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# HYPER RL-IDEALS IN HYPER RESIDUATED LATTICES

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**Abstract**

In this paper, we introduce the notion of a (strong) hyper RL-ideal in hyper residuated lattices and give some properties and characterizations of them. Next, we characterize the (strong) hyper RL-ideals generated by a subset and give some characterizations of the lattice of these hyper RL-ideals. Particularly, we prove that this lattice is algebraic and compact elements are finitely generated hyper RL-ideals, and obtain some isomorphism theorems. Finally, we introduce the notion of nodal hyper RL-ideals in a hyper residuated lattice and investigate their properties. We prove that the set of nodal hyper RL-ideals is a complete Brouwerian lattice and under suitable operations is a Heyting algebra.

**Keywords:** residuated lattice, MV-algebra, BL-algebra, hyper residuated lattice, hyper ideal.

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## 1. Introduction and preliminaries

Residuated lattices were introduced by Ward and Dilworth [10] as a bounded lattice $L$ endowed with a residuated operation '$\star$' with the residual '$\to$' such that $(L;\star)$ forms a commutative monoid and also satisfies the adjoint property

$$a \star b \leq c \;\Leftrightarrow\; a \leq b \to c$$

The main examples of residuated lattices are MV-algebras [5] and BL-algebras [7].

The hyperstructure theory was introduced by Marty at the 8th Congress of Scandinavian Mathematicians. After that many researchers applied it to algebraic structures. It was Borzooei and their co-authors that applied the hyperstructures to algebraic logics and introduced hyper K-algebras [4]. After that

Ghorbani *et al.* introduced hyper MV-algebras [6]. Mittas *et al.* [8] introduced the notion of a hyperlattice and superlattice. A superlattice [8] is a partially ordered set $\langle A; \leq \rangle$ endowed with two binary hyperoperations $\vee$ and $\wedge$ satisfying the following properties: for all $a, b, c \in A$,

- $a \in (a \vee a) \cap (a \wedge a)$,
- $a \vee b = b \vee a$, $a \wedge b = b \wedge a$,
- $a \vee (b \vee c) = (a \vee b) \vee c$, $a \wedge (b \wedge c) = (a \wedge b) \wedge c$,
- $a \in (a \vee (a \wedge b)) \cap (a \wedge (a \vee b))$,
- $a \leq b$ implies $b \in a \vee b$ and $a \in a \wedge b$,
- if $a \in a \wedge b$ or $b \in a \vee b$, then $a \leq b$.

If $(X, \leq)$ is a partially ordered set, the ordering $\leq$ can be extended to a binary relation $\ll$ in the power set of $X$ as

$$A \ll B \Leftrightarrow (\exists a \in A)(\exists b \in B)\ a \leq b.$$

Zahiri *et al.* applied the hyperstructure theory to residuated lattices and introduced hyper residuated lattices as a generalization of residuated lattices and hyper MV-algebras as a structure $L = \langle L, \vee, \wedge, \odot, \rightarrow, 0, 1 \rangle$, where

- $\langle L, \vee, \wedge, 0, 1 \rangle$ is a bounded superlattice (with the induced order $\leq$)
- $\langle L, \odot, 1 \rangle$ is a commutative semihypergroup with 1 as the identity,
- the pair $(\odot, \rightarrow)$ satisfies the condition $a \odot b \ll c$ if and only if $b \ll a \rightarrow c$

In a hyper residuated lattice an auxiliary hyperoperation '$\oplus$' is defined as $x \oplus y = \neg x \rightarrow y$, where $\neg x = x \rightarrow 0$.

**Proposition 1.1.** *In any hyper residuated lattice $L$, for any $a, b, c \in L$ and $A, B, C \subseteq L$, the following properties hold:*

(1) $1 \ll A$ *implies that* $1 \in A$,

(2) $a \leq b$ *implies that* $1 \in a \rightarrow b$. *Particularly,* $1 \in a \rightarrow 1$, $1 \in 0 \rightarrow a$ *and* $1 \in a \rightarrow a$,

(3) *if 1 is a scalar element,* $a \in 1 \rightarrow a$ *and if* $1 \in a \rightarrow b$, *then* $a \leq b$,

(4) $A \ll B \rightarrow C$ *if and only if* $A \odot B \ll C$ *if and only if* $B \ll A \rightarrow C$,

(5) $A \subseteq B$ *implies that* $A \ll B$,

(6) $0 \in a \odot 0$, $0 \in 0 \oplus 0$,

(7) $a \ll \neg\neg a$,

(8) $a \leq b$ *implies that* $b \rightarrow c \ll a \rightarrow c$,

(9) *If* $A \cap B \neq \emptyset$, *then* $A \ll B$.

(10) *If $B$ is a down-set, then $A \ll B$ implies that $A \cap B \neq \emptyset$.*

We recall that a down-set in a partially ordered set $(P; \leq)$ is a subset $A$ satisfying (HI), where (HI) is the following condition if $x \leq y$ and $y \in A$, then $x \in A$.

A closure operator on a set $A$ is a mapping $C : 2^A \to 2^A$ with the following properties:

(C1) $X \subseteq C(X)$,

(C2) $C(C(X)) = C(X)$,

(C3) $X \subseteq Y$ implies that $C(X) \subseteq C(Y)$.

If $X \subseteq A$ satisfies $C(X) = X$, $X$ is said to be closed with respect to $C$ and the set of these subsets are denoted by $A_C$. Closure operator $C$ is said to be algebraic if $C(X) = \bigcup\{C(Y) : Y \subseteq X \text{ is finite}\}$. For more details on closure operators, we refer the reader to [9].

In the sequel, in this paper, $L$ will denote a hyper residuated lattice and $\mathcal{L}$ a complete lattice.

## 2. Main results

**Definition 2.1.** A down-set $I$ of $L$ is called a

- *strong hyper* RL-*ideal* if $x \oplus y \subseteq I$, for all $x, y \in I$,
- *hyper* RL-*ideal* if $x \oplus y \ll I$, for all $x, y \in I$.

By $\mathcal{SHI}(L)$ ($\mathcal{HI}(L)$) we mean the set of strong hyper RL-ideals (hyper RL-ideals) of $L$.

As a direct consequence of the definition and Proposition 1.1(5) we get that

**Theorem 2.2.** *Any strong hyper RL-ideal is a hyper RL-ideal.*

**Example 2.3.** Because of Proposition 1.1(6), it is obvious that in any hyper residuated lattice $L$, the singleton $\{0\}$ is a hyper RL-ideal but not necessary a strong hyper RL-ideal (see Example 2.8). Moreover, $L$ itself is a (strong) hyper RL-ideal.

**Example 2.4.** Consider the hyper residuated lattice $(L; \vee, \wedge, \odot, \to, 0, 1)$, where $L = \{0, a, b, 1\}$ is a chain with the ordering $0 < a < b < 1$ and the hyperoperations $\vee$, $\wedge$, $\odot$ and $\to$ and the auxiliary hyperoperation $\oplus$ are defined as in Tables 1-3 (see [3]). It is easy to verify that the singleton $\{0\}$ is a strong hyper RL-ideal and so a hyper RL-ideal of $L$. This example shows that proper hyper RL-ideals may not be exist, in general.

| $\vee$ | $0$ | $a$ | $b$ | $1$ |
|---|---|---|---|---|
| $0$ | $L$ | $\{a,b,1\}$ | $\{b,1\}$ | $\{1\}$ |
| $a$ | $\{a,b,1\}$ | $\{a,b,1\}$ | $\{b,1\}$ | $\{1\}$ |
| $b$ | $\{b,1\}$ | $\{b,1\}$ | $\{b,1\}$ | $\{1\}$ |
| $1$ | $\{0,1\}$ | $\{1\}$ | $\{1\}$ | $\{1\}$ |

| $\wedge$ | $0$ | $a$ | $b$ | $1$ |
|---|---|---|---|---|
| $0$ | $\{0\}$ | $\{0\}$ | $\{0\}$ | $\{0\}$ |
| $a$ | $\{0\}$ | $\{0,a\}$ | $\{0,a\}$ | $\{0,a\}$ |
| $b$ | $\{0\}L$ | $\{0,a\}$ | $\{0,a,b\}$ | $\{0,a,b\}$ |
| $1$ | $\{0\}$ | $\{0,a\}$ | $\{0,a,b\}$ | $L$ |

Table 1. Cayley tables of $\vee$ and $\wedge$.

| $\odot$ | $0$ | $a$ | $b$ | $1$ |
|---|---|---|---|---|
| $0$ | $\{0\}$ | $\{0\}$ | $\{0\}$ | $\{0\}$ |
| $a$ | $\{0\}$ | $\{0,a\}$ | $\{a\}$ | $\{a\}$ |
| $b$ | $\{0\}$ | $\{a\}$ | $\{b\}$ | $\{b\}$ |
| $1$ | $\{0\}$ | $\{a\}$ | $\{b\}$ | $\{1\}$ |

| $\rightarrow$ | $0$ | $a$ | $b$ | $1$ |
|---|---|---|---|---|
| $0$ | $\{1\}$ | $\{1\}$ | $\{1\}$ | $\{1\}$ |
| $a$ | $\{0,a\}$ | $\{1\}$ | $\{1\}$ | $\{1\}$ |
| $b$ | $\{0\}$ | $\{0,a\}$ | $\{1\}$ | $\{1\}$ |
| $1$ | $\{0\}$ | $\{a\}$ | $\{b\}$ | $\{1\}$ |

Table 2. Cayley Tables of $\oplus$ and $\rightarrow$.

| $\oplus$ | $0$ | $a$ | $b$ | $1$ |
|---|---|---|---|---|
| $0$ | $\{0\}$ | $\{a\}$ | $\{b\}$ | $\{1\}$ |
| $a$ | $\{0,a,1\}$ | $\{1\}$ | $\{1\}$ | $\{1\}$ |
| $b$ | $\{1\}$ | $\{1\}$ | $\{1\}$ | $\{1\}$ |
| $1$ | $\{1\}$ | $\{1\}$ | $\{1\}$ | $\{1\}$ |

Table 3. Cayley table of $\oplus$ of Example 2.4.

**Example 2.5.** Let $L = \{0, a, b, c, 1\}$ be a lattice whose Hasse diagram is below (Figure 1), and let $x \wedge y$ and $x \vee y$ be the set of all lower bounds and upper bounds (respectively) of $\{x, y\}$. Define the hyperoperations $\odot$ and $\rightarrow$ as in Table 4. Then $L$ is a hyper residuated lattice (see [3]). It is not difficult to verify that $\{0\}$, $\{0, c\}$ and $\{0, a, b\}$ are strong hyper $\mathsf{RL}$-ideals of $L$.

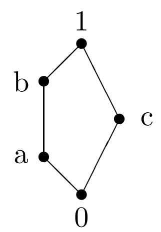


Figure 1. The Hasse diagram of $L$.

The next example shows that the converse of Theorem 2.2 does not hold, in general.

| $\odot$ | $0$ | $a$ | $b$ | $c$ | $1$ |
|---|---|---|---|---|---|
| $0$ | $\{0\}$ | $\{0\}$ | $\{0\}$ | $\{0\}$ | $\{0\}$ |
| $a$ | $\{0\}$ | $\{a\}$ | $\{a\}$ | $\{0\}$ | $\{a\}$ |
| $b$ | $\{0\}$ | $\{a\}$ | $\{a,b\}$ | $\{0\}$ | $\{a,b\}$ |
| $c$ | $\{0\}$ | $\{0\}$ | $\{0\}$ | $\{c\}$ | $\{c\}$ |
| $1$ | $\{0\}$ | $\{a\}$ | $\{a,b\}$ | $\{c\}$ | $\{1\}$ |

| $\to$ | $0$ | $a$ | $b$ | $c$ | $1$ |
|---|---|---|---|---|---|
| $0$ | $\{1\}$ | $\{1\}$ | $\{1\}$ | $\{1\}$ | $\{1\}$ |
| $a$ | $\{c\}$ | $\{1\}$ | $\{1\}$ | $\{c\}$ | $\{1\}$ |
| $b$ | $\{b\}$ | $\{a,b,c\}$ | $\{1\}$ | $\{c\}$ | $\{1\}$ |
| $c$ | $\{a,b\}$ | $\{a,b\}$ | $\{a,b\}$ | $\{1\}$ | $\{1\}$ |
| $1$ | $\{0\}$ | $\{1\}$ | $\{a,b\}$ | $\{c\}$ | $\{1\}$ |

Table 4. Cayley Tables of $\odot$ and $\to$.

**Example 2.6.** Consider the hyper residuated lattice $(L; \vee, \wedge, \odot, \to, 0, 1)$, where $L = \{0, a, b, 1\}$ is a chain with the ordering $0 < a < b < 1$ and the hyperoperations $\vee$, $\wedge$, $\odot$, $\to$ and $\oplus$ are defined as the tables 5–6 (see [3]). Routine calculations show that the subsets $I = \{0, a\}$, $K = \{0, b\}$ and $J = \{0, a, b\}$ are hyper RL-ideals of $L$, which are not strong hyper RL-ideals because $a, 0 \in I \cap J$, while $a \oplus 0 = L$, which is not a subset of $I$ and $J$. And $0 \in K$, while $0 \oplus 0 = \{0, 1\} \nsubseteq K$. Also, the singleton $\{0\}$ is a hyper RL-ideal, which is not a strong hyper RL-ideal. Indeed, $L$ does not contain any proper strong hyper RL-ideals.

| $\vee$ | $0$ | $a$ | $b$ | $1$ |
|---|---|---|---|---|
| $0$ | $L$ | $\{a,b,1\}$ | $\{b,1\}$ | $\{1\}$ |
| $a$ | $\{a,b,1\}$ | $\{a,b,1\}$ | $\{b,1\}$ | $\{1\}$ |
| $b$ | $\{b,1\}$ | $\{b,1\}$ | $\{b,1\}$ | $\{1\}$ |
| $1$ | $\{0,1\}$ | $\{1\}$ | $\{1\}$ | $\{1\}$ |

| $\wedge$ | $0$ | $a$ | $b$ | $1$ |
|---|---|---|---|---|
| $0$ | $\{0\}$ | $\{0\}$ | $\{0\}$ | $\{0\}$ |
| $a$ | $\{0\}$ | $\{0,a\}$ | $\{0,a\}$ | $\{0,a\}$ |
| $b$ | $\{0\}$ | $\{0,a\}$ | $\{0,a,b\}$ | $\{0,a,b\}$ |
| $1$ | $\{0\}$ | $\{0,a\}$ | $\{0,a,b\}$ | $L$ |

Table 5.

| $\to$ | $0$ | $a$ | $b$ | $1$ |
|---|---|---|---|---|
| $0$ | $\{1\}$ | $\{1\}$ | $\{1\}$ | $\{1\}$ |
| $a$ | $\{a,b,1\}$ | $\{a,1\}$ | $\{1\}$ | $\{1\}$ |
| $b$ | $\{a,1\}$ | $\{a\}$ | $\{b,1\}$ | $\{1\}$ |
| $1$ | $\{0,1\}$ | $\{a\}$ | $\{b,1\}$ | $\{1\}$ |

| $\oplus$ | $0$ | $a$ | $b$ | $1$ |
|---|---|---|---|---|
| $0$ | $\{0,1\}$ | $\{a\}$ | $\{b,1\}$ | $\{1\}$ |
| $a$ | $L$ | $\{a,1\}$ | $\{b,1\}$ | $\{1\}$ |
| $b$ | $L$ | $\{a,1\}$ | $\{b,1\}$ | $\{1\}$ |
| $1$ | $\{0,1\}$ | $\{a,1\}$ | $\{b,1\}$ | $\{1\}$ |

Table 6. Cayley table of $\to$ and $\oplus$.

**Proposition 2.7.** *The intersection of any nonempty family of strong hyper* RL-*ideals is again a strong hyper* RL-*ideal.*

***Proof.*** Let $\{I_\alpha : \alpha \in \Lambda\}$ be a nonempty indexed family of strong hyper RL-ideals of hyper residuated lattice $L$ and let $x, y \in L$. If $x \leq y$ and $y \in \bigcap_{\alpha\in\Lambda} I_\alpha$, so $y \in I_\alpha$, for each $\alpha \in \Lambda$, whence $x \in I_\alpha$, for each $\alpha \in \Lambda$. Hence $x \in \bigcap_{\alpha\in\Lambda} I_\alpha$, showing that $\bigcap_{\alpha\in\Lambda} I_\alpha$ is a down-set. Now, assume that $x, y \in \bigcap_{\alpha\in\Lambda} I_\alpha$. Then

$x, y \in I_\alpha$, for each $\alpha \in \Lambda$ and since $I_\alpha$ is a strong hyper $\mathsf{RL}$-ideal of $L$, then $x \oplus y \subseteq I_\alpha$, for each $\alpha \in \Lambda$. This implies that $x \oplus y \subseteq \bigcap_{\alpha\in\Lambda} I_\alpha$. ∎

**Example 2.8.** Consider the hyper residuated lattice $L$ in which $L = \{x_i : i \in \mathbb{N}\} \cup \{0, 1\}$ is a lattice whose Hasse diagram is below (Figure 2) and the hyper-operations $\vee$, $\wedge$, $\odot$ and $\to$ are defined as follows (see [11]):

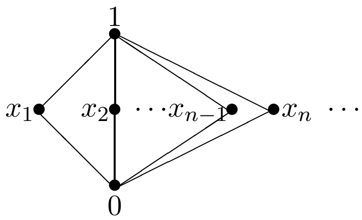


Figure 2. Hasse diagram of $L$.

$$a \vee b = \{c \in L : a \leq c, b \leq c\},\ a \wedge b = \{c \in L : c \leq a, c \leq b\}$$

$$a \odot b = a \wedge b \text{ and } a \to b = \begin{cases} \{1\} & : \ a \leq b \\ \{x_i : i \in \mathbb{N}\} & : \ a = 1, b \in L \setminus \{1\} \\ \{x_j : j \in \mathbb{N}, j \leq i\} \cup \{1\} & : \ a, b \in \{x_i : i \in \mathbb{N}\}, \\ & \phantom{:}\ \ a = x_i, a \neq b \\ \{x_j : j \in \mathbb{N}, j \leq i\} \cup \{1\} & : \ a \in \{x_i : i \in \mathbb{N}\}, b = 0. \end{cases}$$

Routine calculations show that the sets $I_j = \{0, x_j\}$ and $J_k = \{0, x_1, \ldots, x_k\}$ ($j \in \mathbb{N}$ and $j \in \mathbb{N} \setminus \{1\}$) are hyper $\mathsf{RL}$-ideals of $L$, which are not strong hyper $\mathsf{RL}$-ideals. Moreover, for all $r, s \in \mathbb{N}$ with $r \neq s$ we have $I_s \cap I_r = \{0, x_s\} \cap \{0, x_r\} = \{0\}$, which is not a hyper $\mathsf{RL}$-ideal, because $0 \oplus 0 = \{x_i : i \in \mathbb{N}\} \not\ll I_r \cap I_s$. This shows that the intersection of hyper $\mathsf{RL}$-ideals is not a hyper $\mathsf{RL}$-ideal, in general.

**Example 2.9.** Consider the hyper residuated lattice $L$ given in Example 2.5. As we see $I = \{0, c\}$ and $J = \{0, a, b\}$ are (strong) hyper $\mathsf{RL}$-ideals of $L$, while $I \cup J = \{0, a, b, c\}$ is not a hyper $\mathsf{RL}$-ideal (and so not a strong hyper $\mathsf{RL}$-ideal), because $a \oplus c = \{1\} \not\ll I \cup J$.

Despite that the union of two (strong) hyper $\mathsf{RL}$-ideal may not be a (strong) hyper $\mathsf{RL}$-ideal, in general, but it is easily proved that

**Proposition 2.10.** *If $C$ is a chain of (strong) hyper $\mathsf{RL}$-ideals of $L$, $\bigcup C$ is again a (strong) hyper $\mathsf{RL}$-ideal of $L$.*

In virtue of Proposition 2.7, any subset of $L$ generates a strong hyper $\mathsf{RL}$-ideal, which is the intersection of all strong hyper $\mathsf{RL}$-ideals of $L$ containing that subset. More precisely, if $X$ is a subset of $L$, the intersection of all strong hyper

RL-ideals of $L$ containing $X$ is a strong hyper RL-ideal, denoted by $\langle X\rangle_s$. We observe that when $X=\emptyset$, $\langle X\rangle_s = L$. Furthermore, $\langle X\rangle_s$ satisfies the following properties:

- $\langle X\rangle_s = X$ if and only if $X$ is a strong hyper RL-ideal of $L$.
- $X \subseteq Y$ implies that $\langle X\rangle_s \subseteq \langle Y\rangle_s$.
- $\langle\langle X\rangle_s\rangle_s = \langle X\rangle_s$.

Hence, we deduce that the mapping $X \mapsto \langle X\rangle_s$ is a closure operator on $L$, where the closed members of $L$ are the strong hyper RL-ideals of $L$, i.e, the set $\mathcal{SHI}(L)$. Indeed, $\mathcal{SHI}(L)$ is a closed set system for $L$ (see [9, Exercise § 5.5]). Hence,

**Theorem 2.11.** *$\mathcal{SHI}(L)$, as a poset with set-inclusion as the partial ordering, is a complete lattice.*

**Theorem 2.12.** *There is a closure operator $C$ on $\mathcal{SHI}(L)$ such that $\mathcal{SHI}(L)$ is order-isomorphic to the lattice of closed subsets of $\mathcal{SHI}(L)$ with respect to $C$.*

***Proof.*** We define the mapping $C$ with $C(X)=\{J\in\mathcal{SHI}(L) : J\subseteq \bigvee X\}$. It is easy to verify that $C$ is a closure operator on $\mathcal{SHI}(L)$ such that the set of closed subsets of $\mathcal{SHI}(L)$ are those sets $I^{\downarrow}=\{J\in\mathcal{SHI}(L) : J\subseteq I\}$, where $I\in\mathcal{SHI}(L)$. Routine investigations show that the mapping $\Phi : \mathcal{SHI}(L)\longrightarrow \mathcal{SHI}(L)_C$ with $I\mapsto I^{\downarrow}$ is a bijection such that $\Phi$ and $\Phi^{-1}$ are isotone. To prove this, it suffices we observe that $I\subseteq J$ if and only if $I^{\downarrow}\subseteq J^{\downarrow}$. Hence $\Phi$ is the desired order-isomorphism. ■

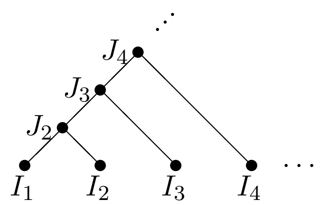


Figure 3. Hasse diagram of $\mathcal{HI}(L)$ in Example 2.8.

One important question is that how we can characterize the elements of generated (strong) hyper RL-ideal by a subset. For strong hyper RL-ideals we can see that for a subset $X$ of $L$

$$\langle X\rangle_s \supseteq \{x\in L : x\ll a_1\oplus a_2\oplus\cdots\oplus a_n;\ \text{for some } n\in\mathbb{N}, a_1,a_2,\ldots,a_n\in X\}. \tag{2.1}$$

As we saw, in Example 2.8, $\mathcal{HI}(L)$ is not closed with respect to the intersection, in general, whence $(\mathcal{HI},\subseteq)$ is not a lattice. Figure 3 shows that the set of all hyper RL-ideals of hyper residuated lattice $L$ given in Example 2.8 is an upper semilattice. Actually, in this example, the supremum of any two elements is the union of them. Obviously, when the intersection is defined, $\mathcal{HI}(L)$ is a complete lattice, as well. Actually, in this case, $(\mathcal{SHI}(L), \subseteq)$ is a complete sublattice of the complete lattice $(\mathcal{HI}(L), \subseteq)$. Furthermore, we can characterize the hyper RL-ideal $\langle X \rangle$ generated by a subset $X$ of $L$. Before we proceed, we give some preliminaries.

**Definition 2.13.** An element $a \in L$ is called a *scalar* with respect to the hyperoperation $\oplus$ (or $\oplus$-scalar) if $|a \oplus b| = |b \oplus a| = 1$, where the symbol $|.|$ denotes the cardinality of the set.

By $SC_\star(L)$, we mean the set of all scalar elements of $L$ with respect the hyperoperation $\star \in \{\odot, \oplus\}$.

Example 2.4 shows that $SC_\odot(L)$ and $SC_\oplus(L)$ may not be equal, in general.

**Example 2.14.** Consider the hyper residuated lattice $L$ given in Example 2.4. Routine calculations show that $SC_\oplus(L) = \{b, 1\}$ (see Table 3).

**Lemma 2.15.** *In any hyper residuated lattice, the following properties hold:*

(1) $x \leq y$ *implies that* $\neg y \ll \neg x$

(2) $x \leq y$ *implies that* $x \oplus z \ll y \oplus z$,

(3) *if* $\oplus$ *is associative, then* $SC_\oplus(L)$ *is closed with respect to* $\oplus$.

***Proof.*** (1) Follows from Proposition 1.1(8).

(2) Let $x \leq y$, for $x, y \in L$. Then $\neg y \ll \neg x$, whence $a \leq b$, for some $a \in \neg y$ and some $b \in \neg x$. Hence $b \to z \ll a \to z \subseteq \neg y \to z = y \oplus z$ and since $b \to z \subseteq \neg x \to z = x \oplus z$, then $x \oplus z \ll y \oplus z$.

(3) Assume that the hyperoperation $\oplus$ is associative and $a, b \in SC(L)$. Then, for any $x \in L$ we have $|b \oplus x| = 1$ and so $|(a \oplus b) \oplus x| = |a \oplus (b \oplus x)| = 1$. Similarly, it is proved that $|x \oplus (a \oplus b)| = 1$. Hence $a \oplus b$ is a scalar element. ■

**Theorem 2.16.** *Assume that the hyperoperation $\oplus$ is associative, $X \subseteq SC_\oplus(L)$ and $\mathcal{HI}(L)$ is closed with respect to the intersection. Then the hyper* RL*-ideal of $L$ generated by $X$ is characterized as*

$$\langle X \rangle = \{x \in L : x \ll a_1 \oplus a_2 \oplus \cdots \oplus a_n, \textit{ for some } n \in \mathbb{N}, a_1, \ldots, a_n \in X\}.$$

*Particularly, if $X = \{a\}$ is singleton, then $\langle a \rangle = \{x \in L : x \ll na$, for some $n \in \mathbb{N}\}$.*

***Proof.*** Let $A = \{x \in L : (\cdots((x \oplus a_1) \oplus a_2) \oplus \cdots) \oplus a_n = \{0\}$, for some $n \in \mathbb{N}, a_1, \ldots, a_n \in X\}$. Reflexivity of $\leq$ implies that $X \subseteq A$. Also, it is obvious that

$0 \in A$. Now, let $x, y \in A$. Then there exist $n, m \in \mathbb{N}$ and $a_1, \ldots, a_n, b_1 \ldots, b_m \in X$ such that

$$x \ll (\cdots (a_1 \oplus a_2) \oplus \cdots) \oplus a_n \tag{2.2}$$

and

$$y \ll (\cdots (b_1 \oplus b_2) \oplus \cdots) \oplus b_m. \tag{2.3}$$

Let $a_1 \oplus a_2 \oplus \cdots a_n = \{u\}$ and $b_1 \oplus b_2 \oplus \cdots \oplus b_m = \{v\}$. (2.2) and (2.3) state that $x \leq u$ and $y \leq v$ and so $x \oplus y \ll u \oplus y \ll u \oplus v$, whence $x \oplus y \ll u \oplus v \subseteq ((\cdots (a_1 \oplus a_2) \oplus \cdots) \oplus a_n) \oplus ((b_1 \oplus b_2) \oplus \cdots) \oplus b_m = (\cdots ((((a_1 \oplus a_2) \oplus \cdots) \oplus a_n) \oplus b_1) \oplus \cdots) \oplus b_m$, whence $x \oplus y \ll A$ showing that $A$ is a hyper $\mathsf{RL}$-ideal of $L$. Now, let $B$ be a hyper $\mathsf{RL}$-ideal of $L$ containing $X$ and $x \in A$. Then $x \ll a_1 \oplus a_2 \oplus \cdots \oplus a_n$ for some $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in X \subseteq B$. Hence $x \ll \{u\} =_{def} a_1 \oplus a_2 \oplus \cdots \oplus a_n \ll B$, whence $x \in B$. Therefore, $A = \langle X \rangle$. ∎

**Theorem 2.17.** *Under the conditions given in Theorem 2.16, the mapping $X \mapsto \langle X \rangle$ is an algebraic closure operator on $L$.*

***Proof.*** Let $X \subseteq L$ and $x \in \langle X \rangle$. Then $x \ll a_1 \oplus \cdots \oplus a_n$, for some $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in X$. If we set $Y = \{a_1, \ldots, a_n\}$, so $Y$ is a finite subset of $X$ and $x \in \langle Y \rangle$. Hence $\langle X \rangle \subseteq \bigcup\{\langle Y \rangle : Y \subseteq X \text{ is finite}\}$, completes the proof. ∎

By Theorem 2.17 and [9, I. Theorem 5.5], we conclude that

**Corollary 2.18.** *Under the conditions given in Theorem 2.16, $(\mathcal{HI}(L), \subseteq)$ as an algebraic lattice whose compact elements are precisely those $\langle X \rangle$, where $X$ is a finite subset of $L$.*

**Theorem 2.19.** *Under the conditions of Theorem 2.16, there exists a set $A$ and an algebraic closure operator $C$ on $A$ such that $\mathcal{HI}(L)$ is order-isomorphic with $\mathcal{HI}(L)_C$.*

***Proof.*** Let $A = \{\langle X \rangle \in \mathcal{HI}(L) : X \text{ is a finite subset of } A\}$. For $X \subseteq A$ we let $C(X) = \{J \in A : J \subseteq \bigvee X\}$. It is easy to verify that $C$ is a closure operator and that the mapping $J \mapsto J^{\downarrow}$ is the desired order-isomorphism. ∎

**Remark 2.20.** We will try to illustrate Theorem 2.16. Consider the hyper residuated lattice $(L = [0, 1], \vee, \wedge, \odot, \rightarrow)$ in which $[0, 1]$ is the real unit interval and the hyperoperations are defined as follows (see [3]):

$$a \odot b = a \wedge b = \min\{a, b\} \quad a \vee b = b \vee a = \begin{cases} L & a = b \\ L \setminus \{a\} & a < b \\ L \setminus \{b\} & b < a \end{cases}$$

$$a \to b = \begin{cases} 1 & a \leq b \\ b & a > b. \end{cases}$$

It is easy to verify that the hyperoperation $\oplus$ is associative and also $SC_{\oplus}(L) = L$. Now, for $X = \{a\}$ (with $a \neq 0$) we have

$$\begin{aligned} \langle X \rangle &= \{x \in L : x \ll na, n \in \mathbb{N}\} \\ &= \{x \in L : x \ll a\} \cup \{x \in L : x \ll a \oplus a\} \cup \cdots \\ &= [0, a] \cup [0, 1] \cup \cdots = [0, 1]. \end{aligned}$$

Obviously, for $a = 0$ we have $\langle 0 \rangle = \{0\}$. This result is expected, because the only hyper $\mathsf{RL}$-ideals of $L$ are $\{0\}$ and $L$. In fact down-sets in $L$ are sub-intervals $[0, a]$, for $a \in [0, 1]$. Routine calculations show that the only sub-interval of $[0, 1]$ which is closed with respect to the hyperoperation $\oplus$ is $[0, 1]$.

## 3. Nodal hyper RL-ideals

We recall that a node in a poset is an element which is comparable with every elements. So, in a bounded poset with 0 and 1 as the least element and the greatest element, 0 and 1 are always nods of $P$.

**Definition 3.1.** In a lattice $L$, an element $a$ is said to be distributive if for all $x, y \in L$ we have $a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y)$.

**Proposition 3.2** [1]**.** *In a lattice, every node is distributive.*

**Definition 3.3.** A *nodal hyper* $\mathsf{RL}$*-ideal* (*nodal strong hyper* $\mathsf{RL}$*-ideal*) is a nod of $\mathcal{HI}(L)$ (resp. $\mathcal{SHI}(L)$).

By $\mathcal{NHI}(L)$ ($\mathcal{NSHI}(L)$), we mean the set of all nodal hyper $\mathsf{RL}$-ideals (nodal strong hyper $\mathsf{RL}$-ideals) of $L$.

**Example 3.4.** From Example 2.3, we know that the singleton $\{0\}$ is a nodal hyper $\mathsf{RL}$-ideal in every hyper residuated lattice. Also, the hyper $\mathsf{RL}$-ideal $J$ in Example 2.6 is a nodal hyper $\mathsf{RL}$-ideal. In Example 2.5, the singleton $\{0\}$ is a nodal strong hyper $\mathsf{RL}$-ideal, while the strong hyper $\mathsf{RL}$-ideal $\{0, c\}$ is not nodal. Moreover, the hyper $\mathsf{RL}$-ideals $I$ and $K$ in the hyper residuated lattice $L$ given in Example 2.6 are not nodal.

**Proposition 3.5.** *If strong hyper* $\mathsf{RL}$*-ideal* $I$ *of* $L$ *is such that for every* $x \in I$ *and* $y \in L \setminus I$*,* $x \ll ny$*, for some* $n \in \mathbb{N}$*, then* $I$ *is a nodal strong hyper* $\mathsf{RL}$*-ideal of* $L$*. Particularly, if* $L$ *is linearly ordered, every strong hyper* $\mathsf{RL}$*-ideal is a nodal strong hyper* $\mathsf{RL}$*-ideal.*

***Proof.*** Assume that $J$ is a strong hyper $\mathsf{RL}$-ideal of $L$ such that $J \not\subseteq I$ and $x \in I$. Then, for $y \in J \setminus I$ we have $x \ll ny$, and since $ny \subseteq J$, we conclude that $x \in J$. This shows that $I \subseteq J$.

Now, if $L$ is linearly ordered and $I$ is a strong hyper $\mathsf{RL}$-ideal of $L$, for every $x \in I$ and $y \in L \setminus I$ we have $x < y$, because $x$ is a node. Hence, $I$ is a nodal strong hyper $\mathsf{RL}$-ideal of $L$. ■

Similarly, for hyper $\mathsf{RL}$-ideals we have the next proposition with a bit modification.

**Proposition 3.6.** *Assume that $I$ is a hyper $\mathsf{RL}$-ideal of $L$ such that $L \setminus I \subseteq SC_{\oplus}(L)$. If for every $x \in I$ and $y \in L \setminus I$ we have $x \ll ny$, for some $n \in \mathbb{N}$, then $I$ is a nodal hyper $\mathsf{RL}$-ideal. Particularly, if $L$ is linearly ordered, every hyper $\mathsf{RL}$-ideal is a nodal hyper $\mathsf{RL}$-ideal.*

**Proposition 3.7.** *Under the conditions given in Theorem 2.16, if $I$ is a nodal hyper $\mathsf{RL}$-ideal $L$, then for any $x \in I$ and $y \in L \setminus I$ we have $x \ll ny$, for some $n \in \mathbb{N}$.*

***Proof.*** Let $I$ be a nodal hyper $\mathsf{RL}$-ideal of $L$, and $x \in I$ and $y \in L \setminus I$. Then $\langle x \rangle \subseteq I \subseteq \langle y \rangle$, whence $x \in \langle y \rangle$. Hence, for some $n \in \mathbb{N}$ we have $x \ll ny$. ■

**Proposition 3.8.** *If $X$ is a set of nodal elements of $L$, then $\langle X \rangle_s$ is a nodal strong hyper $\mathsf{RL}$-ideal of $L$. Particularly, if $x$ is a node, $\langle x \rangle_s$ is a nodal strong hyper $\mathsf{RL}$-ideal of $L$. A similar result holds for hyper $\mathsf{RL}$-ideals.*

***Proof.*** Let $I$ be a strong hyper $\mathsf{RL}$-ideal of $L$. If $X \subseteq I$, clearly $\langle X \rangle_s \subseteq I$. Otherwise, there exists $x \in X$ such that $x \notin I$, whence $i < x$, for all $i \in I$. By (2.1) we get that $i \in \langle x \rangle_s \subseteq \langle X \rangle_s$, whence $I \subseteq \langle X \rangle_s$.

The proof for hyper $\mathsf{RL}$-ideals follows from (2.1). ■

**Remark 3.9.** We observe that $\mathcal{NHI}(L)$ (resp. $\mathcal{NSHI}(L)$ if is nonempty) together with set-inclusion as the partial ordering is a bounded chain which is closed with respect to the intersection and the union. Hence, by Proposition 3.2 and the observations just before in Definition 2.13 we get the next theorem.

**Theorem 3.10.** *$(\mathcal{NHI}(L), \subseteq)$ is a complete Brouwerian sublattice of $(\mathcal{HI}(L), \subseteq)$, which is itself a chain. Similarly, if $L$ contains strong hyper $\mathsf{RL}$-ideal, $(\mathcal{NSHI}(L), \subseteq)$ is a complete Brouwerian sublattice of $(\mathcal{SHI}(L), \subseteq)$, which is itself a chain.*

It is well-known that Heyting lattices are exactly complete Brouwerian lattices (see [2]). Hence, we have

**Corollary 3.11.** *The structure* $(\mathcal{NHI}(L), \cap, \cup, \rightharpoonup, \{0\}, L)$ *is a Heyting algebra, where the operation* $\rightharpoonup$ *is defined as* $I \rightharpoonup J = \cup\{K \in \mathcal{NHI}(L) : I \cap J \subseteq K\}$. *More closely, the set of nodal strong hyper* $\mathsf{RL}$*-ideals of* $L$*, if is nonempty, together with the same operations forms a Heyting subalgebra of* $(\mathcal{NHI}(L), \cap, \cup, \rightharpoonup, \{0\}, L)$.

**Theorem 3.12.** *In a hyper residuated lattice, any proper hyper* $\mathsf{RL}$*-ideal is contained in a maximal hyper* $\mathsf{RL}$*-ideal.*

***Proof.*** Let $I$ be a proper hyper $\mathsf{RL}$-ideal of $L$ and $S$ be the collection of all hyper $\mathsf{RL}$-ideals of $L$ containig $I$. Obviously, $S \neq \emptyset$. Now, let $C$ be a chain in $S$. Clearly, $\bigcup C$ is a proper hyper $\mathsf{RL}$-ideal containing $I$ and so it belongs to $S$. So, by Zorn's lemma, $S$ have a maximal element such as $M$. Now, we shall show that $M$ is a maximal hyper $\mathsf{RL}$-ideal of $L$. Assume that $J$ is a proper hyper $\mathsf{RL}$-ideal of $L$ such that $M \subseteq J$. Then $J \in S$ and so we have $J \subseteq M$, by maximality. Thus $M$ is maximal hyper $\mathsf{RL}$-ideal of $L$. ■

**Corollary 3.13.** *Every hyper residuated lattice contain a maximal hyper* $\mathsf{RL}$*-ideal.*

## 4. Conclusions

Hyper residuated lattices are a generalization of residuated lattices to hyperstructures. Their study is interesting for algebraic and logical reasons, especially from ideal theory point of view. In this paper, we introduced some types of hyper ideals such as hyper $\mathsf{RL}$-ideals, strong hyper $\mathsf{RL}$-ideals and nodal (strong) hyper $\mathsf{RL}$-ideals and investigated their properties and characterizations of them. Particularly, we studied the lattice structure of them and proved that the lattice of (strong) hyper $\mathsf{RL}$-ideals is complete and the lattice of nodal (strong) hyper $\mathsf{RL}$-ideals is a Heyting lattice.

There are still many topics to strudy such as some other types of hyper $\mathsf{RL}$-ideals and their structure, category of these hyper $\mathsf{RL}$-ideals, applying the fuzzy set theory to hyper residuated lattices and introducing types of fuzzy hyper $\mathsf{RL}$-ideals.

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