, we should have $(M \vee [x]) \cap S \neq \emptyset$ and $(M \vee [y]) \cap S \neq \emptyset$. Choose $a \in (M \vee [x]) \cap S$ and $b \in (M \vee [y]) \cap S$. Since $a, b \in S$, we get $[a] \cap [b] \subseteq S$ because of S is $\cap$-closed. Now

$$[a] \cap [b] \subseteq (M \vee [x]) \cap (M \vee [y]) = M \vee ([x] \cap [y]).$$

If $[x] \cap [y] \subseteq M$, then $[a] \cap [b] \subseteq M$. Hence $[a] \cap [b] \subseteq M \cap S$, which is a contradiction. Thus $[a] \cap [b] \nsubseteq M$. Therefore M is a prime ideal of X. ■

In the following, some properties of prime ideals are discussed with respect to cartesian products or direct products of BE-algebras. For this, we first observe the following basic properties.

Lemma 3.17. *Let X_1 and X_2 be two transitive BE-algebras. For any $a \in X_1, b \in X_2$, we have*

(1) $[(a,b)] = [a] \times [b]$,

(2) $([a] \times [b]) \cap ([c] \times [d]) = ([a] \cap [c]) \times ([b] \cap [d])$,

(3) $[(a,b)] \cap [(c,d)] = ([a] \cap [c]) \times ([b] \cap [d])$.

Proof. (1) Let $x \in X_1$ and $y \in X_2$. Then $(x,y) \in X_1 \times X_2$. Hence we have

$$\begin{aligned}
(x,y) \in [(a,b)] &\Leftrightarrow ((a,b)N)^n * (x,y)N = (1,1) \quad \text{for some } n \in \mathbb{N} \\
&\Leftrightarrow ((aN, bN))^n * (xN, yN) = (1,1) \\
&\Leftrightarrow ((aN)^n * xN, (bN)^n * yN) = (1,1) \\
&\Leftrightarrow (aN)^n * xN = 1 \text{ and } (bN)^n * yN = 1 \\
&\Leftrightarrow x \in [a] \text{ and } y \in [b] \\
&\Leftrightarrow (x,y) \in [a] \times [b].
\end{aligned}$$

Therefore $[(a,b)] = [a] \times [b]$.

(2) Let $x \in X_1$ and $y \in X_2$. Then $(x,y) \in X_1 \times X_2$. Hence we have

$$\begin{aligned}
(x,y) \in ([a] \times [b]) \cap ([c] \times [d]) &\Leftrightarrow (x,y) \in [a] \times [b] \text{ and } (x,y) \in [c] \times [d] \\
&\Leftrightarrow x \in [a], y \in [a] \text{ and } x \in [c], y \in [d] \\
&\Leftrightarrow x \in [a] \cap [c] \text{ and } y \in [b] \cap [d] \\
&\Leftrightarrow (x,y) \in ([a] \cap [c]) \times ([b] \cap [d]).
\end{aligned}$$

Therefore $([a] \times [b]) \cap ([c] \times [d]) = ([a] \cap [c]) \times ([b] \cap [d])$.

(3) It is straight forward from (1) and (2). ∎

Theorem 3.18. *Let X_1 and X_2 be two transitive BE-algebras, P_1 and P_2 be the prime ideals of X_1 and X_2 respectively. Then $P_1 \times X_2$ and $X_1 \times P_2$ are prime ideals of $X_1 \times X_2$*

Proof. Let P_1 and P_2 be the prime ideals of X_1 and X_2 respectively. It is easy to verify that $P_1 \times X_2$ and $X_1 \times P_2$ are ideals of $X_1 \times X_2$. Let $(a, b), (c, d) \in X_1 \times X_2$. Suppose $[(a, b)] \cap [(c, d)] \subseteq P_1 \times X_2$. Then by the above lemma 3.17, we get $([a] \cap [c]) \times ([b] \cap [d]) \subseteq P_1 \times X_2$. Hence $[a] \cap [c] \subseteq P_1$. Since P_1 is a prime ideal of X_1, we get $a \in P_1$ or $c \in P_1$. Thus $(a, b) \in P_1 \times X_2$ or $(c, d) \in P_1 \times X_2$. Therefore $P_1 \times X_2$ is a prime ideal of $X_1 \times X_2$. Similarly, we can prove that $X_1 \times P_2$ is also a prime ideal of $X_1 \times X_2$. ∎

Theorem 3.19. *Let X_1 and X_2 be two transitive BE-algebras and P be a prime ideal of $X_1 \times X_2$. Then P is of the form $P_1 \times X_2$ or $X_1 \times P_2$, where P_i is a prime ideal of X_i for $i = 1, 2$.*

Proof. Let P be a prime ideal of $X_1 \times X_2$. Consider the projections $\pi_1(P)$ and $\pi_2(P)$ of P as

$$P_1 = \pi_1(P) = \{x_1 \in X_1 \mid (x_1, x_2) \in P, \text{ for some } x_2 \in X_2\}$$

$$P_2 = \pi_2(P) = \{x_2 \in X_2 \mid (x_1, x_2) \in P, \text{ for some } x_1 \in X_1\}.$$

It is easy to verify that P_1 and P_2 are ideals of X_1 and X_2 respectively. We first show that P_1 and P_2 are prime ideals of X_1 and X_2 respectively. Suppose $P_1 = X_1$ and $P_2 = X_2$. Let $(a, b) \in X_1 \times X_2$. Then there exist $x \in X_1$ and $y \in X_2$ such that $(a, y) \in P$ and $(x, b) \in P$. Since $(a, 0) \leq (a, y)$ and $(0, b) \leq (x, b)$, we get $(a, 0) \in P$ and $(0, b) \in P$. Since $(0, b) \in P$, $(0, bNN) = (0, b)NN \in P$. Now

$$\begin{aligned} ((a,0)N * (a,b)N)N &= ((aN, 0N) * (aN, bN))N \\ &= (aN * aN, 0N * bN)N \\ &= (1, bN)N \\ &= (0, bNN) \in P. \end{aligned}$$

Since $(a, 0) \in P$ and P is an ideal, it gives $(a, b) \in P$. Hence $P = X_1 \times X_2$, which is a contradiction to that P is proper. Next suppose that $P_1 \neq X_1$ and $P_2 \neq X_2$. Choose $a \in X_1 - P_1$ and $b \in X_2 - P_2$. Then $(a, 0) \notin P$ and $(0, b) \notin P$. Since P is prime, we get

$$[(0, 0)] = [0] \times [0] = ([a] \cap [0]) \times ([0] \cap [b]) = [(a, 0)] \cap [(0, b)] \nsubseteq P$$

which is a contradiction. From the above observations, we get that either $P_1 = X_1$ and $P_2 \neq X_2$ or $P_1 \neq X_1$ and $P_2 = X_2$.

Case (i) Suppose $P_1 = X_1$ and $P_2 \neq X_2$. Let $x_2, y_2 \in X_2$ and $[x_2] \cap [y_2] \subseteq P_2$. Then there exists $a \in X_1 = P_1$ such that $[a] \times ([x_2] \cap [y_2]) \subseteq P$. Therefore

$$\begin{aligned}[(a, x_2)] \cap [(a, y_2)] &= ([a] \cap [a]) \times ([x_2] \cap [y_2]) \\ &= [a] \times ([x_2] \cap [y_2]) \subseteq P.\end{aligned}$$

Since P is prime, we get $(a, x_2) \in P$ or $(a, y_2) \in P$. Hence $x_2 \in P_2$ or $y_2 \in P_2$. Therefore P_2 is a prime ideal of X_2. We now show that $P = X_1 \times P_2$. Clearly $P \subseteq X_1 \times P_2$. On the other hand, suppose $(a, y) \in X_1 \times P_2$. Since $P_1 = X_1$, there exists $b \in X_2$ such that $(a, b) \in P$ and there exists $x \in X_1$ such that $(x, y) \in P$. Since $(a, 0) \leq (a, b)$ and $(0, y) \leq (x, y)$, we get $(a, 0) \in P$ and $(0, y) \in P$. Since $(0, y) \in P$, we get $(0, yNN) = (0, y)NN \in P$. Now

$$\begin{aligned}((a, 0)N * (a, y)N)N &= ((aN, 0N) * (aN, yN))N \\ &= (aN * aN, 0N * yN)N \\ &= (1, yN)N \\ &= (0, yNN) \in P.\end{aligned}$$

Since $(a, 0) \in P$ and P is an ideal, it gives $(a, y) \in P$. Hence $X_1 \times P_2 \subseteq P$. Therefore $P = X_1 \times P_2$.

Case (ii) Suppose $P_1 \neq X_1$ and $P_2 = X_2$. Similarly, we can prove that P_1 is prime ideal of X_1 and $P = P_1 \times X_2$. ∎

The following corollary is an extension of the above theorem.

Corollary 3.20. *Let $\{X_i\}_{i=1}^n$ be a finite family of transitive BE-algebras. Let P be an ideal of $\prod_{i=1}^n X_i$. Then P is prime if and only if P is of the form $\prod_{i=1}^n P_i$, where $P_i = X_i$ for all except one i,in this case P_i is a prime ideal of X_i.*

Theorem 3.21. *Let X_1 be a subalgebra of a transitive BE-algebra X and P_1 is a prime ideal of X_1. Then there exists a prime ideal P of X such that $P \cap X_1 = P_1$.*

Proof. Let P_1 be a prime ideal of X_1. Then $X_1 - P_1$ is a $\cap$-closed subset of X. Write $I = [P_1]$, the ideal generated by P_1. Then $P_1 \subseteq I \cap X_1$. Suppose $I \cap (X_1 - P_1) \neq \emptyset$. Choose $x \in I \cap (X_1 - P_1)$. Then $x \in I$ and $x \in (X_1 - P_1)$. Since $x \in I = [P_1]$, there exists $a_1, a_2, \ldots, a_n \in P_1, n \in \mathbb{N}$ such that $a_1N * (a_2N * (\cdots (a_nN * xN) \cdots)) = 1$. Then

$$\begin{aligned}&(a_1N * (a_2N * (\cdots (a_nN * xN) \cdots)))N \\ &= (a_1N * (a_2N * (\cdots (a_nN * xN) \cdots))NN)N \\ &= 1N \\ &= 0 \in P_1.\end{aligned}$$

Since $a_1 \in P_1$, we get $(a_2N*(\cdots(a_nN*xN)\cdots))N \in P_1$. Continuing in this way, finally we get $x \in P_1$. Since $x \in (X_1 - P_1)$, we have arrived at a contradiction. Hence $I \cap (X_1 - P_1) = \emptyset$. Then by Prime ideal theorem, there exists a prime ideal P of X such that $I \subseteq P$ and $P \cap (X_1 - P_1) = \emptyset$. Since $I \subseteq P$, we get $I \cap X_1 \subseteq P \cap X_1$. Since $P \cap (X_1 - P_1) = \emptyset$, we get $P \subseteq P_1$. Hence both observations lead to

$$P_1 \subseteq I \cap X_1 \subseteq P \cap X_1 \subseteq P_1 \cap X_1 \subseteq P_1.$$

Therefore $P_1 = P \cap X_1$. ■

4. Minimal prime ideals

In this section, the notion of minimal prime ideals is introduced in transitive BE-algebras. It is derived that every proper ideal of a transitive BE-algebra can expressed as a decomposition of distinct minimal prime ideals. The notion of finite $\cap$-structure is introduced and investigated its relation with the minimal prime ideal.

Definition 4.1. Let I be an ideal and P a prime ideal of a transitive BE-algebra X such that $I \subseteq P$. Then P is called a *minimal prime ideal* belonging to I if there exists no prime ideal Q such that $I \subseteq Q \subset P$.

Example 4.2. Let $X = \{0, a, b, c, d, 1\}$. Define an operation $*$ on X as follows:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	1	1	d	d
b	1	c	1	c	d	c
c	1	b	b	1	d	b
d	1	a	b	c	1	a
0	1	1	1	1	1	1

Clearly $(X, *, 0, 1)$ is a bounded BE-algebra. Clearly $I_1 = \{0\}$, $I_2 = \{0, a\}$,$I_3 = \{0, b\}$,$I_4 = \{0, c\}$,$I_5 = \{0, a, b\}$ and $I_6 = \{0, a, c\}$ are ideals of X, in which I_2, I_3, I_4, I_5 and I_6 are proper ideals. Here I_5 and I_6 are prime ideals of X. Also I_5 is a minimal prime ideal of I_2, I_3 and I_6 is a minimal prime ideal of I_2, I_4.

In a BE-algebra X, the minimal prime ideals belonging to $\{0\}$ are simply called minimal prime ideals of X. In the other version, a minimal prime ideal of a BE-algebra is the minimal element of the partial order set of all prime ideals. Thus a prime ideal P of X is a minimal prime ideal if for any prime ideal I of X such that $I \subseteq P$, then $P = I$. Using the Zorn's lemma, we have the following proposition.

Proposition 4.3. *Let I be a proper ideal of a transitive BE-algebra X. Then every prime ideal of X, containing I, contains at least a minimal prime ideal belonging to I.*

Proof. Let P be a prime ideal of X such that $I \subseteq P$. Consider the collection

$$\mathfrak{T} = \{Q \mid Q \text{ is a prime ideal of } X \text{ such that } I \subseteq Q \subseteq P\}.$$

Clearly $P \in \mathfrak{T}$ and hence $\mathfrak{T} \neq \emptyset$. Let $\{Q_\alpha\}_{\alpha\in\Delta}$ be a chain of elements in $\mathfrak{T}$. Since $\{Q_\alpha\}_{\alpha\in\Delta}$ is a chain, we get that $\bigcap_{\alpha\in\Delta} Q_\alpha$ is a prime ideal of X. Since $I \subseteq Q_\alpha \subseteq P$ for all $\alpha \in \Delta$, it is clear that $I \subseteq \bigcap_{\alpha\in\Delta} Q_\alpha \subseteq P$. Hence $\bigcap_{\alpha\in\Delta} Q_\alpha$ is a lower bound for $\{Q_\alpha\}_{\alpha\in\Delta}$. Therefore by Zorn's lemma, $\mathfrak{T}$ has a minimal element, say Q_0. Therefore Q_0 is a minimal prime ideal such that $I \subseteq Q_0 \subseteq P$. ∎

By taking $I = \{0\}$, we get the following easy consequence.

Corollary 4.4. *Every prime ideal of a transitive BE-algebra X contains at least a minimal prime ideal.*

Proposition 4.5. *Let I be a proper ideal of a transitive BE-algebra X. Then I is the intersection of all minimal prime ideals of X, belonging to I.*

Proof. Since I is contained in every minimal prime ideal of X, belonging to I and so contained in the intersection of all minimal prime ideals belonging to I. To prove the converse, let $x \notin I$. Then by Corollary 3.9, there exists a prime ideal P of X such that $I \subseteq P$ and $x \notin P$. Then there exists a minimal prime ideal M of X such that $I \subseteq M \subseteq P$. Since $x \notin P$, we get $x \notin M$. Hence M is a minimal prime ideal of X, belonging to I, such that $x \notin M$. Thus x is not in the intersection of all minimal prime ideals of X, belonging to I. ∎

If we take $I = \{0\}$ in the above proposition, the following is a direct consequence.

Corollary 4.6. *Let X be a transitive BE-algebra. Then the intersection of all minimal prime ideals of X is equal to $\{0\}$.*

Corollary 4.7. *Let I be a proper ideal of a transitive BE-algebra X. Then the intersection of all minimal prime ideals of X, belonging to I, coincides with that of all prime ideals of X, containing I.*

By considering $I = \{0\}$ in the above corollary, we get the following.

Corollary 4.8. *In any transitive BE-algebra X, the intersection of all minimal prime ideals of X coincides with that of all prime ideals of X.*

By Corollary 3.8, it is observed that every proper ideal of a BE-algebra X can be decomposed as the intersection of all minimal prime ideals of X, belonging to I.

Theorem 4.9 (Unique decomposition theorem)**.** *Let I be a proper ideal of a transitive BE-algebra X. If there exist positive integers m and n such that*

$$I = P_1 \cap P_2 \cap \cdots \cap P_m \quad \text{and} \quad I = Q_1 \cap Q_2 \cap \cdots \cap Q_n$$

are two representations of distinct minimal prime ideals of X, belonging to I, then $m = n$, and for any P_i in the first expression there is Q_j in the second expression such that $P_i = Q_j$.

Proof. Let $P_i (i = 1, 2, \ldots, m)$ be a minimal prime ideal in the first representation. Clearly $I \subseteq P_i$. By the second representation, we have $Q_1 \cap (Q_2 \cap \cdots \cap Q_n) \subseteq P_i$. Since P_i is prime, we get

$$Q_1 \subseteq P_i \ \text{ or } \ Q_2 \cap \cdots \cap Q_n \subseteq P_i.$$

If $Q_1 \subseteq P_i$, then the minimality of P_i provides that $P_i = Q_1$. If $Q_1 \nsubseteq P_i$, then $Q_2 \cap \cdots \cap Q_n \subseteq P_i$. Repeating the same argument, we finally get that there exists $j \in \{2, 3, \ldots, m\}$ such that $P_i = Q_j$. It remains to show that $m = n$. Note that $P_1, P_2, \ldots, P_m$ are distinct, the preceding argument actually implies $m \leq n$. If we begin with the second representation, by the entirely similar argument, we will obtain $n \leq m$. Therefore $m = n$. ∎

Corollary 4.10. *If a proper ideal I of a transitive BE-algebra X can be expressed as the intersection of a finite number of distinct minimal prime ideals of X, belonging to I, then such representation is unique except their occurring order.*

Corollary 4.11. *If the ideal $\{0\}$ of a transitive BE-algebra X can be expressed as the intersection of a finite number of distinct minimal prime ideals of X, then such representation is unique except their occurring order.*

Definition 4.12. A nonempty subset S of a BE-algebra X is called a *finite $\cap$-structure*, if $([x] \cap [y]) \cap S \neq \emptyset$ for all $x, y \in S$.

Example 4.13. Let $X = \{0, a, b, c, d, 1\}$. Define an operation $*$ on X as follows:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Clearly $(X, *, 0, 1)$ is a bounded BE-algebra.

Recall that $[\alpha] = \{x \in X \mid (\alpha N)^n * xN = 1 \text{ for some } n \in \mathbb{N}\ \}$. Then we have $[a] = \{0, a, b, d\}$, $[b] = \{0, b\}$, $[c] = \{0, c, d\}$, $[d] = \{0, d\}$, $[0] = \{0\}$ and $[1] = X$.

Also $[a]\cap[b] = \{0,b\}, [a]\cap[c] = \{0,d\}, [a]\cap[d] = \{0,d\}, [b]\cap[c] = \{0\}, [b]\cap[d] = \{0\}, [c]\cap[d] = \{0,d\}$.

Take $S_1 = \{1,a,b\}$ then $([a]\cap[b])\cap S_1 = \{b\}, ([a]\cap[1])\cap S_1 = \{a,b\}$ and $([1]\cap[b])\cap S_1 = \{b\}$. Therefore $([x]\cap[y])\cap S_1 \neq \emptyset$ for all $x,y\in S_1$. Hence S_1 is a *finite* $\cap$*-structure*. Similarly we can observe that the set $S_2 = \{0,c,d\}$ is also a *finite* $\cap$*-structure*.

However the set $S_3 = \{a,c\}$ is not a *finite* $\cap$*-structure*, because of $([a]\cap[c])\cap S_3 = \emptyset$.

Lemma 4.14. *Every ideal of a transitive BE-algebra is a finite $\cap$-structure.*

Proof. Let I be an ideal of a transitive BE-algebra X. Let $x,y\in X$. Suppose that $x,y\in I$. Then $[x]\subseteq I$ and $[y]\subseteq I$. Hence $[x]\cap[y]\subseteq I$. Thus $([x]\cap[y])\cap I\neq\emptyset$. Therefore I is a finite $\cap$-structure. ■

Example 4.15. From the above Example 4.13, we can easily observe that $S_2 = \{0,c,d\}$ is an ideal and also a *finite* $\cap$*-structure*.

Proposition 4.16. *Let P be a proper ideal of a transitive BE-algebra X. Then P is prime if and only if $X-P$ is finite $\cap$-structure.*

Proof. Let P be an ideal of X. Assume that P is prime. Let $x,y\in X-P$. Then $x\notin P$ and $y\notin P$. Suppose $([x]\cap[y])\cap(X-P)=\emptyset$. Then $[x]\cap[y]\subseteq P$. Since P is prime, we get $x\in P$ or $y\in P$, which is a contradiction. Hence $([x]\cap[y])\cap(X-P)\neq\emptyset$.

Conversely, assume that $X-P$ is finite $\cap$-structure. Let $x,y\in X$ be such that $[x]\cap[y]\subseteq P$. Suppose $x\notin P$ and $y\notin P$. Then $x,y\in X-P$. Since $X-P$ is finite $\cap$-structure, we get $([x]\cap[y])\cap(X-P)\neq\emptyset$. Hence $[x]\cap[y]\nsubseteq P$, which is a contradiction. Thus $x\in P$ or $y\in P$. Therefore P is a prime ideal of X. ■

Example 4.17. From the above Example 4.13, we can easily observe that $S_2 = \{0,c,d\}$ is a prime ideal and $X-S_2 = \{1,a,b\} = S_1$ is a *finite* $\cap$*-structure*.

Theorem 4.18 (Prime ideal theorem)**.** *Let I be an ideal of a transitive BE-algebra X. If S is a finite $\cap$-structure such that $I\cap S=\emptyset$, then there exists a prime ideal P of X such that $I\subseteq P$ and $P\cap S=\emptyset$.*

Proof. Let I be an ideal of X and S be a finite $\cap$-structure such that $I\cap S=\emptyset$. Consider $\mathcal{F} = \{J\in\mathcal{I}(X)\mid I\subseteq J \text{ and } J\cap S=\emptyset\}$. Clearly $I\in\mathcal{F}$ and so $\mathcal{F}\neq\emptyset$. Let $\{J_\alpha\}_{\alpha\in\Delta}$ be a chain of elements of $\mathcal{F}$. Then clearly $\bigcup_{\alpha\in\Delta} J_\alpha$ is an upper bound of $\{J_\alpha\}_{\alpha\in\Delta}$. Hence the hypothesis of Zorn's lemma is satisfied. Thus $\mathcal{F}$ has a maximal element, say M. Clearly M is an ideal such that $I\subseteq M$ and $M\cap S=\emptyset$. We now prove that M is prime. Let I and J be two ideals of X such that $I\nsubseteq M$ and $J\nsubseteq M$. Then $M\subset[M\cup I]$ and $M\subset[M\cup J]$. By the maximality of M, we should have $[M\cup I]\cap S\neq\emptyset$ and $[M\cup J]\cap S\neq\emptyset$. Choose $a\in[M\cup I]\cap S$ and $b\in[M\cup J]\cap S$. Since $a\in[M\cup I]$ and $b\in[M\cup J]$, we get

$$[a] \cap [b] \subseteq [M \cup I] \cap [M \cup J].$$

Since $a, b \in S$, we get $([a] \cap [b]) \cap S \neq \emptyset$ because of S is finite $\cap$-structure. Hence

$$([M \cup I] \cap [M \cup J]) \cap S \neq \emptyset.$$

Since $M \in \mathcal{F}$, we get $M \cap S = \emptyset$. Comparing this with the last relation, we get $M \neq [M \cup I] \cap [M \cup J]$. By Theorem 3.4, it gives $I \cap J \nsubseteq M$. Therefore M is prime. ∎

Proposition 4.19. *Let I be an ideal and P a prime ideal of a transitive BE-algebra X such that $I \subseteq P$. Then P is a minimal prime ideal belonging to I if and only if $X - P$ is a maximal finite $\cap$-structure with respect to the property that $(X - P) \cap I = \emptyset$.*

Proof. Assume that P is a minimal prime ideal belonging to I. Then by Proposition 4.16, $X - P$ is a finite $\cap$-structure such that $(X - P) \cap I = \emptyset$. Suppose Q is another finite $\cap$-structure such that $Q \cap I = \emptyset$ and $X - P \subseteq Q$. Hence $I \subseteq X - Q \subseteq P$. By the minimality of P, we get $X - Q = P$. Hence $X - P$ is a maximal $\cap$-structure with respect to the property $(X - P) \cap I = \emptyset$.

Conversely, assume that $X - P$ be a maximal finite $\cap$-structure with respect to the property $(X - P) \cap I = \emptyset$. Suppose Q is a prime ideal of X such that $I \subseteq Q \subset P$. Then by Proposition 4.16, we get that $X - Q$ is a finite $\cap$-structure such that $X - P \subseteq X - Q$ and $(X - Q) \cap I = \emptyset$, which contradicts the maximality of $X - P$. Hence P is the minimal prime ideal belonging to I. ∎

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