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ON THE STRUCTURE SPACE OF A Γ-SEMIGROUP VIA ITS LEFT OPERATOR SEMIGROUP

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Abstract

The structure space of a semigroup endowed with hull kernel topology is introduced and studied. Also the structure space of a Γ-semigroup is defined and a homeomorphism has been established between structure space of a Γ-semigroup and the structure space of its left operator semigroup. Moreover, various properties of structure space of a Γ-semigroup are studied via its left operator semigroup.

Keywords: operator semigroup, Γ-semigroup, prime ideal, hull-kernel topology.

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1. Introduction

A semigroup is an algebraic structure consisting of a non-empty set S together with an associative binary operation[3]. The notion of a Γ-semigroup was introduced by Sen and Saha [13] as a generelisation of semigroup and ternary semigroup, some works on Γ-semigroup may be found in [9, 10, 12, 13].

The space of prime ideals of a ring was studied by Kohls in [7] and Gillman studied Rings with Hausdorff structure space in [6]. The structure space of Semiring was studied by Adhikari and Das in [1] while the structure space of uniformly strongly prime ideals of a Γ-semigroup by Chattopadhyay and Kar in [2]. In this paper we study structure space of prime ideals of a semigroup as well as structure space of prime ideals of a Γ-semigroup via its left operator semigroup.

We consider the collection $\mathcal{A}$ of all prime ideals of a semigroup and define a topology $\tau_{\mathcal{A}}$ on $\mathcal{A}$ in Definition 3.3 on $\mathcal{A}$ in terms of closure operator, we call the topological space $(\mathcal{A}, \tau_{\mathcal{A}})$ as structure space of the semigroup S and studied various topological properties such as separation axioms in Theorem 3.8, 3.11, 3.13, 3.15, compactness property in Theorem 3.16, 3.17, 3.20. Then we define the structure space of prime ideals of a Γ-semigroup in Definition 4.6 and we establish a homeomorphism in Theorem 4.8 between structure space of a Γ-semigroup and structure space of its left operator semigroup. Moreover, necessary and sufficient conditions for the structure space of a Γ-semigroup to be T_1, T_2, T_3, compact are obtained via left operator semigroup in Theorem 4.10, 4.12, 4.21, Corollary 4.20, 4.25.

2. Preliminaries

In this section we discuss some elementary preliminaries that we use in the sequel.

Definition 2.1 [13]**.** Let $S = \{a, b, c, \ldots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \ldots\}$ be two non-empty sets. Then S is called a Γ-semigroup if there exists a mapping $S \times \Gamma \times S \rightarrow S$ (images to be denoted by $a\alpha b$) satisfying

(1) $a\gamma b \in S$,

(2) $(a\beta b)\gamma c = a\beta(b\gamma c), \forall a, b, c \in S, \forall \gamma \in \Gamma$.

Example 2.2. Let $S = \{-i, 0, i\}$ and $\Gamma = S$. Then S is a Γ-semigroup under the multiplication over complex numbers while S is not a semigroup under complex number multiplication.

Example 2.3. Let S be the set of all $m \times n$ matrices with entries from a field F and Γ be a set of $n \times m$ matrices with entries from F. Then S is a Γ-semigroup with the usual product of matrices.

Definition 2.4 [3]**.** A non-empty subset I of the semigroup S is said to be an ideal if $SI \subseteq I$ and $IS \subseteq I$. An ideal I of S is called a proper ideal if $I \neq S$.

Definition 2.5 [11]**.** A proper ideal P of the semigroup S is said to be a prime ideal if $AB \subseteq P$ then either $A \subseteq P$ or $B \subseteq P$ for any two ideals A, B of S.

Definition 2.6 [13]**.** A non-empty subset I is said to be an ideal of the Γ-Semigroup S if $I\Gamma S \subseteq I$ and $S\Gamma I \subseteq I$ where for subsets U, V of S and Q of Γ, $UQV = \{uqv : u \in U, v \in V, q \in Q\}$. An ideal I of S is called a proper ideal if $I \neq S$.

Definition 2.7 [13]**.** Let S be Γ-semigroup. A proper ideal P of S is called a prime ideal if for any two ideals I and J of S, $I\Gamma J \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$.

Definition 2.8 [10]**.** Let S be a Γ-semigroup. Define a relation ρ on $S \times \Gamma$ as follows: $(x,\alpha)\rho(y,\beta) \iff x\alpha s = y\beta s, \forall s \in S$. Obviously ρ is an equivalence relation. Let $[x,\alpha]$ denotes the equivalence class containing (x,α). Let $L = \{[x,\alpha] : x \in S \text{ and } \alpha \in \Gamma\}$. Then L is a semigroup under the binary operation defind as $[x,\alpha][y,\beta] = [x\alpha y,\beta]$, for all $x, y \in S$ and $\alpha, \beta \in \Gamma$. The semigroup L is called the left operator semigroup of S. Similarly, right operator semigroup R of a Γ-semigroup S is defind as $R = \{[\alpha,x] : \alpha \in \Gamma, x \in S\}$, where $[\alpha,x][\beta,y] = [\alpha,x\beta y]$, for all $x, y \in S$ and $\alpha, \beta \in \Gamma$.

Let S be a Γ-semigroup with left operator semigroup L. For $P \subseteq L$ and $Q \subseteq S$ we define $P^+ = \{x \in S : [x,\alpha] \in P \text{ for all } \alpha \in \Gamma\}$ and $Q^{+'} = \{[x,\alpha] \in L : x\alpha s \in Q \text{ for all } s \in S\}$.

Theorem 2.9 [4]**.** *Let S be a Γ-semigroup with left and right unities and L be its left operator semigroup. Then if P is a prime ideal of L then P^+ is a prime ideal of S and if Q is a prime ideal of S then $Q^{+'}$ is a prime ideal of L. Moreover $(P^{+'})^+ = P$ and $(Q^+)^{+'} = Q$.*

The proof is same as the proof of Theorem 3.1.11 *and* 3.1.12 *of* [4]*. So we omit it.*

Theorem 2.10 [4]**.** *Let S be a Γ- semigroup with left and right unities and let L and R be its left operator semigroup and right operator semigroup, respectively. Then there is an inclusion preserving bijection between the set of all prime ideals of a Γ-semigroup S and that of its left operator semigroup L (respectively, right operator semigroup R), via the mapping $P \longrightarrow P^{+'}$(resp $P \longrightarrow P^*$), where P is a prime ideal of S, $P^{+'} = \{[x,\alpha] \in L : x\alpha s \in P \text{ for all } s \in S\}$ and $P^{*\prime} = \{[\alpha,x] \in R : s\alpha x \in P \text{ for all } s \in S\}$. The proof is same as Theorem* 3.1.13 *of* [4]*. So we omit it.*

Definition 2.11 [14]**.** Let (X,τ_1) and (Y,τ_2) be two topological spaces. Then a bijection $f : X \longrightarrow Y$ is said to be a homeomorphism if both f and f^{-1} are continuous.

3. Structure space of Semigroup

Definition 3.1. Let S be a semigroup and $\mathcal{A}$ be the collection of all prime ideals of the semigroup S. For any subset A of $\mathcal{A}$, we define

$$\overline{A} = \left\{ I \in \mathcal{A} : \bigcap_{I_\alpha \in A} I_\alpha \subseteq I \right\}.$$

Throughout this section unless otherwise mentioned the semigroup under considerations denoted by S and $\mathcal{A}$ denotes the collection of all prime ideals of S and for any $A \subseteq \mathcal{A}$, $\overline{A}$ has the meaning as in the definition.

Note: $\overline{\emptyset} = \emptyset$.

Theorem 3.2. *Let A,B be any two subsets of $\mathcal{A}$. Then*

(1) $A \subseteq \overline{A}$

(2) $\overline{\overline{A}} = \overline{A}$

(3) $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$

(4) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Proof. (1) Clearly, $\bigcap_{I_\alpha \in A} I_\alpha \subseteq I_\alpha$ for each α and hence $A \subseteq \overline{A}$.

(2) By (1), we have $\overline{A} \subseteq \overline{\overline{A}}$. For converse part, let $I_\beta \in \overline{\overline{A}}$. Then $\bigcap_{I_\alpha \in \overline{A}} I_\alpha \subseteq I_\beta$. Now $I_\alpha \in \overline{A}$ implies that $\bigcap_{I_\gamma \in A} I_\gamma \subseteq I_\alpha$ for all $I_\alpha \in \overline{A}$. Thus

$$\bigcap_{I_\gamma \in A} I_\gamma \subseteq \bigcap_{I_\alpha \in \overline{A}} I_\alpha \subseteq I_\beta \text{ i.e., } \bigcap_{I_\gamma \in A} I_\gamma \subseteq I_\beta.$$

So $I_\beta \in \overline{A}$ and hence $\overline{\overline{A}} \subseteq \overline{A}$. Consequently, $\overline{\overline{A}} = \overline{A}$.

(3) Suppose that $A \subseteq B$. Let $I_\alpha \in \overline{A}$. Then $\bigcap_{I_\beta \in A} I_\beta \subseteq I_\alpha$. Since $A \subseteq B$, so

$$\bigcap_{I_\beta \in B} I_\beta \subseteq \bigcap_{I_\beta \in A} I_\beta \subseteq I_\alpha.$$

This implies that $I_\alpha \in \overline{B}$ and hence $\overline{A} \subseteq \overline{B}$.

(4) Clearly, $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$. For the converse part, let $I_\alpha \in \overline{A \cup B}$. Then $\bigcap_{I_\beta \in A \cup B} I_\beta \subseteq I_\alpha$. It is easy to see that

$$\bigcap_{I_\beta \in A \cup B} I_\beta = \left(\bigcap_{I_\beta \in A} I_\beta \right) \cap \left(\bigcap_{I_\beta \in B} I_\beta \right).$$

Since $\bigcap_{I_\beta \in A} I_\beta$ and $\bigcap_{I_\beta \in B} I_\beta$ are ideals of S, We have

$$\left(\bigcap_{I_\beta \in A} I_\beta \right) \left(\bigcap_{I_\beta \in B} I_\beta \right) \subseteq \left(\bigcap_{I_\beta \in A} I_\beta \right) \cap \left(\bigcap_{I_\beta \in B} I_\beta \right) = \bigcap_{I_\beta \in A \cup B} I_\beta \subseteq I_\alpha.$$

As I_α is a prime ideal of S, either $\bigcap_{I_\beta \in A} I_\beta \subseteq I_\alpha$ or $\bigcap_{I_\beta \in B} I_\beta \subseteq I\alpha$ i.e., either $I_\alpha \in \overline{A}$ or $I_\alpha \in \overline{B}$ i.e., $I_\alpha \in \overline{A} \cup \overline{B}$. Consequently, $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$ and hence $\overline{A \cup B} = \overline{A} \cup \overline{B}$. ■

Definition 3.3. The closure operator $A \mapsto \overline{A}$ gives a topology $\tau_{\mathcal{A}}$ is called the hull-kernel topology and the topological space $(\mathcal{A}, \tau_{\mathcal{A}})$ is called the structure space of the semigroup S.

Definition 3.4. Let I be an ideal of a semigroup S. We define

$$\Delta(I) = \{I' \in \mathcal{A} : I \subseteq I'\} \text{ and } C\Delta(I) = \{I' \in \mathcal{A} : I \nsubseteq I'\}.$$

For the results of the rest of this section we use the same notation as of Definition 3.3 and 3.4.

Proposition 3.5. *Any closed set in $\mathcal{A}$ is of the form $\Delta(I)$, where I is an ideal of the semigroup S.*

Proof. Let $\overline{A}$ be any closed set in $\mathcal{A}$, where $A \subseteq \mathcal{A}$. Let $A = \{I_\alpha : \alpha \in \Lambda\}$, where Λ is an index set and $I = \bigcap_{I_\alpha \in A} I_\alpha$. Then I is an ideal of S. Let $I' \in \overline{A}$. Then $\bigcap_{I_\alpha \in A} I_\alpha \subseteq I'$. This implies that $I \subseteq I'$. Consequently, $I' \in \Delta(I)$. So $\overline{A} \subseteq \Delta(I)$. Again, let $I' \in \Delta(I)$. Then $I \subseteq I'$ i.e., $\bigcap_{I_\alpha \in A} I_\alpha \subseteq I'$. Consequently, $I' \in \overline{A}$ and hence $\Delta(I) \subseteq \overline{A}$. Thus $\overline{A} = \Delta(I)$. ■

Corollary 3.6. *Any open set in $\mathcal{A}$ is of the form $C\Delta(I)$, where I is an ideal of S.*

Let S be a semigroup and $a \in S$. We define $\Delta(a) = \{I \in \mathcal{A} : a \in I\}$ and $C\Delta(a) = \{I \in \mathcal{A} : a \notin I\}$.

Proposition 3.7. *$\{C\Delta(a) : a \in S\}$ forms an open base for the hull-kernel topology $\tau_{\mathcal{A}}$ on $\mathcal{A}$.*

Proof. Let $U \in \tau_{\mathcal{A}}$. Then $U = C\Delta(I)$, where I is an ideal of S. Let $J \in U = C\Delta(I)$. Then $I \nsubseteq J$. This implies that there exists $a \in I$ such that $a \notin J$. Thus $J \in C\Delta(a)$. Now it remains to show that $C\Delta(a) \subset U$. Let $K \in C\Delta(a)$. Then $a \notin K$. This implies that $I \nsubseteq K$. Consequently, $K \in U$ and hence $C\Delta(a) \subset U$. So we find that $J \in C\Delta(a) \subset U$. Thus $C\Delta(a)$ is an open base for the hull-kernel topology $\tau_{\mathcal{A}}$ on $\mathcal{A}$. ■

Theorem 3.8. *The structure space $(\mathcal{A}, \tau_{\mathcal{A}})$ is a T_0 space.*

Proof. Let I_1 and I_2 be two distinct elements of $\mathcal{A}$. Then there is an element a either in $I_1 \backslash I_2$ or in $I_2 \backslash I_1$. Suppose that $a \in I_1 \backslash I_2$. Then $C\Delta(a)$ is a neighbourhood of I_2 not containing I_1. Hence $(\mathcal{A}, \tau_{\mathcal{A}})$ is a T_0 space. ■

Example 3.9 [11]**.** Let $S = Z_6$, the classes of residues of integers modulo 6 i.e., $S = \{\bar{0}, \bar{1}, \ldots, \bar{5}\}$. Then S forms a semigroup with respect to multiplication modulo n. Here $P_1 = \{\bar{0}, \bar{2}, \bar{3}, \bar{4}\}$ is a prime ideal and $P_2 = \{\bar{0}, \bar{3}\}$ is another prime ideal, which is contained in P_1. So this two element P_1 and P_2 violating the T_1 axiom and hence the space $(\mathcal{A}, \tau_{\mathcal{A}})$ is not T_1.

Example 3.10. Consider the Prime ideal $I =< 2 >$ and $J =< 3 >$ of the semigroup S of Natural number with usual multiplication. Then $I \cup J$ is also a prime ideal of S. So this two element violating the T_1 axiom and hence the space $(\mathcal{A}, \tau_{\mathcal{A}})$ is not T_1.

We derive a necessary and sufficient condition for the space $(\mathcal{A}, \tau_{\mathcal{A}})$ to be T_1 as follows.

Theorem 3.11. *$(\mathcal{A}, \tau_{\mathcal{A}})$ is a T_1 space if and only if no element of $\mathcal{A}$ is contained in any other element of $\mathcal{A}$.*

Proof. Let $(\mathcal{A}, \tau_{\mathcal{A}})$ be a T_1 space. Suppose that I_1 and I_2 be any two distinct elements of $\mathcal{A}$. Then each of I_1 and I_2 has a neighbourhood not containing the other. Since I_1 and I_2 are arbitary elements of $\mathcal{A}$, it follows that no element of $\mathcal{A}$ is contained in any other element of $\mathcal{A}$.

Conversely, suppose that no element of $\mathcal{A}$ is contained in any other element of $\mathcal{A}$. Let I_1 and I_2 be any two distinct elements of $\mathcal{A}$. Then by hypothesis, $I_1 \not\subset I_2$ and $I_2 \not\subset I_1$. This implies that there exists $a, b \in S$ such that $a \in I_1$ but $a \notin I_2$ and $b \in I_2$ but $b \notin I_1$. Consequently, we have $I_1 \in C\Delta(b)$ but $I_1 \notin C\Delta(a)$ and $I_2 \in C\Delta(a)$ but $I_2 \notin C\Delta(b)$ i.e., each of I_1 and I_2 has a neighbourhood not containing the other. Hence $(\mathcal{A}, \tau_{\mathcal{A}})$ is a T_1 space. ∎

Example 3.12 [11]**.** Let G be a simple semigroup without idempotent. Adjoin an identity element e and consider the semigroup $S = G \cup \{e\}$. In this semigroup S every prime ideal is maximal and so the corresponding structure space $(\mathcal{A}, \tau_{\mathcal{A}})$ is T_1.

The followings are necessary and sufficient condition for a structure space $(\mathcal{A}, \tau_{\mathcal{A}})$ of a semigroup to be Housdorff and regular, the proofs are analogues to (Theorem 3.9, 3.11, [2]), hence we omit the proof

Theorem 3.13. *$(\mathcal{A}, \tau_{\mathcal{A}})$ is a Housdorff space if and only if for any two distinct pair of elements I, J of $\mathcal{A}$, there exists $a, b \in S$ such that $a \notin I$ and $b \notin J$ and there does not exist any element L such that $a \notin L$ and $b \notin L$.*

Theorem 3.14. *$(\mathcal{A}, \tau_{\mathcal{A}})$ is a regular space if and only if for any $I \in \mathcal{A}$ and $a \notin I$, $a \in S$, there exist an ideal J of S and $b \in S$ such that $I \in C\Delta(b) \subseteq \Delta(J) \subseteq C\Delta(a)$.*

The space $(\mathcal{A}, \tau_{\mathcal{A}})$ is a T_0 space and every regular T_0 space is a T_3 space, so we have the following corollary:

Corollary 3.15. *$(\mathcal{A}, \tau_{\mathcal{A}})$ is a T_3 space if and only if for any $I \in \mathcal{A}$ and $a \notin I$, $a \in S$, there exist an ideal J of S and $b \in S$ such that $I \in C\Delta(b) \subseteq \Delta(J) \subseteq C\Delta(a)$.*

Theorem 3.16. *$(\mathcal{A}, \tau_{\mathcal{A}})$ is a compact space if and only if for any collection $\{a_\alpha\}_{\alpha\in\Lambda} \subset S$ (where Λ is an index set) there exists a finite subcollection $\{a_i : i = 1, 2, \ldots, n\}$ in S such that for any $I \in \mathcal{A}$, there exists a_i such that $a_i \notin I$.*

Proof. Let $(\mathcal{A}, \tau_{\mathcal{A}})$ be a compact space. Then the open cover $\{C\Delta(a_\alpha) : a_\alpha \in S\}$ of $(\mathcal{A}, \tau_{\mathcal{A}})$ has a finite subcover $\{C\Delta(a_i) : i = 1, 2, \ldots, n\}$. Let I be any element of $\mathcal{A}$. Then $I \in C\Delta(a_i)$ for some $a_i \in S$. This implies that $a_i \notin I$. Hence $\{a_i : i = 1, 2, \ldots, n\}$ is the required finite subcollection of elements of S such that for any $I \in \mathcal{A}$, there exists a_i such that $a_i \notin I$.

Conversely, suppose that the given condition holds. Let $\{C\Delta(a_\alpha) : a_\alpha \in S\}$ be an open cover of $\mathcal{A}$. Suppose to the contrary that no finite subcollection of $\{C\Delta(a_\alpha) : a_\alpha \in S\}$ covers $\mathcal{A}$. This means that for any finite set $\{a_1, a_2, \ldots, a_n\}$ of elements of S, $C\Delta(a_1) \cup C\Delta(a_2) \cup \cdots \cup C\Delta(a_n) \neq \mathcal{A}$.

$\Rightarrow \Delta(a_1) \cap \Delta(a_2) \cap \cdots \cap \Delta(a_n) \neq \phi$.
$\Rightarrow$ there exists $I \in \mathcal{A}$ such that $I \in \Delta(a_1) \cap \Delta(a_2) \cap \cdots \cap \Delta(a_n)$
$\Rightarrow a_1, a_2, \ldots, a_n \in I$, which contradicts our hypothesis.

So the open cover $\{C\Delta(a_\alpha) : a_\alpha \in S\}$ has a finite subcover and hence $(\mathcal{A}, \tau_{\mathcal{A}})$ is compact. ■

Corollary 3.17. *If S is finitely generated, then $(\mathcal{A}, \tau_{\mathcal{A}})$ is a compact space.*

Proof. Let $\{a_i : i = 1, 2, \ldots, n\}$ be a finite set of generators of S. Then for any $I \in \mathcal{A}$, there exists a_i such that $a_i \notin I$, since I is a proper prime ideal of S. Hence by Theorem 3.16, $(\mathcal{A}, \tau_{\mathcal{A}})$ is a compact space. ■

Definition 3.18. A semigroup S is called a Noetherian semigroup if it satisfies ascending chain condition on ideals i.e., if $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$ is an ascending chain of ideals of S, then there exists a positive integer m such that $I_n = I_m$ for all $n \geq m$.

Theorem 3.19. *If S is a Noetherian semigroup, then $(\mathcal{A}, \tau_{\mathcal{A}})$ is countably compact.*

Proof. Let $\{\Delta(I_n)\}_{n=1}^{\infty}$ be a countable collection of closed sets in $\mathcal{A}$ with finite intersection property(FIP). Let us consider the following ascending chain of prime ideals of $S :< I_1 >\subseteq< I_1 \cup I_2 >\subseteq< I_1 \cup I_2 \cup I_3 >\subseteq \cdots$. Since S is a Noetherian semigroup, there exists a positive integers m such that $< I_1 \cup I_2 \cup \cdots \cup I_m >=< I_1 \cup I_2 \cup \cdots \cup I_{m+1} >= \cdots$.

Thus it follows that $< I_1 \cup I_2 \cup \cdots \cup I_m >\in \bigcap_{n=1}^{\infty} \Delta(I_n)$. Consequently, $\bigcap_{n=1}^{\infty} \Delta(I_n) \neq \phi$ and hence $(\mathcal{A}, \tau_{\mathcal{A}})$ is countably compact.

Since countably compact second countable topological space is compact, the following is an obvious consequence of the above result. ■

Corollary 3.20. *If S is a Noetherian semigroup and $(\mathcal{A}, \tau_{\mathcal{A}})$ is second countable then $(\mathcal{A}, \tau_{\mathcal{A}})$ is compact.*

4. Structure space of Γ-semigroup

Let S be a Γ- semigroup with left and right unities and L be its left operator semigroup. Let $\mathcal{A}_L$ be the collection of all prime ideals of L and τ_L be the hull-kernel topology as defined in § 3 (Definition 3.3). We call the topological space $(\mathcal{A}_L, \tau_L)$ to be the structure space of the operator semigroup L.

Let $\mathcal{A}_S$ be the collection of all prime ideals of the Γ-semigroup S. Let $f : \mathcal{A}_S \mapsto \mathcal{A}_L$ be the inclusion preserving bijection defined as $f(P) = P^{+'}$, where $P \in \mathcal{A}_S$ and $P^{+'} \in \mathcal{A}_L$ [Theorem 2.10]. We define a map from $\mathcal{A}_L$ to $\mathcal{A}_S$ by $f' : Q \mapsto Q^+$, where $Q \in \mathcal{A}_L$ and $Q^+ \in \mathcal{A}_S$ [Theorem 2.9]. Let $A = \{P_1, P_2, \ldots, P_n, \ldots\} \subseteq \mathcal{A}_S$. We define $g : \rho(\mathcal{A}_S) \mapsto \rho(\mathcal{A}_L)$ by $g(A) = A^{+'} = \{P_1^{+'}, \ldots, P_n^{+'}, \ldots\} \subseteq \mathcal{A}_L$, where $P_i^{+'} \in \mathcal{A}_L$ are images of $P_i \in \mathcal{A}_S$ by the map f and $\rho(\mathcal{A}_S)$ and $\rho(\mathcal{A}_L)$ are power set of $\mathcal{A}_S$ and $\mathcal{A}_L$ respectively. In a similar way, for any $B \subseteq \mathcal{A}_L$ we define $B^+ = \{Q_1^+, Q_2^+, \ldots, Q_n^+, \ldots\} \subseteq \mathcal{A}_S$, the images of $Q_i \in \mathcal{A}_L$ by the map f'.

Theorem 4.1. *There is an inclusion preserving bijection from $\rho(\mathcal{A}_S)$ to $\rho(\mathcal{A}_L)$ via the mapping defined by $g(A) = A^{+'}$, where $A \in \rho(\mathcal{A}_S)$.*

Proof. Let $A \in \rho(\mathcal{A}_S)$. we shall now prove that$(A^{+'})^+ = A$. Let $I^+ \in (A^{+'})^+$. This implies $I \in A^{+'}$. So there exists J in A such that $J^{+'} = I \in A^{+'}$. Now $I^+ = (J^{+'})^+ = J \in A$. This implies $(A^{+'})^+ \subseteq A$.

Let $I \in A$. Then $I^{+'} \in A^{+'}$. This implies $I = (I^{+'})^+ \in (A^{+'})^+$, i.e., $A \subseteq (A^{+'})^+$. So $(A^{+'})^+ = A$.

Similarly we can prove that $(B^+)^{+'} = B$ for all $B \in \rho(\mathcal{A}_L)$. Hence the mapping $g : A \mapsto A^{+'}$ is a bijection.

Let $A \subseteq B \in \rho(\mathcal{A}_S)$. Now $I^{+'} \in A^{+'}$ implies $I \in A \subseteq B$. So $I^{+'} \in B^{+'}$ and hence $A^{+'} \subseteq B^{+'}$. This completes the proof. ∎

Definition 4.2. Let S be a Γ-semigroup and $\mathcal{A}_S$ denote the set of all prime ideals of S. For any subset A of $\mathcal{A}_S$, we define $\overline{A} = \{I \in \mathcal{A}_S : \bigcap_{I_\alpha \in A} I_\alpha \subseteq I\}$.

Note: $\overline{\emptyset} = \emptyset$.

Lemma 4.3. *For any $A, B \subseteq \mathcal{A}_L$, we have*

(1) $(A \cap B)^+ = A^+ \cap B^+$

(2) $\overline{(B^+)} = (\overline{B})^+$

(3) $(A \cup B)^+ = A^+ \cup B^+$.

Proof. (1) Let $I^+ \in A^+ \cap B^+$.

$\iff I^+ \in A^+$ and $I^+ \in B^+ \iff I \in A$ and $I \in B \iff I \in A \cap B$

$\iff I^+ \in (A \cap B)^+$. This completes the proof.

(2) Let $I \in \overline{B^+}$. Then $I \in \mathcal{A}_S$ and $\bigcap_{J \in B} J^+ \subseteq I \iff (\bigcap_{J \in B} J^+)^{+'} = ((\bigcap_{J \in B} J)^+)^{+'} \subseteq I^{+'}$ [by (1)] $\iff \bigcap_{J \in B} J \subseteq I^{+'} \iff I^{+'} \in \overline{B} \iff (I^{+'})^+ \in (\overline{B})^+ \iff I \in (\overline{B})^+$.

(3) Let $I^+ \in (A \cup B)^+ \iff I \in A \cup B \iff I \in A$ or $I \in B \iff I^+ \in A^+$ or $I^+ \in B^+ \iff I^+ \in A^+ \cup B^+$. This completes the proof. ∎

Lemma 4.4. *Similarly for any $A, B \subseteq \mathcal{A}_S$, we can prove the following*

(1) $(A \cap B)^{+'} = A^{+'} \cap B^{+'}$

(2) $(\overline{A})^{+'} = \overline{(A^{+'})}$

(3) $(A \cup B)^{+'} = A^{+'} \cup B^{+'}$.

Theorem 4.5. *Let A, B be any two subsets of $\mathcal{A}_S$. Then*

(1) $A \subseteq \overline{A}$

(2) $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$

(3) $\overline{\overline{A}} = \overline{A}$

(4) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Proof. (1) $A^{+'} \subseteq \overline{A^{+'}} = (\overline{A})^{+'} \Rightarrow (A^{+'})^+ \subseteq ((\overline{A})^{+'})^+ = \overline{A} \Rightarrow A \subseteq \overline{A}$

(2) Let $A \subseteq B$. Then $A^{+'} \subseteq B^{+'} \Rightarrow \overline{(A^{+'})} \subseteq \overline{(B^{+'})}$ [by Theorem 3.2(3)].
$\Rightarrow (\overline{(A^{+'})})^+ \subseteq (\overline{(B^{+'})})^+$ [by Theorem 4.1].
$\Rightarrow \overline{(A^{+'})^+} \subseteq \overline{(B^{+'})^+}$ [by Lemma 4.3(2)].$\Rightarrow \overline{A} \subseteq \overline{B}$.

(3) As $A^{+'} \in \mathcal{A}_L$, so $\overline{\overline{(A^{+'})}} = \overline{(A^{+'})}$ [by Theorem 3.2(2)].
$\Rightarrow \overline{((\overline{A})^{+'})} = \overline{(A^{+'})}$ [by Lemma 4.4(2)] $\Rightarrow (\overline{\overline{A}})^{+'} = \overline{(A^{+'})}$
$\Rightarrow ((\overline{\overline{A}})^{+'})^+ = (\overline{(A^{+'})})^+ \Rightarrow \overline{\overline{A}} = ((\overline{A})^{+'})^+ = \overline{A}$.

(4) $\overline{A \cup B} = ((\overline{A \cup B})^{+'})^+$ [by Theorem 4.1]
$= (\overline{A^{+'} \cup B^{+'}})^+$[by Lemma 4.3(2) and 4.3(3)]$= (\overline{A^{+'}} \cup \overline{B^{+'}})^+$[by Theorem 3.2(4)]
$= (\overline{A^{+'}})^+ \cup (\overline{B^{+'}})^+ = \overline{(A^{+'})^+} \cup \overline{(B^{+'})^+} = \overline{A} \cup \overline{B}$. ∎

Definition 4.6. The closure operator $A \mapsto \overline{A}$ gives a topology τ_S on $\mathcal{A}_S$. This topology is called the hull-kernel topology and the topological space $(\mathcal{A}_S, \tau_S)$ is called the structure space of the Γ-semigroup S.

Definition 4.7. Let I be an ideal of a Γ-semigroup S. We define $\Delta(I) = \{I' \in \mathcal{A}_S : I \subseteq I'\}$ and $C\Delta(I) = \{I' \in \mathcal{A}_S : I \nsubseteq I'\}$. For any $a \in \mathcal{A}_S$, we define $\Delta(a) = \{I \in \mathcal{A}_S : a \in I\}$ and $C\Delta(a) = \{I \in \mathcal{A}_S : a \notin I\}$.

Theorem 4.8. *The structure spaces $(\mathcal{A}_S, \tau_S)$ and $(\mathcal{A}_L, \tau_L)$ are homeomorphic.*

Proof. Let $f : (\mathcal{A}_S, \tau_S) \mapsto (\mathcal{A}_L, \tau_L)$ be the map defind by $f(P) = P^{+'}$ where $P \in \mathcal{A}_S$ and $P^{+'} \in \mathcal{A}_L$. We know that f is an inclusion preserving bijection (Theorem 2.10).

Let $A \subseteq \mathcal{A}_S$. Then $A \subseteq \overline{A} \Rightarrow A^{+'} \subseteq (\overline{A})^{+'} \Rightarrow \overline{(A^{+'})} \subseteq \overline{((\overline{A})^{+'})} = (\overline{\overline{A}})^{+'} = (\overline{A})^{+'}$. So $\overline{f(A)} \subseteq f(\overline{A}) \Rightarrow f$ is a closed map. Since f is closed bijection, so f is open map. Hence f^{-1} is continuous. Similarly, let $B \subseteq \mathcal{A}_L$.

Then $B \subseteq \overline{B} \Rightarrow B^{+} \subseteq (\overline{B})^{+} \Rightarrow \overline{(B^{+})} \subseteq \overline{((\overline{B})^{+})} = (\overline{\overline{B}})^{+} = (\overline{B})^{+}$ i.e., $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$. Since B is arbitary, f^{-1} is a closed map and also bijection, so f^{-1} is an open map, and hence f is continuous.

So f is a homeomorphism. This completes the proof. ■

Theorem 4.9. *The structure space $(\mathcal{A}_S, \tau_S)$ is a T_0 space.*

Proof. We know the space $(\mathcal{A}_L, \tau_L)$ is T_0 and is homeomorphic to $(\mathcal{A}_S, \tau_S)$, so $(\mathcal{A}_S, \tau_S)$ is a T_0 space. ■

Theorem 4.10. *$(\mathcal{A}_S, \tau_S)$ is a T_1 space if and only if no element of $\mathcal{A}_S$ is contained in any other element of $\mathcal{A}_S$.*

Proof. Let $(\mathcal{A}_S, \tau_S)$ be a T_1 space, then $(\mathcal{A}_L, \tau_L)$ is a T_1 space. Then no element of $\mathcal{A}_L$ is contained in any other element of $\mathcal{A}_L$. Since there is a inclusion preserving bijection between $\mathcal{A}_S$ and $\mathcal{A}_L$, so no element of $\mathcal{A}_S$ is contained in any other element of $\mathcal{A}_S$.

Conversely, let no element of $\mathcal{A}_S$ be contained in any other element of $\mathcal{A}_S$. Then no element of $\mathcal{A}_L$ is contained in any other element of $\mathcal{A}_L$ which implies $(\mathcal{A}_L, \tau_L)$ is T_1 by Theorem 3.11 and so $(\mathcal{A}_S, \tau_S)$ is T_1. ■

Lemma 4.11. *For $P, Q \in \mathcal{A}_S$, $P \neq Q$ there exist $p, q \in S$ such that $p \notin P$ and $q \notin Q$ and there does not exists any element $F \in \mathcal{A}_S$ such that $p \notin F, q \notin F$. Then there exist some $\gamma_1, \gamma_2 \in \Gamma$ such that $[p, \gamma_1] \notin P^{+'}$, $[q, \gamma_2] \notin Q^{+'}$ and there exist no $F' \in \mathcal{A}_L$ such that $[p, \gamma_1] \notin F'$, $[q, \gamma_2] \notin F'$ and conversely.*

Proof. Let $P, Q \in \mathcal{A}_S$ such that $P \neq Q$. Then $P^{+'}, Q^{+'} \in \mathcal{A}_L$ and there exist some $\gamma_1, \gamma_2 \in \Gamma$ such that $[p, \gamma_1] \notin P^{+'}$, $[q, \gamma_2] \notin Q^{+'}$. Hence there exist no $F' \in \mathcal{A}_L$ such that $[p, \gamma_1] \notin F'$ and $[q, \gamma_2] \notin F'$, otherwise $(F')^{+} \in \mathcal{A}_S$ such that $p \notin (F')^{+}$, $q \notin (F')^{+}$. Similarly we can prove the converse. ■

By using the Lemma 4.11, Theorems 4.8 and 3.13 we deduce the following result.

Theorem 4.12. *$(\mathcal{A}_S, \tau_S)$ is T_2 if and only if $P, Q \in \mathcal{A}_S, P \neq Q$ there exists $p, q \in S$ such that $p \notin P$, $q \notin Q$ and there does not exist any element $F \in \mathcal{A}_S$ such that $p \notin F$ and $q \notin F$.*

Proposition 4.13. *Let I be an ideal of a Γ-semigroup S. Then $(\Delta(I))^{+'} = \Delta(I^{+'})$.*

Proof. Let $P^{+'} \in (\Delta(I))^{+'}$. Then $P \in \Delta(I) \subseteq \mathcal{A}_S$ which implies $P \in \mathcal{A}_S$ and $I \subseteq P$. Thus $I^{+'} \subseteq P^{+'} \in \mathcal{A}_L$ and $I^{+'}$ is an ideal of L. So $P^{+'} \in \Delta(I^{+'})$. Therefore $(\Delta(I))^{+'} \subseteq \Delta(I^{+'})$. Similarly we can prove $\Delta(I^{+'}) \subseteq (\Delta(I))^{+'}$ (cf. Theorem 4.1). Hence the result follows. ■

Similarly we can prove that $(\Delta(I))^{+} = \Delta(I^{+})$.

Proposition 4.14. *Any closed set in $\mathcal{A}_S$ is of the form $\Delta(I)$ where I is an ideal of S.*

Proof. Let $\overline{A}$ be any closed set in $\mathcal{A}_S$. Then $(\overline{A})^{+'}$ is a closed set (cf. Lemma 4.3(2)) in $\mathcal{A}_L$, so $(\overline{A})^{+'} = \Delta(I^{+'}) = (\Delta(I))^{+'}$ (using 4.13)for some ideal I of S. So $((\overline{A})^{+'})^{+} = ((\Delta(I))^{+'})^{+}$ implies $\overline{A} = \Delta(I)$. Hence the result follows. ■

In view of the relevant definitions the following is an easy consequence of the above result.

Corollary 4.15. *Any open set in $\mathcal{A}_S$ is of the form $C\Delta(I)$, where I is an ideal of S.*

Proposition 4.16. *The family of open sets $\{C\Delta(a) : a \in S)$ forms a base for the hull kernel topology τ_S on $\mathcal{A}_S$.*

Proof. The following result follows by applying arguments similar to those of semigroup. ■

Proposition 4.17. *Let S be a Γ-semigroup and $a \in S$, then $[C\Delta(a)]^{+'} = C\Delta([a,\gamma])$ for some $\gamma \in \Gamma$.*

Proof. Let $I^{+'} \in [C\Delta(a)]^{+'}$. Then $I \in C\Delta(a)$, so $a \notin I$ and hence there exists some $\gamma \in \Gamma$ such that $[a,\gamma] \notin I^{+'}$ implies $I^{+'} \in C\Delta([a,\gamma])$. Therefore $[C\Delta(a)]^{+'} \subseteq C\Delta([a,\gamma])$.

Again let $I \in C\Delta([a,\gamma]) \Rightarrow [a,\gamma] \notin I \Rightarrow a \notin I^{+} \Rightarrow I^{+} \in C\Delta(a) \Rightarrow I \in [C\Delta(a)]^{+'}$. So $C\Delta([a,\gamma]) \subseteq [C\Delta(a)]^{+'}$. Therefore $[C\Delta(a)]^{+'} = C\Delta([a,\gamma])$. ■

Corollary 4.18. $(C\Delta([a,\gamma]))^{+} = C\Delta(a)$ *where $a \in S$ and for some $\gamma \in \Gamma$.*

Theorem 4.19. *$(\mathcal{A}_S, \tau_S)$ is a regular space if and only if for any $P \in \mathcal{A}_S$ and $p \notin P$, $p \in S$, there exists an ideal Q of S and $q \in S$ such that $P \in C\Delta(q) \subseteq \Delta(Q) \subseteq C\Delta(p)$.*

Proof. Let $(\mathcal{A}_S, \tau_S)$ is a regular space. Then $(\mathcal{A}_L, \tau_L)$ is regular. Let $P \in \mathcal{A}_S$ and $p \in S$ such that $p \notin P$. Therefore $P^{+'} \in \mathcal{A}_L$ and $[p, \gamma] \notin P^{+'}$ for some $\gamma \in \Gamma$. so there exists an ideal K of L and an element $[q, \gamma_1] \in L$ such that $P^{+'} \in C\Delta[q, \gamma_1] \subseteq \Delta(K) \subseteq C\Delta[p, \gamma] \Rightarrow P^{+'} \in [C\Delta(q)]^{+'} \subseteq \Delta(K) \subseteq [C\Delta(p)]^{+'} \Rightarrow I \in C\Delta(q) \subseteq \Delta(Q) \subseteq C\Delta(p)$, where $Q = K^+$.

Similarly we can prove the converse. ∎

Since every regular T_0 space is a T_3 space, we have the following corollary.

Corollary 4.20. *$(\mathcal{A}_S, \tau_S)$ is a T_3 space if and only if for any $P \in \mathcal{A}_S$ and $p \notin P$, $p \in S$, there exists an ideal Q of S and $q \in S$ such that $P \in C\Delta(q) \subseteq \Delta(Q) \subseteq C\Delta(p)$.*

Theorem 4.21. *$(\mathcal{A}_S, \tau_S)$ is compact space if and only if for any collection $\{a_\alpha\}_{\alpha \in \Lambda} \subset S$, (where Λ is an index set) there exists a finite subcollection $\{a_i : i = 1, 2, \ldots, n\}$ in S such that for any $I \in \mathcal{A}_S$, there exists a_i such that $a_i \notin I$.*

Proof. Let $(\mathcal{A}_S, \tau_S)$ be compact. So $(\mathcal{A}_L, \tau_L)$ is compact. Let $\{a_\alpha\}_{\alpha \in \Lambda}$ be any collection of subsets of S. Then $\{[a_\alpha \, \gamma] : \alpha \in \Lambda\}$ is a collection in L. Since $(\mathcal{A}_L, \tau_L)$ is compact, there exists a finite subcollection $\{[a_i, \gamma] \in L : i = 1, 2, \ldots, n\}$ such that for any $I \in \mathcal{A}_S$ there exists some a_i with $[a_i, \gamma] \notin I^{+'}$ which implies that $a_i \notin I$ and $\{a_i \in S : i = 1, 2, \ldots, n\}$ is a finite subcollection in S. Converse follows similarly. ∎

Definition 4.22. A Γ-semigroup S is called Noetherian Γ-semigroup if it satisfies ascending chain condition on ideals i.e., if $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq ...$ is an ascending chain of ideals of S, then there exists a positive integer m such that $I_n = I_m$ for all $n \geq m$.

Since there exists a inclusion preserving bijection between ideals of S and L, so we have the following result.

Theorem 4.23. *A Γ-semigroup S is Noetherian if and only if L is Noetherian.*

Theorem 4.24. *If S is Noetherian Γ-semigroup, then $(\mathcal{A}_S, \tau_S)$ is countably compact.*

Proof. Let S is Noetherian Γ- semigroup. Then L is Noetherian semigroup. Then $(\mathcal{A}_L, \tau_L)$ is countably compact and hence $(\mathcal{A}_S, \tau_S)$ is countably compact. ∎

Since countably compact second countable topological space is compact, the following is an obvious consequence of the above result:

Corollary 4.25. *If S is Noetherian Γ-semigroup and $(\mathcal{A}_S, \tau_S)$ is second countable then $(\mathcal{A}_S, \tau_S)$ is compact.*

Concluding Remark. We can obtain analogous results on structure space of Γ-semigroup via right operator semigroup instead of left operator semigroup by applying similar arguments as above and Theorem 2.10.

There is a scope to study the structure space of uniformly strongly prime ideals of a Γ-semigroup via operator semigroup to verify results of S. Kar [2].

If S is a commutative Γ-semigroup, then one can study minimal prime ideal of a commutative Γ -semigroup via operator semigroup by using results of "Minimal prime ideal of a commutative semigroup" of J. Kist [8].

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References

[1] M.R. Adhikari and M.K. Das, *Structure spaces of semirings*, Bull. Cal. Math. Soc. **86** (1994) 313–317.

[2] S. Chattopadhyay and S. Kar, *On structure space of* Γ*-semigroup*, Acta Univ. Palacki Olomuc Fac. Rer. Nat. Math. **47** (2008) 37–46.

[3] A.H. Clifford and G.B. Preston, The Algebraic Theory of Semigroups, American Math. Soc. (Providence, R.I., 1961).
https://doi.org/10.1090/surv/007.1

[4] T.K. Dutta and N.C. Adhikary, *On* Γ*-semigroup with right and left unities*, Soochow J. Math. **19** (4) (1993) 461–474.

[5] T.K. Dutta and S. Chattopadhyay, On Uniformly Strongly Prime Γ-Semigroup, Annale Stiintifice Ale Universita TII "AL.I.CUZA" IASI Tomul LII, s.I, Mathematica, (2006).

[6] L. Gillman, *Rings with Housdorff structure space*, Fund. Math. **45** (1957) 1–16.
https://doi.org/10.4064/fm-45-1-1-16

[7] C.W. Kohls, *The space of prime ideals of a ring*, Fund. Math. **45** (1957) 17–27.
https://doi.org/10.4064/fm-45-1-17-27

[8] J. Kist, *Minimal prime ideal of a commutative semigroup*, Proc. London. Math. Soc. **13** (3) (1963) 31–50.
https://doi.org/10.1112/plms/s3-13.1.31

[9] N.K. Saha, *On* Γ*-semigroup* II, Bull. Cal. Math. Soc. **79** (1987) 331–335.

[10] N.K. Saha, *On* Γ*-semigroup* III, Bull. Cal. Math. Soc. **80** (1988) 1–12.

[11] S. Schwartz, *Prime ideals and maximal ideals in semigroups*, Czechhoslovak Math. J. **19** (1969) 72–79.
https://doi.org/10.21136/CMJ.1969.100877

[12] M.K. Sen, *On* Γ*-semigroups*, in: Int. Conf. on Algebra and Its Appl., Decker Publication (Ed(s)), (New York, 1981).

[13] M.K. Sen and N.K. Saha, *On* Γ*-semigroup* I, Bull. Cal. Math. Soc. **78** (1986) 180–186.

[14] J.R. Munkrees, *Topology*, Pearson Education India.

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