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ORDER OF FINITE SOFT QUASIGROUPS WITH APPLICATION TO EGALITARIANISM

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Abstract

In this work, a soft set (F, A) was introduced over a quasigroup $(Q, \cdot)$ and the study of finite soft quasigroup was carried out, motivated by the study of algebraic structures of soft sets. By introducing the order of a finite soft quasigroup, various inequality relationships that exist between the order of a finite quasigroup, the order of its soft quasigroup and the cardinality of its set of parameters were established. By introducing the arithmetic mean $\mathcal{AM}(F, A)$ and geometric mean $\mathcal{GM}(F, A)$ of a finite soft quasigroup (F, A), a sort of Lagrange's Formula $|(F, A)| = |A|\mathcal{AM}(F, A)$ for finite soft quasigroup was gotten. Some of the inequalities gotten gave an upper bound for the order of a finite soft quasigroup in terms of the order of its quasigroup and cardinality of its set of parameters, and a lower bound for the order of the quasigroup in terms of the arithmetic mean of the finite soft quasigroup. A chain of inequalities called the Maclaurin's inequality for any finite soft quasigroup $(F, A)_{(Q,\cdot)}$ was shown to exist. A necessary and sufficient condition for a type of finite soft quasigroup to be extensible to a finite super soft quasigroup was established. This result is of practical use

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whenever a larger set of parameters is required. The results therein were illustrated with examples. Application to uniformity, equality and equity in distribution for social living is considered.

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1. Introduction

Soft set theory introduced by Molodtsov [16] offers a more overall mathematical tool for dealing with uncertainty, fuzzy and inscrutable objects, because of its freedom from parametrization inadequacies.

In this current study, soft sets theory will be defined over a non-associative algebra called quasigroups. Different algebraic notions such as soft quasigroups and soft subquasigroups are introduced and studied.

Quasigroups are algebraic structures which fail to be groups when restricted to be of non-associativity. The study of quasigroups and loops started over two centuries ago. It started with Euler's 1782 postulations about certain pairs of mutually orthogonal Latin squares that does not exist. Along the line, Albert [1], introduced non-associative algebraic structures called quasigroups between 1939 and 1944.

Bruck [9] prepared the ground the development of the theory of quasigroup by defining multiplication group and the inner mapping group of a quasigroup (loop), hence linking theories of quasigroup and group.

However, Molodtsov [16] launched the concept of soft sets theory. He established better features for soft sets theory over fuzzy sets [26] and rough sets [21] so that true information and membership grade are excluded. This particularity enhances some applications because in most three dimensional settings, the basic data, probabilities and membership grades are unknown to justify the use of mathematical estimations.

Many algebraic operations and applications on soft sets in decision making were introduced by Maji *et al.* [15]. Chen *et al.* [11] introduced unique definitions such as restricted union and intersection, restricted difference and extended union and intersection of soft set parametrization reduction. Aktas and Ozlu [3] defined soft groups and their basic attributes which includes order of soft group, cyclic soft group, and properties of homomorphism and normalistic soft groups, and they corrected some invalid propositions cited in Aktas and Cagman [2]. The properties of De Morgan Laws was introduced by Sezgin and Atagun [20] into soft sets theory.

Some recent works on soft sets, soft structures and their applications can be found in [7, 14, 17–19, 24].

Due to our interest in the study of algebraic properties of soft sets, our purpose in this paper is to pioneer research on the connections between the order of a finite quasigroup and the order of soft finite quasigroups. In group theory for example, the order of a subgroup divides the order of its group, hence obeys Lagrange's Theorem, but quasigroup theory does not inevitably obey Lagrange's Theorem.

2. Preliminaries

We start this section by examining some definitions and results concerning quasigroups and soft sets.

Definition 2.1 (Groupoid, Quasigroup)**.** Consider a non-empty set G with a binary operation $(\cdot)$ defined on it. Whenever $x \cdot y \in G$ for all $x, y \in G$, then the pair $(G, \cdot)$ is called a magma or *groupoid*. A groupoid $(G, \cdot)$ is called a *quasigroup* whenever each of the equations:

$$a \cdot x = b \qquad \text{and} \qquad y \cdot c = d \tag{1}$$

has unique solution in G for x and y respectively. A quasigroup $(G, \cdot)$ is called a *loop* whenever there exists a unique element $e \in G$ called the *identity element* such that for all $x \in G$, $x \cdot e = e \cdot x = x$.

Note that we shall use juxtaposition among factors to be multiplied, that is xy will mean of $x \cdot y$, and $(\cdot)$ will have lower priority in the sense that $x \cdot yz$ will mean $x(yz)$.

Fix an element x in a groupoid $(G, \cdot)$. The left translation and right translation maps of x in G, denoted by L_x and R_x respectively are defined by

$$yL_x = x \cdot y = xy \qquad \text{and} \qquad yR_x = y \cdot x = yx. \tag{2}$$

Thus, going by (1), a groupoid $(G, \cdot)$ is a quasigroup if its left translation and right translation mappings are permutations. The fact that for quasigroup, the mappings in (2) are bijective means that their inverse mappings L_x^{-1} and R_x^{-1} respectively, exist and are defined in

$$x \backslash y = yL_x^{-1} \qquad \text{and} \qquad x/y = xR_y^{-1} \tag{3}$$

where the binary operations $(\backslash)$ and $(/)$ in (3) are related to the binary operation $(\cdot)$ in the following manner for all $x, y \in G$:

$$x \backslash y = z \Leftrightarrow x \cdot z = y \qquad \text{and} \qquad x/y = z \Leftrightarrow z \cdot y = x.$$

For more on quasigroups, readers can check [1, 9, 11, 12, 22, 25].

Definition 2.2 (Subgroupoid, Subquasigroup)**.** Let $(Q, \cdot)$ be a groupoid (quasigroup) and $\emptyset \neq H \subseteq Q$. Then, H is called a subgroupoid (subquasigroup) of Q if $(H, \cdot)$ is a groupoid (quasigroup). This is often expressed as $H \leq Q$.

We shall now introduce the notion of soft sets and operations defined on quasigroup. We refer readers to [2,3,6,10,13,15,16,20,21,23,26] for earlier works on soft sets, soft groups and their operations.

Definition 2.3 (Soft Sets, Soft Subset)**.** Let Q be a set and E be a set of parameters. For $A \subset E$, the pair (F, A) is called a soft set over Q if $F(a) \subset Q$ for all $a \in A$, where F is a function mapping A into the set of all non-empty subsets of Q, i.e., $F : A \longrightarrow 2^Q \backslash \{\emptyset\}$. Let (F, A) and (H, B) be two soft sets over a common universe U, then (H, B) is called a soft subset of (F, A) if

1. $B \subseteq A$; and
2. $H(x) \subseteq F(x)$ for all $x \in B$.

This is usually expressed as $(H, B) \subset (F, A)$ or $(F, A) \supset (H, B)$, and (F, A) is said to be a soft super set of (H, B).

Definition 2.4 (Restricted Intersection)**.** Let (F, A) and (G, B) be two soft sets over a common universe U such that $A \cap B \neq \emptyset$. Then their restricted intersection is $(F, A) \cap (G, B) = (H, C)$ where (H, C) is defined as $H(c) = F(c) \cap G(c)$ for all $c \in C$, where $C = A \cap B$.

Definition 2.5 (Extended Intersection)**.** The extended intersection of two soft sets (F, A) and (G, B) over a common universe U is the soft set (H, C), where $C = A \cup B$, and for all $x \in C$, $H(x)$ is defined as

$$H(x) = \begin{cases} F(x) & \text{if } x \in A - B \\ G(x) & \text{if } x \in B - A \\ F(x) \cap G(x) & \text{if } x \in A \cap B. \end{cases}$$

Definition 2.6 (Union)**.** The union of two soft sets (F, A) and (G, B) over U is denoted by $(F, A) \bigcup (G, B)$ and is a soft set (H, C) over U, such that $C = A \cup B$, $\forall x \in C$ and

$$H(x) = \begin{cases} F(x) & \text{if } x \in A - B \\ G(x) & \text{if } x \in B - A \\ F(x) \cup G(x) & \text{if } x \in A \cap B. \end{cases}$$

Lemma 2.1 (Wall [25])**.** *Let $(Q, \cdot)$ be a finite quasigroup.*

1. *If $X \subset Q$ and $a \in Q$, then $|X| = |a \cdot X| = |X \cdot a|$.*
2. *If $X \subset Q$ and $(X, \cdot)$ is a groupoid, then $X \leq Q$.*
3. *If $X \leq Q$, then $a \in X$ and $b \notin X$ imply $ab \notin X$.*

Theorem 2.1 (Wall [25]). *Let Q be a finite quasigroup with a proper subquasigroup X. Then,*

$$2|X| \leq |Q|.$$

Theorem 2.2 (Wall [25]). *Let Q be a finite quasigroup with non-disjoint proper subquasigroups X_1 and X_2. Then,*

$$|Q| \geq |X_1| + |X_2| + \max(X_1, X_2) - 2|X_1 \cap X_2|.$$

Theorem 2.3 (Wall [25]). *Let Q be a finite quasigroup with non-disjoint proper subquasigroups X_1 and X_2. If*

$$|Q| = |X_1| + |X_2| + \max(X_1, X_2) - 2|X_1 \cap X_2|,$$

then, $|X_1| = |X_2|$ if and only if $(X_1 \cap X_2) \cup (Q \backslash (X_1 \cup X_2)) \leq Q$.

3. MAIN RESULTS

3.1. Soft groupoid and soft quasigroup

Definition 3.1 (Soft: Groupoid and Quasigroup). Let Q be a groupoid (quasigroup) and E be a set of parameters. For $A \subset E$, the pair $(F, A)_Q$ will be called a soft groupoid (quasigroup) over Q if $F(a)$ is a subgroupoid (subquasigroup) of Q for all $a \in A$, where $F : A \longrightarrow 2^Q \backslash \{\emptyset\}$.

Remark 3.1. Based on Definition 3.1, a soft quasigroup is a soft groupoid, but the converse is not necessarily true. A soft groupoid (quasigroup) will be considered to be finite if its underlying groupoid (quasigroup) is finite.

$\cdot$	i	j	k	l	m	n	o	p
i	i	j	k	l	m	n	o	p
j	j	i	l	k	n	m	p	o
k	k	l	i	j	o	p	n	m
l	l	k	j	i	p	o	m	n
m	n	m	p	o	j	i	l	k
n	m	n	o	p	i	j	k	l
o	p	o	m	n	k	l	i	j
p	o	p	n	m	l	k	j	i

Table 1. Quasigroup $(Q, \cdot)$ of order 8.

Example 3.1. Let Table 1 represent the Latin square of a finite quasigroup $(Q, \cdot)$, $Q = \{i, j, k, l, m, n, o, p\}$ and let $A = \{\gamma_1, \gamma_2, \gamma_3\}$ be any set of parameters. Let $F : A \longrightarrow 2^Q \backslash \{\emptyset\}$ be defined by

$$F(\gamma_1) = \{i, j\}, \ F(\gamma_2) = \{i, j, k, l\}, \ F(\gamma_3) = \{i, j, o, p\}.$$

Then, the pair (F, A) is called a soft quasigroup over quasigroup Q because each of $F(\gamma_i) \leq Q$, $i = 1, 2, 3$ based on their representations in the Latin squares in Table 2.

·	i	j
i	i	j
j	j	i

$\equiv F(\gamma_1)$

·	i	j	k	l
i	i	j	k	l
j	j	i	l	k
k	k	l	i	j
l	l	k	j	i

$\equiv F(\gamma_2)$

·	i	j	o	p
i	i	j	o	p
j	j	i	p	o
o	p	o	i	j
p	o	p	j	i

$\equiv F(\gamma_3)$

Table 2. Soft quasigroup (F, A) over $(Q, \cdot)$.

Example 3.2. Let Table 1 represent the Latin square of a finite quasigroup $(Q, \cdot)$, $Q = \{i, j, k, l, m, n, o, p\}$ and let $B = \{\gamma_1, \gamma_2, \gamma_3\}$ be any set of parameters. Let $F : B \longrightarrow 2^Q \backslash \{\emptyset\}$ be defined by

$$G(\gamma_1) = \{i\}, \ G(\gamma_2) = \{i, j\}, \ G(\gamma_3) = \{i, j, m, n\}, \ G(\gamma_4) = Q.$$

Then, the pair (G, B) is called a soft quasigroup over quasigroup Q because each of the $G(\gamma_i) \leq Q$, $i = 1, 2, 3$ based on their representations in the Latin squares in Table 3.

·	i
i	i

$\equiv G(\gamma_1)$

·	i	j
i	i	j
j	j	i

$\equiv G(\gamma_2)$

·	i	j	m	n
i	i	j	m	n
j	j	i	n	n
m	n	m	i	j
n	m	n	j	i

$\equiv G(\gamma_3)$

Table 3. Soft quasigroup (G, B) over $(Q, \cdot)$.

Example 3.3. Let Table 4 represent the Latin square of a finite quasigroup $(R, \circ)$, $R = \{1, 2, 3, 4, 5\}$ and let $C = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ be any set of parameters. Let $H : C \longrightarrow 2^R \backslash \{\emptyset\}$ be defined by

$$H(\gamma_1) = \{1, 2\}, \ H(\gamma_2) = \{1, 3\}, \ H(\gamma_3) = \{1\}, \ G(\gamma_4) = R.$$

Then, the pair (H, C) is a soft quasigroup over quasigroup R because each of the $H(\gamma_i) \leq Q$, $i = 1, 2, 3, 4$.

$\circ$	1	2	3	4	5
1	1	2	3	5	4
2	2	1	5	4	3
3	3	4	1	2	5
4	4	5	2	3	1
5	5	3	4	1	2

Table 4. Quasigroup $(R, \circ)$ of order 5.

Example 3.4. Let Table 5 represent the Latin square of a finite quasigroup $(S, \bullet)$, $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and let $D = \{a, b, c, d\}$ be any set of parameters. Let $J : D \longrightarrow 2^S \backslash \{\emptyset\}$ be defined by

$$J(a) = \{1, 2\},\ J(b) = \{1, 2, 3, 4\},\ J(c) = \{1, 2, 5, 6\},\ J(d) = \{1, 2, 7, 8\}.$$

$\bullet$	1	2	3	4	5	6	7	8
1	1	2	3	4	6	5	7	8
2	2	1	4	3	5	6	8	7
3	3	4	1	2	7	8	6	5
4	4	3	2	1	8	7	5	6
5	6	5	8	7	2	1	4	3
6	5	6	7	8	1	2	3	4
7	8	7	5	6	3	4	1	2
8	7	8	6	5	4	3	2	1

Table 5. Quasigroup $(S, \bullet)$ for soft quasigroup (J, D).

Then, the pair (J, D) is a soft quasigroup over quasigroup Q because each of the $J(x) \leq Q$, $x \in D$ based on their representations in the Latin squares when extracted from in Table 5.

3.2. Finite soft quasigroup: order, arithmetic and geometric means

In group theory, the order of a finite group G is its cardinality and the order of its subgroup H divides the order of the group G. In [22] and [25] it was stated that a finite quasigroup does not necessarily obey Lagrange's theorem; hence the introduction of Lagrange-like property in quasigroup (loop) theory. In this section, some of the results of [25] on subquasigroup are extended to soft quasigroup theory. The definition of the order of soft group in [2] is dependent of the existence of the identity element. Thus, would need a new definition for the order of a soft quasigroup. To this effect, we introduce a new definition for the order of a soft quasigroup (F, A) over a finite quasigroup Q with the intention of checking if for divisibility between $|Q|$ and $|(F, A)|$ and also the relationships that could exist between the orders of a quasigroup Q and it's soft quasigroup.

Definition 3.2 (Order of Soft Quasigroup)**.** Let (F, A) be a soft quasigroup over a finite quasigroup Q. The order of the soft quasigroup (F, A) will be defined as

$$|(F, A)|_Q = |(F, A)| = \sum_{a \in A} |F(a)|, \quad for \quad F(a) \in (F, A) \quad and \quad a \in A.$$

where the sum is over distinct proper subquasigroups $F(a) \in (F, A),\ a \in A$.

Definition 3.3 (Arithmetic and Geometric Means of Finite Soft Quasigroup)**.** Let (F, A) be a soft quasigroup over a finite quasigroup Q. The arithmetic mean and geometric mean of (F, A) will be defined respectively as

$$\mathcal{AM}(F, A) = \frac{1}{|A|} \sum_{a \in A} |F(a)| \quad \text{and} \qquad \mathcal{GM}(F, A) = \sqrt[|A|]{\prod_{a \in A} |F(a)|}$$

Remark 3.2. 1. Let (F, A) be the soft quasigroup over a finite quasigroup Q in Example 3.1. Then it can be observed that $|F(a)|\Big| |(F, A)|, |Q|$ for just one case of $a \in A$, $|F(a)|\Big| |(F, A)|$ for just one case of $a \in A$ and $|F(a)|\Big| |Q|$ for all cases of $a \in A$.

2. Let (G, B) be the soft quasigroup over a finite quasigroup Q in Example 3.2. Then it can be observed that $|G(a)|\Big| |(G, B)|, |Q|$ for just one case of $a \in B$, $|G(a)|\Big| |(G, B)|$ for just one case of $a \in B$ and $|G(a)|\Big| |Q|$ for all cases of $a \in B$.
3. Let (H, C) be the soft quasigroup over a finite quasigroup R in Example 3.3. Then it can be observed that $|H(a)|\Big| |(H, C)|, |R|$ for just one case of $a \in C$, $|H(a)|\Big| |(H, C)|$ for all cases of $a \in C$ and $|F(a)|\Big| |Q|$ just two cases of $a \in C$.

Lemma 3.1. *Let $(Q, \cdot)$ be a finite quasigroup.*

1. *Let (F, A) be a soft set over Q. For any $a \in Q$, $|F(a)| = |x \cdot F(a)| = |F(a) \cdot x|$ for all $x \in Q$.*
2. *Let (F, A) be a soft set over Q. $(F, A)_{(Q,\cdot)}$ is a soft quasigroup if and only if $(F, A)_{(Q,\cdot)}$ is a soft groupoid.*
3. *Let $(F, A)_{(Q,\cdot)}$ be a soft quasigroup. Then,*

   (a) *for any $a \in A$, $x \in F(a)$ and $y \notin F(a)$ imply $xy \notin X$.*

(b) $F(a) \cdot Q \backslash F(a) \subset Q \backslash F(a)$ *for all* $a \in A$.

Proof. 1. Let (F, A) be a soft set over a finite quasigroup Q. Based on 1 of Lemma 2.1, since $F(a) \subset Q$ for all $a \in A$, then for any $x \in Q$, $|F(a)| = |x \cdot F(a)| = |F(a) \cdot x|$ for all $x \in Q$.

2. Let (F, A) be a soft set over Q. If $(F, A)_{(Q,\cdot)}$ is a soft quasigroup, then $(F, A)_{(Q,\cdot)}$ is a soft groupoid. Conversely, if $(F, A)_{(Q,\cdot)}$ is a soft groupoid, then $F(a)$ is a subgroupoid of $(Q, \cdot)$ for all $a \in A$. Hence, by 2 of Lemma 2.1, $F(a)$ is a subquasigroup of $(Q, \cdot)$ for all $a \in A$. Thence, $(F, A)_{(Q,\cdot)}$ is a soft quasigroup.

3. Let $(F, A)_{(Q,\cdot)}$ be a soft quasigroup.

(a) Then, $F(a) \leq Q$ for all $a \in A$ and so by 3 of Lemma 2.1, for any $a \in A$, $x \in F(a)$ and $y \notin F(a)$ imply $xy \notin X$.

(b) This follows from (a). ■

We now establish some results which reveal the relationships between the order of a finite quasigroup and the order of its soft quasigroups.

Theorem 3.1. *Let* $(F, A)_{(Q,\cdot)}$ *be a finite soft quasigroup. Then*

$$|(F, A)| = |A|\mathcal{AM}(F, A), \quad 2|(F, A)| \leq |A||Q| \quad \text{and} \quad |Q| \geq 2\mathcal{AM}(F, A).$$

Proof. The first part follows by the definitions of $|(F, A)|$ and $\mathcal{AM}(F, A)$. If $(F, A)_{(Q,\cdot)}$ is a finite soft quasigroup, then $F(a) \leq Q$ for all $a \in A$. Thus, by Theorem 2.1, $2|F(a)| \leq |Q|$ for all $a \in A$. Hence, with $A = \{a_1, a_2, \dots, a_n\}$,

$$2|F(a_1)| + 2|F(a_2)| + \cdots + 2|F(a_n)| \leq |A||Q|$$

$$\Rightarrow 2\sum_{a \in A} |F(a)| \leq |A||Q| \Rightarrow 2|(F, A)| \leq |A||Q|.$$

Also,

$$2\sum_{a \in A} |F(a)| \leq |A||Q| \Rightarrow |Q| \geq \frac{2}{|A|}\sum_{a \in A} |F(a)| \Rightarrow |Q| \geq 2\mathcal{AM}(F, A).$$

■

Remark 3.3. 1. In Theorem 3.1, the formula $|(F, A)| = |A|\mathcal{AM}(F, A)$ can be viewed as a sort of Lagrange's Formula for finite soft quasigroup where $|A|$ and $\mathcal{AM}(F, A)$ (which is not necessarily an integer) play the roles of the order of subgroup and index of the subgroup in the group.

2. The first and second inequalities in Theorem 3.1 respectively give an upper bound for the order of a finite soft quasigroup in terms of the order of its quasigroup and cardinality of its set of parameters, and a lower bound for the order of the quasigroup in terms of the arithmetic mean of the finite soft quasigroup.

The second part of Theorem 3.1 can also be proved in the following manner. By 1 of Lemma 3.1, for any $a \in Q$, $|F(a)| = |x \cdot F(a)| = |F(a) \cdot x|$ for all $x \in Q$. Now, choose $x \in Q$ such that $x \notin F(a)$, then, going by 3 of Lemma 3.1,

$$|(F,A)| \leq \sum_{a\in A} |Q \backslash F(a)| \Rightarrow |(F,A)| \leq \sum_{a\in A} (|Q| - |F(a)|) = \sum_{a\in A} |Q| - \sum_{a\in A} |F(a)|$$

$$\Rightarrow |(F,A)| \leq |A||Q| - |(F,A)| \Rightarrow 2|(F,A)| \leq |A||Q|.$$

Let (F,A) be the soft quasigroup over a finite quasigroup Q in Example 3.1. Then it can be observed that $|A| = 3,\ |Q| = 8,\ |(F,A)| = 10$. Thus, the inequality in Theorem 3.1 will be satisfied. But if (F,A) has the improper subquasigroup Q so that $|A| = 4,\ |Q| = 8,\ |(F,A)| = 10$, then the inequality in Theorem 3.1 will not be satisfied. Hypothetically, Theorem 3.1 should not be applied to the finite soft quasigroups in Example 3.2 and Example 3.3 because of the improper subquasigroups in them. But they actually obey the inequality therein.

Theorem 3.2. *Let $(F,A)_{(Q,\cdot)}$ be a finite soft quasigroup. Then,*

$$|Q| \geq 2 \times \sqrt[|A|]{\prod_{a\in A} |F(a)|},\quad |Q| \geq 2\mathcal{AM}(F,A) \quad \text{and} \quad |Q| \geq \mathcal{AM}(F,A) + \mathcal{GM}(F,A).$$

Proof. Using Theorem 2.1, we have,

$$\prod_{a\in A} 2|F(a)| \leq \prod_{i=1}^{|A|} |Q| \Rightarrow 2^{|A|} \times \prod_{a\in A} |F(a)| \leq \prod_{i=1}^{|A|} |Q|$$

$$\Rightarrow 2^{|A|} \times \prod_{a\in A} |F(a)| \leq |Q|^{|A|} \Rightarrow \left(\frac{|Q|}{2}\right)^{|A|} \geq \prod_{a\in A} |F(a)|$$

$$\Rightarrow \frac{|Q|}{2} \geq \sqrt[|A|]{\prod_{a\in A} |F(a)|} \Rightarrow |Q| \geq 2 \times \sqrt[|A|]{\prod_{a\in A} |F(a)|} \Rightarrow |Q| \geq 2\mathcal{GM}(F,A).$$

From Theorem 3.1, $|Q| \geq 2\mathcal{AM}(F,A)$, and so $2|Q| \geq 2\mathcal{AM}(F,A) + 2\mathcal{GM}(F,A)$ $\Rightarrow |Q| \geq \mathcal{AM}(F,A) + \mathcal{GM}(F,A)$. ∎

Remark 3.4. The second and third inequalities in Theorem 3.2 give lower bounds for the order of the quasigroup in terms of the arithmetic and geometric means of the finite soft quasigroup.

Let (F,A) be the soft quasigroup over a finite quasigroup Q in Example 3.1. Then it can be observed that

$$|A| = 3,\ |Q| = 8,\ |(F,A)| = 10,\ \mathcal{AM}(F,A) = \frac{10}{3},\ \mathcal{GM}(F,A) = \sqrt[3]{32}.$$

Thus, the inequalities in Theorem 3.2 are all satisfied.

Theorem 3.3. *Let $(F, A)_{(Q,\cdot)}$ be a finite soft quasigroup. Then, there exists a chain of inequalities*

$$\frac{|Q|}{2} \geq S_1 = \mathcal{AM}(F, A) \geq \sqrt{S_2} \geq \sqrt[3]{S_3} \geq \cdots \geq \sqrt[|A|]{S_{|A|}} = \mathcal{GM}(F, A) \quad where$$

$$S_i = \frac{\sum\limits_{1\leq j_1<\cdots<j_i\leq|A|} |F(a_{j_1})||F(a_{j_2})|\cdots|F(a_{j_i})|}{\binom{|A|}{i}} \quad and\ A = \{a_i\}_{i=1}^{|A|}.$$

Proof. Note that the numerator of the RHS of S_i is the elementary symmetric polynomial of degree i in the $|A|$ variables $|F(a_1)|, |F(a_2)|, \ldots, |F(a_{|A|})|$. That is, the sum of all products of i of the numbers $|F(a_1)|, |F(a_2)|, \ldots, |F(a_{|A|})|$. Using the Newton's inequality, the result follows. ∎

Remark 3.5. The chain of inequalities in Theorem 3.3 will be called the Maclaurin's inequality for a finite soft quasigroup $(F, A)_{(Q,\cdot)}$. Its consequence for a family of finite soft quasigroups $(F_i, A_i)_{(Q,\cdot)}$ will be interesting.

Theorem 3.4. *Let $(F, A)_{(Q,\cdot)}$ be a finite soft quasigroup. Then,*

1. $|Q| \geq \dfrac{2}{1 + |A|}\left[|(F, A)| + \mathcal{GM}(F, A)\right]$.
2. $|Q| \geq \dfrac{2}{1 + |A|}\left[|A|\mathcal{AM}(F, A) + \mathcal{GM}(F, A)\right]$.
3. $|(F, A)| \leq \dfrac{|Q|(1 + |A|)}{2} - \mathcal{GM}(F, A)$.
4. $|Q| \geq 2 \times \sqrt{\dfrac{|(F, A)|\mathcal{GM}(F, A)}{|A|}}$.
5. $|Q| \geq 2 \times \sqrt{\mathcal{AM}(F, A)\mathcal{GM}(F, A)}$.
6. $|(F, A)| \leq \dfrac{|A||Q|^2}{4\mathcal{GM}(F, A)}$.

Proof. By Theorem 3.1 and Theorem 3.2: $|A||Q| \geq 2|(F, A)|$ and $|Q| \geq 2 \times \sqrt[|A|]{\prod\limits_{a\in A} |F(a)|}$.

1. Adding these two inequalities, we have,

$$|A||Q| + |Q| \geq 2\sum_{a\in A} |F(a)| + 2 \times \sqrt[|A|]{\prod_{a\in A} |F(a)|}$$

$$\Rightarrow |Q|(1 + |A|) \geq 2\left[\sum_{a\in A} |F(a)| + \sqrt[|A|]{\prod_{a\in A} |F(a)|}\right]$$

$$\Rightarrow |Q| \geq \frac{2}{1+|A|}\left[\sum_{a\in A}|F(a)| + \sqrt[|A|]{\prod_{a\in A}|F(a)|}\right]$$

$$\Rightarrow |Q| \geq \frac{2}{1+|A|}\left[|(F,A)| + \mathcal{GM}(F,A)\right].$$

2. Using 1, $|Q| \geq \frac{2}{1+|A|}\left[|A|\mathcal{AM}(F,A) + \mathcal{GM}(F,A)\right]$.
3. From 1, $|(F,A)| \leq \frac{|Q|(1+|A|)}{2} - \mathcal{GM}(F,A)$.
4. Multiplying these two inequalities, we have

$$|A||Q|^2 \geq 2\sum_{a\in A}|F(a)| \times \left(2\times \sqrt[|A|]{\prod_{a\in A}|F(a)|}\right)$$

$$\Rightarrow |A||Q|^2 \geq 4\times \sum_{a\in A}|F(a)| \times \sqrt[|A|]{\prod_{a\in A}|F(a)|}$$

$$\Rightarrow |Q| \geq 2\times \sqrt{\frac{\left(\sum_{a\in A}|F(a)|\right)\left(\sqrt[|A|]{\prod_{a\in A}|F(a)|}\right)}{|A|}}$$

$$\Rightarrow |Q| \geq 2\times \sqrt{\frac{|(F,A)|\mathcal{GM}(F,A)}{|A|}}.$$

5. Using 4, $|Q| \geq 2\times\sqrt{\mathcal{AM}(F,A)\mathcal{GM}(F,A)}$.
6. Following 4, $|(F,A)| \leq \frac{|A||Q|^2}{4\mathcal{GM}(F,A)}$. ■

Remark 3.6. Let (F,A) be the soft quasigroup over a finite quasigroup Q in Example 3.1. Then it can be observed that

$$|A| = 3,\ \ |Q| = 8,\ \ |(F,A)| = 10,\ \ \mathcal{AM}(F,A) = \frac{10}{3},\ \ \mathcal{GM}(F,A) = \sqrt[3]{32}.$$

Thus, the inequalities in Theorem 3.4 are all satisfied.

3.3. Finite soft quasigroup with non-disjointed part

Lemma 3.2. Let $(F,A)_{(Q,\cdot)}$ be a finite soft quasigroup such that $F(a)\cap F(b)\neq\emptyset$ for any $a,b\in A$. Then:

1. $|Q| \geq |F(a)| + |F(b)| + \max\left(|F(a)|,|F(b)|\right) - 2|F(a)\cap F(b)|$ for any $a,b\in A$.
2. $|Q| \geq |F(a)\cup F(b)| - |F(a)\cap F(b)| + \max\left(|F(a)|,|F(b)|\right)$ for any $a,b\in A$.

3. if $F(a) \cup F(b) = Q$ for any $a, b \in A$, $|F(a) \cap F(b)| \geq \max(|F(a)|, |F(b)|)$ for any $a, b \in A$.

Proof. 1. Let $(F, A)_{(Q,\cdot)}$ be a finite soft quasigroup, then $F(a), F(b) \leq Q$ for any $a, b \in A$. Thus, by Theorem 2.2, $|Q| \geq |F(a)| + |F(b)| + \max(|F(a)|, |F(b)|) - 2|F(a) \cap F(b)|$.
2. Recall that $|F(a) \cup F(b)| = |F(a)| + |F(b)| - |F(a) \cap F(b)|$. So, from 1, $|Q| \geq |F(a) \cup F(b)| - |F(a) \cap F(b)| + \max(|F(a)|, |F(b)|)$ for any $a, b \in A$.
3. If $F(a) \cup F(b) = Q$, then by 2, the conclusion follows. ∎

Remark 3.7. Let (F, A) be the soft quasigroup over a finite quasigroup Q in Example 3.1. Then it can be observed that

$$|A| = 3,\ |Q| = 8,\ |(F, A)| = 10,\ |F(\gamma_1)| = 2,\ |F(\gamma_2)| = |F(\gamma_3)| = 4,$$
$$|F(\gamma_1) \cap F(\gamma_2)| = 2,\ |F(\gamma_2) \cap F(\gamma_3)| = 2,\ |F(\gamma_1) \cup F(\gamma_2)| = 4,$$
$$|F(\gamma_2) \cup F(\gamma_3)| = 6.$$

Thus, the inequalities in Lemma 3.2 are all satisfied for the pairs $(a, b) \in \{(\gamma_1, \gamma_2), (\gamma_2, \gamma_3)\}$.

Theorem 3.5. *Let $(F, A)_{(Q,\cdot)}$ be a finite soft quasigroup such that $F(a) \cap F(b) \neq \emptyset$ for any $a, b \in A$. Then:*

1. $|Q| \geq \frac{1}{|A|-1}(2|(F, A)| - |F(a) \cup F(b)| - 3|F(a) \cap F(b)| + \max(|F(a)|, |F(b)|))$.
2. $|(F, A)| \leq \frac{1}{2}((|A|-1)|Q| + |F(a) \cup F(b)| + 3|F(a) \cap F(b)| - \max(|F(a)|, |F(b)|))$.
3. *if $F(a) \cup F(b) = Q$ for any $a, b \in A$,*
$$|Q| \geq \frac{1}{|A|}\left(2|(F, A)| - 3|F(a) \cap F(b)| + \max(|F(a)|, |F(b)|)\right).$$
4. *if $F(a) \cup F(b) = Q$ for any $a, b \in A$,*
$$|(F, A)| \leq \frac{1}{2}\left((|A|\,|Q| + 3|F(a) \cap F(b)| - \max(|F(a)|, |F(b)|)\right).$$

Proof. 1. By Lemma 3.2(1), for any $a, b \in A$,

$$|Q| \geq |F(a)| + |F(b)| + \max(|F(a)|, |F(b)|) - 2|F(a) \cap F(b)|.$$

For any $c \in A \backslash \{a, b\}$, $|Q| \geq 2|F(c)|$ by Theorem 2.2. So, using this,

$$|Q| \geq |F(a)| + |F(b)| + \max(|F(a)|, |F(b)|) - 2|F(a) \cap F(b)|$$
$$\Rightarrow (|A| - 1)|Q| \geq \sum_{c \in A \backslash \{a,b\}} |F(c)| + |F(a)| + |F(b)|$$
$$+ \sum_{c \in A \backslash \{a,b\}} |F(c)| + \max(|F(a)|, |F(b)|) - 2|F(a) \cap F(b)|$$

$$\Rightarrow (|A|-1)|Q| \geq \sum_{d\in A}|F(d)| + \sum_{d\in A}|F(d)| - |F(a)| - |F(b)|$$
$$+ \max(|F(a)|, |F(b)|) - 2|F(a)\cap F(b)|$$
$$\Rightarrow (|A|-1)|Q| \geq 2|(F,A)| - |F(a)| - |F(b)| - 2|F(a)\cap F(b)|$$
$$+ \max(|F(a)|, |F(b)|)$$
$$\Rightarrow |Q| \geq \frac{1}{|A|-1}(2|(F,A)| - |F(a)\cup F(b)| - 3|F(a)\cap F(b)|$$
$$+ \max(|F(a)|, |F(b)|)).$$

2. This follows from 1.
3. If $F(a)\cup F(b) = Q$ for only $a, b \in A$, then by 1, the result follows.
4. This following from 3. ■

Remark 3.8. Let (F, A) be the soft quasigroup over a finite quasigroup Q in Example 3.1. Then it can be observed that $|A| = 3,\ |Q| = 8,$

$$|(F,A)| = 10,\ |F(\gamma_1)| = 2,\ |F(\gamma_2)| = |F(\gamma_3)| = 4,$$
$$|F(\gamma_1)\cap F(\gamma_2)| = |F(\gamma_1)\cap F(\gamma_3)| = 2, |F(\gamma_2)\cap F(\gamma_3)| = 2,$$
$$|F(\gamma_1)\cup F(\gamma_2)| = |F(\gamma_1)\cup F(\gamma_3)| = 4,\ |F(\gamma_2)\cup F(\gamma_3)| = 6.$$

Thus, the inequalities 1, 2 in Theorem 3.5 are satisfied for any of the pairs $(a, b) \in \{(\gamma_1, \gamma_2), (\gamma_2, \gamma_3), (\gamma_1, \gamma_3)\}$ because $F(a)\cap F(b) \neq \emptyset$ is true for those pairs (a, b).

Theorem 3.6. *Let $(F, A)_{(Q,\cdot)}$ be a finite soft quasigroup such that $F(a_i) \cap F(a_{i+1}) \neq \emptyset$ for all $i = 1, 2, \ldots, n-1$, where $A = \{a_i\}_{i=1}^n$. Then,*

1. $|Q| \geq \dfrac{1}{|A|-1}\left[|F(a_1)| + |F(a_n)| + 2\sum_{i=2}^{n-1}|F(a_i)| + \sum_{i=1}^{n-1}\max(|F(a_i)|, |F(a_{i+1})|) - 2\sum_{i=1}^{n-1}|F(a_i)\cap F(a_{i+1})|\right].$
2. $|Q| \geq \dfrac{1}{|A|-1}\left[2|(F,A)| + \sum_{i=1}^{n-1}\max(|F(a_i)|, |F(a_{i+1})|) - \left(|F(a_1)| + |F(a_n)| + 2\sum_{i=1}^{n-1}|F(a_i)\cap F(a_{i+1})|\right)\right].$

3. $|(F,A)| \leq \frac{1}{2}\left[(|A|-1)|Q| + |F(a_1)| + |F(a_n)| + 2\sum_{i=1}^{n-1}|F(a_i)\cap F(a_{i+1})| - \sum_{i=1}^{n-1}\max(|F(a_i)|,|F(a_{i+1})|)\right].$

4. $|Q| \geq \frac{1}{|A|-1}\left[\sum_{i=1}^{n-1}|F(a_i)\cup F(a_{i+1})| - \sum_{i=1}^{n-1}|F(a_i)\cap F(a_{i+1})| + \sum_{i=1}^{n-1}\max(|F(a_i)|,|F(a_{i+1})|)\right].$

5. $\sum_{i=1}^{n-1}|F(a_i)\cap F(a_{i+1})| \geq \sum_{i=1}^{n-1}\max(|F(a_i)|,|F(a_{i+1})|)$ *if* $F(a_i)\cup F(a_{i+1}) = Q$ *for all* $i = 1,2,\ldots,n-1$.

Proof. 1. Let $A = \{a_i\}_{i=1}^n$ such that $F(a_i)\cap F(a_{i+1}) \neq \emptyset$ for all $i = 1,2,\ldots,n-1$. Then by Lemma 3.2,

$$\begin{aligned}
|Q| &\geq |F(a_1)| + |F(a_2)| + \max(|F(a_1)|,|F(a_2)|) - 2|F(a_1)\cap F(a_2)|\\
|Q| &\geq |F(a_2)| + |F(a_3)| + \max(|F(a_2)|,|F(a_3)|) - 2|F(a_2)\cap F(a_3)|\\
|Q| &\geq |F(a_3)| + |F(a_4)| + \max(|F(a_3)|,|F(a_4)|) - 2|F(a_3)\cap F(a_4)|\\
&\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots\\
|Q| &\geq |F(a_{n-1})| + |F(a_n)| + \max(|F(a_{n-1})|,|F(a_n)|) - 2|F(a_{n-1})\cap F(a_n)|.
\end{aligned}$$

Adding up gives,

$$\begin{aligned}
(|A|-1)|Q| &\geq |F(a_1)| + |F(a_n)| + 2\sum_{i=1}^{n-1}|F(a_i)| + \sum_{i=1}^{n-1}\max(|F(a_i|,|F(a_{i+1})\\
&\quad - 2\sum_{i=1}^{n-1}|F(a_i)\cap F(a_{i+1}|)\\
\Rightarrow |Q| &\geq \frac{1}{|A|-1}\left[|F(a_1)| + |F(a_n)| + 2\sum_{i=2}^{|A|-1}|F(a_i)| + \sum_{i=1}^{n-1}\max(|F(a_i)|,|F(a_{i+1})|)\right.\\
&\quad \left. - 2\sum_{i=1}^{n-1}|F(a_i)\cap F(a_{i+1})|\right].
\end{aligned}$$

2. Use 1.
3. Use 2.
4. Use 1.
5. Use 4. ∎

Remark 3.9. Let (J, D) be the soft quasigroup over a finite quasigroup S in Example 3.4. Then it can be observed that $|D| = 4,\ |S| = 8$. Thus, the inequalities 1, 2, 3, 4 in Theorem 3.6 are satisfied with $a_1 = a,\ a_2 = b,\ a_3 = c,\ a_4 = d$ because $J(a_i) \cap J(a_{i+1}) \neq \emptyset$ for all $i = 1, 2, 3$.

Theorem 3.7. *Let $(F, A)_{(Q,\cdot)}$ be a finite soft quasigroup such that $F(a_i) \cap F(a_{i+1}) \neq \emptyset$ for all $i = 1, 2, \ldots, n-1$ where $\{a_i\}_{i=1}^{n} \subseteq A$. Then*

1. $$|Q| \geq \frac{1}{|A|-1}\left(2|(F,A)| - \frac{1}{n-1}\sum_{i=1}^{n-1}|F(a_i) \cup F(a_{i+1})| - \frac{3}{n-1}\sum_{i=1}^{n-1}|F(a_i) \cap F(a_{i+1})| + \frac{1}{n-1}\sum_{i=1}^{n-1}\max\left(|F(a_i)|, F(a_{i+1})|\right)\right).$$

2. $$|(F,A)| \leq \frac{1}{2}\left((|A|-1)|Q| + \frac{1}{n-1}\sum_{i=1}^{n-1}|F(a_i) \cup F(a_{i+1})| + \frac{3}{n-1}\sum_{i=1}^{n-1}|F(a_i) \cap F(a_{i+1})| - \frac{1}{n-1}\sum_{i=1}^{n-1}\max\left(|F(a_i)|, F(a_{i+1})|\right)\right).$$

3. $$|Q| \geq \frac{1}{|A|}\left(2|(F,A)| - \frac{3}{n-1}\sum_{i=1}^{n-1}|F(a_i) \cap F(a_{i+1})| + \frac{1}{n-1}\sum_{i=1}^{n-1}\max\left(|F(a_i)|, F(a_{i+1})|\right)\right) \textit{if } F(a_i) \cup F(a_{i+1}) = Q\ \forall\ i = 1, 2, \ldots, n-1.$$

4. $$|(F,A)| \leq \frac{1}{2}\left((|A|\,|Q| + \frac{3}{n-1}\sum_{i=1}^{n-1}|F(a_i) \cap F(a_{i+1})| - \frac{1}{n-1}\sum_{i=1}^{n-1}\left(|F(a_i)|, F(a_{i+1})|\right)\right) \textit{if } F(a_i) \cup F(a_{i+1}) = Q\ \forall\ i = 1, 2, \ldots, n-1.$$

Proof. This is achieved with the judicious use of Theorem 3.5 for each pair $(a, b) = (a_i, a_{i+1})$ for all $i = 1, 2, \ldots, n-1$. ■

Remark 3.10. Let (J, D) be the soft quasigroup over a finite quasigroup S in Example 3.4. Then it can be observed that $|D| = 4,\ |S| = 8$, Thus, the inequalities 1, 2 in Theorem 3.7 are satisfied with $a_1 = b,\ a_2 = c,\ a_3 = d$ because $J(a_i) \cap J(a_{i+1}) \neq \emptyset$ for all $i = 1, 2$.

Lemma 3.3. *Let $(F, A)_{(Q,\cdot)}$ be a finite soft quasigroup such that $F(a) \cap F(b) \neq \emptyset$ and $|F(a)| = |F(b)|$ for any $a, b \in A$. Then, $|Q| \geq 3|F(a)| - 2|F(a) \cap F(b)|$.*

Proof. From Lemma 3.2, $|Q| \geq |F(a)| + |F(b) + \max(|F(a)|, |F(b)|) - 2|F(a) \cap$

$F(b)|$. Since $|F(a) = F(b)$, it implies that,

$$|Q| \geq |F(a)| + |F(a)| + \max(|F(a)|, |F(a)|) - 2|F(a) \cap F(b)| \Rightarrow$$
$$|Q| \geq |F(a)| + |F(a)| + |F(a)| - 2|F(a) \cap F(b)| \Rightarrow |Q| \geq 3|F(a)| - 2|F(a) \cap F(b)|.$$

■

Remark 3.11. Let (J, D) be the soft quasigroup over a finite quasigroup S in Example 3.4. Then it can be observed that $|D| = 4,\ |S| = 8$, Thus, the inequality $|S| \geq 3|J(x)| - 2|J(x) \cap J(y)|$ in Lemma 3.3 is satisfied for any $x, y \in \{b, c, d\} \subset D$.

Theorem 3.8. *Let $(F, A)_{(Q,\cdot)}$ be a finite soft quasigroup such that $F(a_i) \cap F(a_{i+1}) \neq \emptyset$ and $|F(a_i)| = |F(a_{i+1})| = k$ for all $i = 1, 2, \ldots, n-1$ where $A = \{a_i\}_{i=1}^n$. Then,*

1. $|Q| \geq \dfrac{1}{|A|+1}\left(3|(F,A)| + k - 2\displaystyle\sum_{i=1}^{n-1} |F(a_i) \cap F(a_{i+1})|\right).$

2. $|(F,A)| \leq \dfrac{1}{3}\left((|A|+1)|Q| - k + 2\displaystyle\sum_{i=1}^{n-1} |F(a_i) \cap F(a_{i+1})|\right).$

Proof. 1. Let $F(a_i) \cap F(a_{i+1}) \neq \emptyset$ and $|F(a_i)| = |F(a_{i+1})|$ for all $i = 1, 2, \ldots, n-1$ where $A = \{a_i\}_{i=1}^n$. Then, by Lemma 3.3 and Theorem 2.1,

$$|Q| \geq 3|F(a_i)| - 2|F(a_i) \cap F(a_{i+1})| \ \forall\ 1 \leq i \leq n-1 \ \text{ and } \ 2|Q| \geq 4|F(a_n)|.$$

Adding these inequalities, we get

$$(n+1)|Q| \geq 3|(F,A)| + F(a_n) - 2\sum_{i=1}^{n-1} |F(a_i) \cap F(a_{i+1})| \Rightarrow$$
$$|Q| \geq \frac{1}{|A|+1}\left(3|(F,A)| + k - 2\sum_{i=1}^{n-1} |F(a_i) \cap F(a_{i+1})|\right).$$

2. This follows from 1. ■

Remark 3.12. Let (J, D) be the soft quasigroup over a finite quasigroup S in Example 3.4 and consider a soft subset (J', D') of (J, D) over S where $D' = \{b, c, d\} \subset D$ and $J : D' \longrightarrow 2^S \backslash \{\emptyset\}$ is defined by

$$J'(b) = \{1, 2, 5, 6\},\ J'(c) = \{1, 2, 3, 4\},\ J'(d) = \{1, 2, 7, 8\}.$$

Then, $(J', D')_{(S,\bullet)}$ is a soft quasigroup and it can be observed that $|D'| = 3,\ |S| = 8$. Thus, the inequalities 1, 2 in Theorem 3.8 are satisfied with $a_1 = b,\ a_2 = c,\ a_3 = d$ because $J(a_i) \cap J(a_{i+1}) \neq \emptyset$ and $|F(a_i)| = |F(a_{i+1})| = 4$ for $i = 1, 2$.

Theorem 3.9. *Let $(F,A)_{(Q,\cdot)}$ be a finite soft quasigroup such that $F(a_i)\cap F(a_{i+1})\neq\emptyset$ and $|F(a_i)|=|F(a_{i+1})|$ for all $i=1,2,\ldots,n-1$ where $\{a_i\}_{i=1}^n\subseteq A$. Then:*

1. $$|Q|\geq\frac{1}{2|A|+1-n}\left(3|(F,A)|+\sum_{c\in A\backslash\{a_i\}_{i=1}^{n-1}}|F(c)|-2\sum_{i=1}^{n-1}|F(a_i)\cap F(a_{i+1})|\right).$$

2. $$|(F,A)|\leq\frac{1}{3}\left((2|A|+1-n)|Q|-\sum_{c\in A\backslash\{a_i\}_{i=1}^{n-1}}|F(c)|+2\sum_{i=1}^{n-1}|F(a_i)\cap F(a_{i+1})|\right).$$

Proof. 1. Let $F(a_i)\cap F(a_{i+1})\neq\emptyset$ and $|F(a_i)|=|F(a_{i+1})|$ for all $i=1,2,\ldots,n-1$ where $\{a_i\}_{i=1}^n\subseteq A$ Then, by Lemma 3.3 and Theorem 2.1, for all $1\leq i\leq n-1$,

$$|Q|\geq 3|F(a_i)|-2|F(a_i)\cap F(a_{i+1})|,\quad 2|Q|\geq 4|F(a_n)|,\quad 2|Q|\geq 4\sum_{c\in A\backslash\{a_i\}_{i=1}^n}|F(c)|.$$

Adding these inequalities, we get

$$(n-1)|Q|+2|Q|+2\sum_{c\in A\backslash\{a_i\}_{i=1}^n}|Q|\geq 3\sum_{i=1}^{n-1}|F(a_i)|+4|F(a_n)|+4\sum_{c\in A\backslash\{a_i\}_{i=1}^n}|F(c)|$$
$$-2\sum_{i=1}^{n-1}|F(a_i)\cap F(a_{i+1})|\Rightarrow[(n-1)+2+2(|A|-n)]\,|Q|\geq 3\sum_{c\in A}|F(c)|+|F(a_n)|$$
$$+\sum_{c\in A\backslash\{a_i\}_{i=1}^n}|F(c)|-2\sum_{i=1}^{n-1}|F(a_i)\cap F(a_{i+1})|\Rightarrow[2|A|+1-n)]\,|Q|\geq 3|(F,A)|$$
$$+\sum_{c\in A\backslash\{a_i\}_{i=1}^{n-1}}|F(c)|-2\sum_{i=1}^{n-1}|F(a_i)\cap F(a_{i+1})|\Rightarrow$$
$$|Q|\geq\frac{1}{2|A|+1-n}\left(3|(F,A)|+\sum_{c\in A\backslash\{a_i\}_{i=1}^{n-1}}|F(c)|-2\sum_{i=1}^{n-1}|F(a_i)\cap F(a_{i+1})|\right).$$

2. This follows from 1. ∎

Remark 3.13. Let (J,D) be the soft quasigroup over a finite quasigroup S in Example 3.4 and observe that $|D|=4,\ |S|=8$. Thus, the inequalities 1,2 in Theorem 3.9 are satisfied with $a_1=b,\ a_2=c,\ a_3=d$ because $J(a_i)\cap J(a_{i+1})\neq\emptyset$ and $|F(a_i)|=|F(a_{i+1})|=4$ for $i=1,2$.

Theorem 3.10. *Let $(F,A)_{(Q,\cdot)}$ be a finite soft quasigroup such that there does not exist $c \in A$ such that $F(c) = (F(a) \cap F(b)) \cup [Q \backslash (F(a) \cup F(b))]$ for any $a, b \in A$. For any $a, b \in A$, let*

$$|Q| = |F(a)| + |F(b)| + \max(|F(a)|, |F(b)|) - 2|F(a) \cap F(b)| \tag{4}$$

Then, $(F,A)_{(Q,\cdot)}$ can be extended to a super soft quasigroup $(G,B)_{(Q,\cdot)}$ where $A \subset B$ and

$$G(x) = \begin{cases} F(x) & \text{if } x \in A \\ (F(a) \cap F(b)) \cup [Q \backslash (F(a) \cup F(b))] & \text{if } x \notin A,\ a, b \in A \end{cases} \tag{5}$$

if and only if $|F(a)| = |F(b)|$ for any $a, b \in A$.

Proof. Let $|Q| = |F(a)| + |F(b)| + \max(|F(a)|, |F(b)|) - 2|F(a) \cap F(b)|$ for any $a, b \in A$. If $(F,A)_{(Q,\cdot)}$ can be extended to a super soft quasigroup $(G,B)_{(Q,\cdot)}$ of the soft quasigroup $(F,A)_{(Q,\cdot)}$ where $A \subset B$ and (5) is true, then, $(F(a) \cap F(b)) \cup [Q \backslash (F(a) \cup F(b))] \leq Q$. So, by Theorem 2.3, $|F(a)| = |F(b)|$ for any $a, b \in A$.

Conversely, if $|F(a)| = |F(b)|$ for any $a, b \in A$, then, by Theorem 2.3, $(F(a) \cap F(b)) \cup [Q \backslash (F(a) \cup F(b))] \leq Q$. Thus, there exists a super soft quasigroup $(G,B)_{(Q,\cdot)}$ of the quasigroup $(F,A)_{(Q,\cdot)}$ where $A \subset B$ and (5) is true. ∎

Remark 3.14. Consider Table 5 representing the Latin square of the finite quasigroup $(S, \bullet)$, $S = \{1,2,3,4,5,6,7,8\}$ and let $E = \{a,b,c\}$ be a set of parameters. Let $K : E \longrightarrow 2^S \backslash \{\emptyset\}$ be defined by

$$K(a) = \{1,2\},\ K(b) = \{1,2,3,4\},\ K(c) = \{1,2,5,6\}.$$

Then, $(K,E)_{(S,\bullet)}$ is a soft quasigroup. When $x = b$ and $y = c$, then (4) of Theorem 3.10 is satisfied. Note that $|K(b)| = |K(c)|$, hence, by Theorem 3.10, there exists an extension of $(F,A)_{(Q,\cdot)}$, that is, a soft quasigroup $(G,B)_{(Q,\cdot)}$ where $E \subset B = \{a,b,c,d\}$ and

$$G(a) = K(a) = \{1,2\},\ G(b) = K(b) = \{1,2,3,4\},\ G(c) = K(c) = \{1,2,5,6\},$$
$$G(d) = (K(b) \cap K(c)) \cup [S \backslash (K(b) \cup K(c))] = \{1,2,7,8\}$$

going by (5).

3.4. Application to uniformity, equality and equity in distribution

According to [8], the equality and equity of distribution of income or wealth or responsibility or task or economic resources are basically objective and basically subjective, respectively. In human life, uniformity, equality and equity come to play for efficient distribution in social living and politics, welfare (e.g. during the Covid-19 pandemic), governance. Equality and equity come in forms. For instance, equality has principles and types. In fact, Aristotle [4,5] formulated a formal equality principle in reference to Plato.

Distribution can be on individual basis (e.g. legal will on properties and belongings), organization basis (e.g. task and responsibility) and federating unit (e.g. allocation of wealth and budgetary allocation, administrative appointment and elections). In each of the cases highlighted, challenges spring up for equity, equality and uniformity in distribution based on differentials like culture, tradition, ethnicity, population, level of contribution etc.

In this guise, some of our results on finite soft quasigroup can be adopted as guide for uniformity, equality and equity in distribution. For a finite soft quasigroup $(F, A)_{(Q,\cdot)}$, the following can be adopted for instance:

$$F = \text{Owner of Legal Will}$$
$$A = \text{Benefitiaries of the Will}$$
$$Q = \text{Wealth, Properties and Belongings.}$$

Then, $F : A \longrightarrow 2^Q \backslash \{\emptyset\}$ defines how the will owner (F) wants to distribute his wealth, properties and belongings (Q) to beneficiaries (A) such that none of them gets null thing and none gets the whole alone. The family $\{F(a)\}_{a\in A}$ forms the possible allocations to beneficiaries (A). It must be noted that Q has a structure (as a quasigroup) which determines the possibilities of $\{F(a)\}_{a\in A}$. For uniformity, equality and equity in distribution, the inequalities in Theorem 3.6, Theorem 3.7, Theorem 3.8 and Theorem 3.9 can be used as guide based on the hypotheses $F(a_i) \cap F(a_{i+1}) \neq \emptyset$ or/and $|F(a_i)| = |F(a_{i+1})|$ for all $i = 1, 2, \ldots, n-1$ where $\{a_i\}_{i=1}^n \subseteq A$ or $A = \{a_i\}_{i=1}^n$.

4. Conclusion

In this work, we have introduced soft set over quasigroup and studied finite soft quasigroup. which is motivated by the study of algebraic structures of soft sets. By introducing the order of a finite soft quasigroup, we established various inequality relationships that exist between the order of a finite quasigroup, the order of its soft quasigroup and the cardinality of its set of parameters. By introducing the arithmetic mean and geometric mean of finite soft quasigroup, we got a sort of Lagrange's Formula $|(F, A)| = |A|\mathcal{AM}(F, A)$ for finite soft quasigroup. Some of the inequalities gotten give an upper bound for the order of a finite soft quasigroup in terms of the order of its quasigroup and cardinality of its set of parameters, and a lower bound for the order of the quasigroup in terms of the arithmetic mean of the finite soft quasigroup.

A chain of inequalities called the Maclaurin's inequality for a finite soft quasigroup $(F,A)_{(Q,\cdot)}$ was shown to exist. We envisage that the consequence of the Maclaurin's inequality for a finite soft quasigroup in the case of a family of finite soft quasigroups $(F_i,A_i)_{(Q,\cdot)}$ will be interesting. A necessary and sufficient condition for a type of finite soft quasigroup to be extensible to a finite super soft quasigroup was established. This result is of practical use whenever a larger set of parameters is required. Our results were illustrated with examples and found to be applicable to Egalitarianism of distribution.

Abbreviations

Not applicable.

Declarations

Availability of data and materials. Not applicable.

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