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ON PARTIAL CLONES OF k-TERMS

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Abstract

The main purpose of this paper is to generalize the concept of linear terms. A linear term is a term in which every variable occurs at most once. K. Denecke defined partial operations on linear terms and partial clones. Moreover, their properties are also studied. In the present paper, a generalized notion of the partial clone of linear terms, which is called k-terms clone, is presented and we also study its properties. We provide a characterization of the k-terms clone being free with respect to itself. Moreover, we attempt to define mappings analogue to the concept of hypersubstitutions.

Keywords: linear term, generalized linear term, superposition of generalized linear term, Menger algebra, hypersubstitution, partial algebra.

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1. Introduction and preliminaries

To describe or to study algebraic properties of algebras, we need an appropriate language as a tool for our desire. In universal algebra, the concept of identities is used to classify algebras of the same type into classes which are called varieties. Identities are made up of terms of the same type. Therefore, in the study of identities we need to know the concept of terms.

Let $n \in \mathbb{N}$, where $\mathbb{N}$ is the set of all positive integers. The n-element set of variables is denoted by $X_n := \{x_1, \ldots, x_n\}$. We denote the set X to be the set $\bigcup_{n\in\mathbb{N}} X_n$. Thus, the set X is a countable set of variables. Let $\{f_i : i \in I\}$ be an indexed set of operation symbols of type $\tau = (n_i)_{i\in I}$, where f_i is of arity $n_i \in \mathbb{N}$.

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In our setting, the set of variables and the set of operation symbols have to be disjoint. For the rest of the paper, we fix the type $\tau = (n_i)_{i \in I}$, unless otherwise stated.

An *n-ary term* of type τ is defined inductively as follows.

1. Any variable in X_n is an n-ary term of type τ.
2. If $t_1, \dots, t_{n_i}$ are n-ary terms of type τ, then $f_i(t_1, \dots, t_{n_i})$ is an n-ary term of type τ.

By $\mathrm{T}_\tau(X_n)$ we denote the set of all n-ary terms of type τ and let

$$\mathrm{T}_\tau(X) := \bigcup_{n \geq 1} \mathrm{T}_\tau(X_n)$$

be the set of all terms of type τ.

To make up new terms, one of the fundamental ways is to compose terms by using superposition operations. Let $m, n \in \mathbb{N}$. *Superposition* operations are mapping

$$S_m^n \colon \mathrm{T}_\tau(X_n) \times (\mathrm{T}_\tau(X_m))^n \to \mathrm{T}_\tau(X_m)$$

defined by the following inductive steps.

1. If $t = x_i \in X_n$, then $S_m^n(x_i, t_1, \dots, t_n) := t_i$.
2. If $t = f_i(s_1, \dots, s_{n_i})$, then, under the assumption that $S_m^n(s_j, t_1, \dots, t_n)$ is already defined for all $1 \leq j \leq n_i$,

$$S_m^n(f_i(s_1, \dots, s_{n_i}), t_1, \dots, t_n) := f_i(S_m^n(s_1, t_1, \dots, t_n), \dots, S_m^n(s_{n_i}, t_1, \dots, t_n)).$$

Then we get a many-sorted algebra

$$\mathbf{clone}(\tau) := \langle (\mathrm{T}_\tau(X_n))_{n \in \mathbb{N}}; (S_m^n)_{m,n \in \mathbb{N}}, (x_i)_{i \leq n, n \in \mathbb{N}} \rangle.$$

Moreover, $\mathbf{clone}(\tau)$ satisfies the following identities.

(C1) $\tilde{S}_m^p(\tilde{Z}, \tilde{S}_m^n(\tilde{Y}_1, \tilde{X}_1, \dots, \tilde{X}_n), \dots, \tilde{S}_m^n(\tilde{Y}_p, \tilde{X}_1, \dots, \tilde{X}_n))$
$\approx \tilde{S}_m^n(\tilde{S}_n^p(\tilde{Z}, \tilde{Y}_1, \dots, \tilde{Y}_p), \tilde{X}_1, \dots, \tilde{X}_n)$, $m, n, p \in \mathbb{N}$.

(C2) $\tilde{S}_m^n(\lambda_i, \tilde{X}_1, \dots, \tilde{X}_n) \approx \tilde{X}_i$ for all $1 \leq i \leq n$, $m, n \in \mathbb{N}$.

(C3) $\tilde{S}_n^n(\tilde{Y}, \lambda_1, \dots, \lambda_n) \approx \tilde{Y}$, $n \in \mathbb{N}$.

Here $\tilde{Z}, \tilde{Y}_i, \tilde{X}_j$ are variables for terms for each $1 \leq i \leq p$, $1 \leq j \leq n$, $\tilde{S}_m^n$ is an operation symbol, and λ_i is a variable for all $1 \leq i \leq n$ (see [9]).

In General Algebra, many scientists study a subject which is called clone theory, or function algebras. This subject plays a vital part of other branches of science, for example, electrical engineering and computer engineering. Especially in computer science, in particular, in switching theory and theory of automata (see [10]).

For any $n \in \mathbb{N}$, let A be a nonempty set. The usual notation for a tuple in A^n is $(a_1, \ldots, a_n)$ where $a_i \in A$ for all $1 \leq i \leq n$. We will use bold letters to denote tuples and corresponding italic letters with subscript to denote their components, that is, $\mathbf{a} := (a_1, \ldots, a_n)$. For any $n \in \mathbb{N}$. Let $\mathcal{O}_A^{(n)}$ and $\mathcal{O}_A$ be the set of all n-ary operations on a nonempty set A and the set of all operations on the set A, respectively. For any $m, n \in \mathbb{N}$, a mapping

$$S_m^{n,A} \colon \mathcal{O}_A^{(n)} \times (\mathcal{O}_A^{(m)})^n \to \mathcal{O}_A^{(m)}$$

defined by $(f, g_1, \ldots, g_n) \mapsto h$, where $h(\mathbf{a}) := f(g_1(\mathbf{a}), \ldots, g_n(\mathbf{a}))$ for all $\mathbf{a} \in A^m$, is called a superposition operation. A nonempty subset X of $\mathcal{O}_A$ is said to be a clone on A if X contains all projections on A and is closed under superposition operations, where the i-th n-ary projection $\mathrm{proj}_i^{n,A}$ on A is a mapping from A^n to A defined by $(a_1, \ldots, a_n) \mapsto a_i$ for all $1 \leq i \leq n$.

We can consider a clone on A as a many-sorted algebra

$$\mathbf{clone}A := \left\langle \left(\mathcal{O}_A^{(n)}\right)_{n\in\mathbb{N}}; \left(S_m^{n,A}\right)_{n,m\in\mathbb{N}}, \left(\mathrm{proj}_i^{n,A}\right)_{i\leq n, n\in\mathbb{N}} \right\rangle.$$

The many-sorted algebra $\mathbf{clone}A$ satisfies the identities (C1)–(C3).

Any many-sorted algebra satisfying (C1)–(C3) is called an *abstract clone*. On the other hand, any clone of operations on a nonempty set A is called a *concrete clone*. One can easily see that every concrete clone is an abstract clone. The following theorem shows how an abstract clone relates to a concrete clone.

Theorem 1 ([16]). *Any abstract clone isomorphics to a concrete one.*

Let $n \in \mathbb{N}$ and $\mathbf{A}$ be an algebra of type τ. Any given $t \in \mathrm{T}_\tau(X_n)$ induces an n-ary operation $t^{\mathbf{A}}$ on A defined by the following steps.

1. If $t = x_i$, then $t^{\mathbf{A}} := \mathrm{proj}_i^{n,A}$.
2. If $t = f_i(s_1, \ldots, s_{n_i})$, then, under the assumption that $s_j^{\mathbf{A}}$ is an n-ary operation on A for all $1 \leq j \leq n_i$, we define $t^{\mathbf{A}} := f_i^{\mathbf{A}}\left(s_1^{\mathbf{A}}, \ldots, s_{n_i}^{\mathbf{A}}\right)$.

The n-ary operation $t^{\mathbf{A}}$ on A is called an *n-ary term operation on* $\mathbf{A}$ induced by t. We denote the set of all n-ary operation on $\mathbf{A}$ by $\mathrm{T}_\tau^{\mathbf{A}}(X_n)$. It turns out that for any algebra $\mathbf{A}$ of type τ we obtain the many-sorted algebra

$$\mathbf{clone}(\mathbf{A}) := \left\langle \left(\mathrm{T}_\tau^{\mathbf{A}}(X_n)\right)_{n\in\mathbb{N}}; \left(S_m^{n,A}\right)_{n,m\in\mathbb{N}}, \left(x_i^{\mathbf{A}}\right)_{i\leq n, n\in\mathbb{N}} \right\rangle.$$

Clearly, $\mathbf{clone}(\mathbf{A})$ is a concrete clone. By the definition of operations induced by terms, we can see that a mapping defined by $t \mapsto t^{\mathbf{A}}$ is a homomorphism from $\mathbf{clone}(\tau)$ onto $\mathbf{clone}(\mathbf{A})$. Therefore, the following result holds.

Corollary 2. *Let* $\mathbf{A}$ *be an algebra of type* τ*. Then* $\mathbf{clone}(\tau)$ *is a homomorphic image of* $\mathbf{clone}(\mathbf{A})$.

Any algebra of type $(n+1, \underbrace{0, \ldots, 0}_{n\text{-times}})$ is called a *unitary Menger algebra of rank* n if it satisfies the following identities.

(C'1) $\circ(z, \circ(y_1, x_1, \ldots, x_n), \ldots, \circ(y_n, x_1, \ldots, x_n))$
$\approx \circ(\circ(z, y_1, \ldots, y_n), x_1, \ldots, x_n)$.

(C'2) $\circ(e_i, x_1, \ldots, x_n) \approx x_i$ for all $1 \leq i \leq n$.

(C'3) $\circ(x, e_1, \ldots, e_n) \approx x$.

Here $\circ$ is an $(n+1)$-ary operation symbol, $e_1, \ldots, e_n$ are nullary operation symbols and $z, x_1, \ldots, x_n, y_1, \ldots, y_n$ are variables. The readers can be found more information and results about unitary Menger algebras of rank n in [15].

For any $n \in \mathbb{N}$ and a nonempty set A, let us now mention

$$n\text{-}\mathbf{clone}(\tau) := \langle \mathrm{T}_\tau(X_n); S_n^n, x_1, \ldots, x_n \rangle$$

and

$$n\text{-}\mathbf{clone}A := \langle \mathcal{O}_A^{(n)}; S_n^{n,A}, \mathrm{proj}_1^{n,A}, \ldots, \mathrm{proj}_n^{n,A} \rangle$$

the unitary Menger algebras of rank n. It is not difficult to see that n-$\mathbf{clone}(\tau)$ and n-$\mathbf{clone}A$ is a particular case of $\mathbf{clone}(\tau)$ and $\mathbf{clone}A$, respectively. In 1963, Dicker showed the following result.

Theorem 3 ([7])**.** *Let* $n \in \mathbb{N}$ *and* A *be a nonempty set. Then* n-$\mathbf{clone}(\tau)$ *is isomorphic to a subalgebra of* n-$\mathbf{clone}A$.

It is clear that if $n = 1$, then n-$\mathbf{clone}A$ is the monoid of all 1-ary functions defined on A. Thus, the above result generalizes the following result: every semigroup is isomorphic to a transformation semigroup.

Let us recall a particular class of terms which is called linear terms. For any term t of type τ, the set $\mathrm{var}(t)$ is the set of all variables occurring in t. A linear term is a term in which any variable in the term occurs only once. Formally, an *n-ary linear term* of type τ is defined inductively as follows.

1. Any variable in X_n is an n-ary linear term of type τ.
2. If $t_1, \ldots, t_{n_i}$ are n-ary linear terms of type τ with $\mathrm{var}(t_j) \cap \mathrm{var}(t_l) = \emptyset$ for all $1 \leq j < l \leq n_i$, then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary linear term of type τ.

We denote the set of all n-ary linear terms of type τ by $\mathrm{T}_\tau^{\mathrm{lin}}(X_n)$ and let

$$\mathrm{T}_\tau^{\mathrm{lin}}(X) := \bigcup_{n\geq 1} \mathrm{T}_\tau^{\mathrm{lin}}(X_n)$$

be the set of all linear terms of type τ (see [2]).

In vector spaces, any vector can be represented as a linear combination between scalars and vectors. Recently, many authors studied various directions concerning linear terms. In 2015, N. Lekkoksung and P. Jampachon studied a class of mappings called non-deterministic linear hypersubstitutions. A non-deterministic linear hypersubstitution is a mapping assigning to each operation symbol a subset of linear terms preserving arity. They showed that the set of all such non-deterministic linear hypersubstitutions together with a particular binary operation forms a monoid (see [12]). In 2016, K. Denecke studied the properties of linear clone, a subalgebra of **clone**(τ). The author obtained that the linear clone satisfies (C1)–(C3) as weak identities (see [3]). In 2017, L. Lohapan and P. Jampachon determined the semigroups properties of linear terms, where the binary operation of this semigroup is induced by a *generalized superposition* (see [11]). They characterized special elements in this semigroup, for example, idempotent elements and regular elements. Moreover, they determined its Green's relations (see [13]). In 2019, D. Phusanga and J. Koppitz defined a binary operation on the set of all linear terms adjoined a special symbol. It turned out that they obtained a new semigroup. Some special elements of this semigroup are characterized. Furthermore, its Green's relations are studied (see [14]). The above-mentioned list of authors shows the interest of a particular kind of terms in some directions.

We now focus on the study of linear clone (see [3]). It is not difficult to see that the set $\mathrm{T}_\tau^{\mathrm{lin}}(X_n)$ is not closed under the superpositions. For example, let $\tau = (2)$ with a binary operation symbol f, and $t = f(x_2, x_1), t_1, t_2 = f(x_1, x_2)$ be 2-ary linear terms of type (2). Then $S_2^2(t, t_1, t_2) = f(f(x_1, x_2), f(x_1, x_2))$ is a term but not linear. K. Denecke introduced a partial superposition operation

$$\overline{S}^{n,m}\colon \mathrm{T}_\tau^{\mathrm{lin}}(X_n) \times (\mathrm{T}_\tau^{\mathrm{lin}}(X_m))^n \multimap\!\!\rightarrow \mathrm{T}_\tau^{\mathrm{lin}}(X_m)$$

as follows (see [3]). For any $m, n \in \mathbb{N}$,

$$\overline{S}^{n,m}(t, s_1, \ldots, s_n) := \begin{cases} S_m^n(t, s_1, \ldots, s_n) & \text{if } \mathrm{var}(s_j) \cap \mathrm{var}(s_l) = \emptyset \\ & \text{for all } 1 \leq j < l \leq n, \\ \text{not defined} & \text{otherwise.} \end{cases}$$

Then he obtained the many-sorted partial algebra

$$\mathbf{clone}_{\mathrm{lin}}(\tau) := \left\langle \left(\mathrm{T}_\tau^{\mathrm{lin}}(X_n)\right)_{n\in\mathbb{N}}; \left(\overline{S}^{m,n}\right)_{m,n\in\mathbb{N}}, (x_i)_{i\leq n, n\in\mathbb{N}} \right\rangle.$$

We see that the operation $\overline{S}^{n,m}$ is defined under the condition

$$\text{(L)} \qquad \operatorname{var}(s_j) \cap \operatorname{var}(s_l) = \emptyset \text{ for all } 1 \leq j < l \leq n.$$

Now, let us consider the following example.

Example 4. Let $\tau = (2)$ with a binary operation symbol f, $t = f(x_1, x_3) \in \mathrm{T}_\tau^{\mathrm{lin}}(X_3)$, $t_1 = f(x_1, x_2), t_2 = f(x_1, x_3), t_3 = f(x_3, x_4) \in \mathrm{T}_\tau^{\mathrm{lin}}(X_4)$. We can see immediately that $\operatorname{var}(t_1) \cap \operatorname{var}(t_2) \neq \emptyset$ and $\operatorname{var}(t_2) \cap \operatorname{var}(t_3) \neq \emptyset$, but

$$S_4^3(t, t_1, t_2, t_3) = f(f(x_1, x_2), f(x_3, x_4)) \in \mathrm{T}_\tau^{\mathrm{lin}}(X_4).$$

By the above example, it seems to mean that the condition (L) is too strong. Our problem is whether we can reduce the condition (L) to a weaker one.

2. THE k-TERMS AND SUPERPOSITION ON THE SET OF k-TERMS

We define a generalized concept of linear terms and also a superposition operation for this concept. To do that, we need a function that counts the occurrences of a specific variable in the term.

Definition ([6])**.** Let $n \in \mathbb{N}$ and $t \in \mathrm{T}_\tau(X_n)$. For any variable $x_j \in X_n$, we define the x_j-variable count $\mathrm{vb}_j(t)$ of t as follows.

1. $\mathrm{vb}_j(x_j) := 1$ and $\mathrm{vb}_j(x_i) := 0$ for all $i \neq j$.
2. If $t = f_i(t_1, \ldots, t_{n_i})$, then $\mathrm{vb}_j(t) := \sum_{l=1}^{n_i} \mathrm{vb}_j(t_l)$.

Example 5. Let $\tau = (2)$ with a binary operation symbol f. We consider a ternary term $t = f(x_2, f(f(x_2, x_1), x_3))$ of type τ. Then $\mathrm{vb}_1(t) = 1$, $\mathrm{vb}_2(t) = 2$ and $\mathrm{vb}_3(t) = 1$.

Now, we are ready to define our concept. Let $n, k \in \mathbb{N}$. An *n-ary k-term* of type τ is defined in the following steps.

1. Any variable in X_n is an n-ary k-term of type τ.
2. If $t_1, \ldots, t_{n_i}$ are n-ary k-terms of type τ with $\sum_{j=1}^{n_i} \mathrm{vb}_l(t_j) \leq k$ for all $1 \leq l \leq n$, then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary k-term of type τ.

By $\mathrm{T}_\tau^{(k)}(X_n)$ we denote the set of all n-ary k-terms of type τ and let

$$\mathrm{T}_\tau^{(k)}(X) := \bigcup_{n \geq 1} \mathrm{T}_\tau^{(k)}(X_n)$$

be the set of all k-terms of type τ.

Example 6.

1. Let $\tau = (3)$ with a ternary operation symbol f. Then $\mathrm{T}_\tau^{(1)}(X_2) = \{x_1, x_2\}$.
2. Let $\tau = (3)$ with a ternary operation symbol f. Then

$$\begin{array}{cccc} x_1 & x_2 & f(x_1,x_1,x_2) & f(x_1,x_2,x_1) \\ f(x_2,x_1,x_1) & f(x_2,x_2,x_1) & f(x_2,x_1,x_2) & f(x_1,x_2,x_2) \end{array}$$

are all elements of $\mathrm{T}_\tau^{(2)}(X_2)$.

Remark 7.

1. For any $k \in \mathbb{N}$, $\mathrm{T}_\tau^{(k)}(X) \subseteq \mathrm{T}_\tau^{(l)}(X)$ for all $l \geq k$.
2. If $k = 1$, then the condition $\sum_{j=1}^{n_i} \mathrm{vb}_l(t_j) \leq k$ for all $1 \leq l \leq n$ of n-ary k-terms of type τ is equivalent to the condition $\mathrm{var}(t_j) \cap \mathrm{var}(t_l) = \emptyset$ for all $1 \leq j < l \leq n_i$ of n-ary linear terms of type τ.

By the above remark, we see that if $k = 1$, then the concept of k-terms and linear terms are coincided. Indeed,

$$\mathrm{T}_\tau^{(1)}(X) = \mathrm{T}_\tau^{\mathrm{lin}}(X).$$

Now, let us consider the following example. Let $\tau = (2)$ with a binary operation symbol f, $t = f(x_1,x_1) \in \mathrm{T}_\tau^{(3)}(X_2)$ and $t_1 = f(x_1,x_1), t_2 = f(x_2,x_3) \in \mathrm{T}_\tau^{(3)}(X_3)$. Then $A := S_3^2(t,t_1,t_2) = f(f(x_1,x_1),f(x_1,x_1)) \notin \mathrm{T}_\tau^{(3)}(X_3)$ since $\mathrm{vb}_1(A) = 4 > 3$. This example shows that the set $\mathrm{T}_\tau^{(k)}(X_n)$ is not closed under the superposition operations. We noticed already that the set of linear terms is also not closed under such operation.

It is of interest which conditions making $\mathrm{T}_\tau^{(k)}(X_n)$ closed under the superposition operations. To see this, let $t \in \mathrm{T}_\tau^{(k)}(X_n)$, $t_1,\ldots,t_n \in \mathrm{T}_\tau^{(k)}(X_m)$, and the superposition

$$S_m^n(t,t_1,\ldots,t_n) := A.$$

It is not difficult to see that for any $1 \leq j \leq m$,

$$\mathrm{vb}_j(A) = \sum_{l=1}^{n} \mathrm{vb}_l(t)\mathrm{vb}_j(t_l).$$

Or in another expression we use the multiplication of matrices, where in the resulting matrix any row represents the $\mathrm{vb}_j(A)$, as follows:

$$\begin{pmatrix} \mathrm{vb}_1(t_1) & \cdots & \mathrm{vb}_1(t_n) \\ \vdots & \ddots & \vdots \\ \mathrm{vb}_m(t_1) & \cdots & \mathrm{vb}_m(t_n) \end{pmatrix} \begin{pmatrix} \mathrm{vb}_1(t) \\ \vdots \\ \mathrm{vb}_n(t) \end{pmatrix}.$$

To see this, let us suppose that $t = x_i$ for some $1 \leq i \leq n$. Then $\mathrm{vb}_j(A) = \mathrm{vb}_j(t_i) = \sum_{l=1}^{n} \mathrm{vb}_l(t)\mathrm{vb}_j(t_l)$. Now, we suppose that $t = f_i(s_1,\ldots,s_j)$ and we assume more that for any $1 \leq p \leq n_i$, we have $\mathrm{vb}_j(S_m^n(s_p,t_1,\ldots,t_n)) = \sum_{l=1}^{n} \mathrm{vb}_l(s_p)\mathrm{vb}_j(t_l)$. Since

$$A = S_m^n(t,t_1,\ldots,t_n) = f_i(S_m^n(s_1,t_1,\ldots,t_n),\ldots,S_m^n(s_{n_i},t_1,\ldots,t_n)),$$

by the definition of vb_j, we have

$$\mathrm{vb}_j(A) = \sum_{p=1}^{n_i} \mathrm{vb}_j(S^n_m(s_p, t_1, \ldots, t_n)) = \sum_{l=1}^{n}\sum_{p=1}^{n_i} \mathrm{vb}_l(s_p)\mathrm{vb}_j(t_l) = \sum_{l=1}^{n} \mathrm{vb}_l(t)\mathrm{vb}_j(t_l).$$

By the above discussion, we obtain the following result.

Proposition 8. *If* $t = f_i(s_1, \ldots, s_{n_i}) \in \mathrm{T}^{(k)}_\tau(X_n)$, $t_1, \ldots, t_n \in \mathrm{T}^{(k)}_\tau(X_m)$, *and*

$$(\mathrm{K}_j) \qquad \mathrm{vb}_j(S^n_m(t, t_1, \ldots, t_n)) = \sum_{l=1}^{n} \mathrm{vb}_l(t)\mathrm{vb}_j(t_l) \leq k$$

for any $1 \leq j \leq m$, *then* $S^n_m(t, t_1, \ldots, t_n) \in \mathrm{T}^{(k)}_\tau(X_m)$.

Remark 9. We note here that if $k = 1$, then the condition (L) implies the condition (K_j) for any $1 \leq j \leq m$, but not the converse. For instance, for $\tau = (2)$ with a binary operation symbol f and $t = x_1, t_1 = f(x_1, x_2), t_2 = f(x_2, x_1) \in \mathrm{T}^{(1)}_\tau(X_2)$. We see that for $i = 1, 2$, $\mathrm{vb}_1(t)\mathrm{vb}_i(t_1) + \mathrm{vb}_2(t)\mathrm{vb}_i(t_2) \leq 1$, but $\mathrm{var}(t_1) \cap \mathrm{var}(t_2) \neq \emptyset$.

Now, we are ready to define the many-sorted partial operation

$$S^{(k)}_{n,m} \colon \mathrm{T}^{(k)}_\tau(X_n) \times (\mathrm{T}^{(k)}_\tau(X_m))^n \multimap \mathrm{T}^{(k)}_\tau(X_m)$$

by

$$S^{(k)}_{n,m}(t, t_1, \ldots, t_n) := \begin{cases} S^n_m(t, t_1, \ldots, t_n) & \text{if } (\mathrm{K}_j) \text{ holds for all } 1 \leq j \leq m, \\ \text{not defined} & \text{otherwise,} \end{cases}$$

for $m, n, k \in \mathbb{N}$.

As a consequence, we obtain a many-sorted partial algebra

$$\mathbf{clone}^{(k)}(\tau) := \left\langle \left(\mathrm{T}^{(k)}_\tau(X_n)\right)_{n\in\mathbb{N}}; \left(S^{(k)}_{n,m}\right)_{m,n\in\mathbb{N}}, (x_i)_{i\leq n, n\in\mathbb{N}} \right\rangle,$$

the *k-terms clone* of type τ.

Remark 10. For any $k, m, n \in \mathbb{N}$, by Example 4 and Remark 9, we see that $\mathrm{dom}(\overline{S}^{n,m}) \subsetneq \mathrm{dom}(S^{(k)}_{n,m})$.

One of the fundamental ways to define the concept of homomorphism on partial algebras is considered in the following sense. Let $\mathbf{A} := \langle A; (f^A_i)_{i\in I}\rangle$ and $\mathbf{B} := \langle B; (f^B_i)_{i\in I}\rangle$ be partial algebras of the same type τ. A mapping $h \colon A \to B$

is called a *weak homomorphism* if for any $i \in I$, $(a_1, \ldots, a_{n_i}) \in \mathrm{dom}(f_i^A)$ implies $(h(a_1), \ldots, h(a_{n_i})) \in \mathrm{dom}(f_i^B)$ and

$$h\left(f_i^A(a_1, \ldots, a_{n_i})\right) = f_i^B\big(h(a_1), \ldots, h(a_{n_i})\big),$$

where $\mathrm{dom}(f_i^A)$ and $\mathrm{dom}(f_i^B)$ are the domains of f_i^A and f_i^B, respectively (see [1, 3]). In particular, a weak homomorphism from A to itself is called a *weak endomorphism* on $\mathbf{A}$.

Each (partial) algebra which has a generating system and any substitution (a mapping from the generating system into the universe of the (partial) algebra) can be extended to an (weak) endomorphism is called *free* with respect to itself. The following result was proved.

Theorem 11 ([8]). *Let $n \in \mathbb{N}$. The algebra*

$$n\text{-}\mathbf{clone}(\tau_n) := \langle \mathrm{T}_{\tau_n}(X_n); S_n^n, x_1, \ldots, x_n \rangle$$

is free with respect to $\mathcal{V}_{M_n}$ the variety of unitary Menger algebras of rank n, and freely generated by $\{f_i(x_1, \ldots, x_n) : i \in I\}$, where $\tau_n = (n_i)_{i\in I}$ such that $n_i = n$ for all $i \in I$.

We will provide a characterization of $\mathbf{clone}^{(k)}(\tau)$ being free with respect to itself. To go to that point let us provide some ingredients.

Let $m, n \in \mathbb{N}$, where $n > m$. Define the set $K_{n,m}$ to be empty set if $\lceil \frac{n}{m} \rceil > m$ and to be the set

$$\{\alpha\colon \{1, \ldots, n\} \to \{1, \ldots, n\} : \mathrm{range}(\alpha) = \{1, \ldots, m_0\}, \left\lceil \frac{n}{m} \right\rceil \leq m_0 \leq m\}$$

if $\lceil \frac{n}{m} \rceil \leq m$. Moreover, we define

$$K_{n,m}^* := \{\alpha\colon \{1, \ldots, n\} \to \{1, \ldots, n\} : \mathrm{range}(\alpha) = \{1, \ldots, m_0\}, m_0 > m\}.$$

Let $n, k \in \mathbb{N}$. We denote the sets

$$X_n \cup \{f_i(x_1, \ldots, x_{n_i}) \in \mathrm{T}_\tau^{(k)}(X_n) : i \in I_n^\tau\}$$

and

$$\begin{aligned}X_n &\cup \big\{f_i(x_1, \ldots, x_{n_i}) \in \mathrm{T}_\tau^{(k)}(X_n) : i \in I_n^\tau\big\}\\ &\cup \big\{f_i(x_{\alpha_i(1)}, \ldots, x_{\alpha_i(n_i)}) \in \mathrm{T}_\tau^{(k)}(X_n) : \alpha_i \in K_{n_i,n}, i \in I_n^{\tau,*}\big\}\\ &\cup \big\{f_i(t_{\alpha_i(1)}, \ldots, t_{\alpha_i(n_i)}) \in \mathrm{T}_\tau^{(k)}(X_n) : \alpha_i \in K_{n_i,n}^*, i \in I_n^{\tau,*}\\ &\quad \text{such that } |\{t_j : 1 \leq j \leq n_i\}| > n\big\},\end{aligned}$$

by $A^{(k)}_{\tau,n}$ and $B^{(k)}_{\tau,n}$, respectively. Here $I^{\tau}_{n} := \{l \in I : n_l \leq n\}$ and $I^{\tau,*}_{n} := \{l \in I : n_l > n\}$.

Next, for any $n \in \mathbb{N}$ we define the set

$$F^{(k)}_{\tau,n} := \begin{cases} A^{(k)}_{\tau,n} & \text{if } n \geq n_i \text{ for all } i \in I \\ B^{(k)}_{\tau,n} & \text{if } n < n_i \text{ for some } i \in I. \end{cases}$$

Example 12. Let $\tau = (3)$ with a ternary operation symbol f. Let us consider $f(f(x_1, x_2, x_1), x_2, x_2), f(f(x_1, x_2, x_1), x_1, x_2) \in \mathrm{T}^{(3)}_{\tau}(X_2)$. We see that the set $\{f_i(x_{\alpha_i(1)}, \dots, x_{\alpha_i(n_i)}) \in \mathrm{T}^{(k)}_{\tau}(X_n) : \alpha_i \in K_{n_i,n}, i \in I^{\tau,*}_{n}\}$ is exactly the set $\{f(x_{\alpha(1)}, x_{\alpha(2)}, x_{\alpha(3)}) \in \mathrm{T}^{(3)}_{\tau}(X_2) : \alpha \in K_{3,2}\}$, and the set $\{f_i(t_{\alpha_i(1)}, \dots, t_{\alpha_i(n_i)}) \in \mathrm{T}^{(k)}_{\tau}(X_n) : \alpha_i \in K^{*}_{n_i,n}, i \in I^{\tau,*}_{n}$ such that $|\{t_j : 1 \leq j \leq n_i\}| > n\}$ is exactly the set $\{f(t_{\alpha(1)}, t_{\alpha(2)}, t_{\alpha(3)}) \in \mathrm{T}^{(3)}_{\tau}(X_2) : \alpha \in K^{*}_{3,2}$ such that $|\{t_j : j = 1, 2, 3\}| > 2\}$. Then $f(x_1, x_2, x_2)$ belongs to the set

$$\left\{f\left(x_{\alpha(1)}, x_{\alpha(2)}, x_{\alpha(3)}\right) \in \mathrm{T}^{(3)}_{\tau}(X_2) : \alpha \in K_{3,2}\right\}$$

such that

$$\begin{aligned} S^{(3)}_{2,3}(f(x_1, x_2, x_2), f(x_1, x_2, x_1), x_2) &= S^{2}_{3}(f(x_1, x_2, x_2), f(x_1, x_2, x_1), x_2) \\ &= f(f(x_1, x_2, x_1), x_2, x_2), \end{aligned}$$

and $f(f(x_1, x_2, x_1), x_1, x_2)$ belongs to the set

$$\left\{f\left(t_{\alpha(1)}, t_{\alpha(2)}, t_{\alpha(3)}\right) \in \mathrm{T}^{(3)}_{\tau}(X_2) : \alpha \in K^{*}_{3,2} \text{ such that } |\{t_j : j = 1, 2, 3\}| > 2\right\}$$

such that

$$\begin{aligned} S^{(3)}_{2,3}(f(f(x_1, x_2, x_1), x_1, x_2), x_1, x_2) &= S^{2}_{3}(f(f(x_1, x_2, x_1), x_1, x_2), x_1, x_2) \\ &= f(f(x_1, x_2, x_1), x_1, x_2). \end{aligned}$$

We can see that $F^{(k)}_{\tau,n} \subseteq \mathrm{T}^{(k)}_{\tau}(X_n)$ for any $k, n \in \mathbb{N}$. By this setting, we obtain the following result.

Lemma 13. *The sequence* $(F^{(k)}_{\tau,n})_{n \in \mathbb{N}}$ *is a generating system of* $\mathbf{clone}^{(k)}(\tau)$ *for any* $k \in \mathbb{N}$.

Proof. Let $k, m, n \in \mathbb{N}$. It is clear that any $x_j \in X_n$ is in the type of $\mathbf{clone}^{(k)}(\tau)$, thus, it is generated. Let $t = f_i(t_1, \dots, t_{n_i}) \in \mathrm{T}^{(k)}_{\tau}(X_m)$ and assume that

$t_1, \ldots, t_{n_i} \in \mathrm{T}_\tau^{(k)}(X_m)$ are generated. This implies that $\sum_{j=1}^{n_i} \mathrm{vb}_l(t_j) \leq k$ for all $1 \leq l \leq m$. If $n_i \leq m$ for all $i \in I$, then

$$S_{n_i,m}^{(k)}(f_i(x_1, \ldots, x_{n_i}), t_1, \ldots, t_{n_i}) = S_m^{n_i}(f_i(x_1, \ldots, x_{n_i}), t_1, \ldots, t_{n_i})$$
$$= f_i(t_1, \ldots, t_{n_i}).$$

If $n_i > m$ for some $i \in I$, then we separate our consideration into two cases. For the first case, if the cardinality of $\{t_1, \ldots, t_{n_i}\}$ is $m_0 \leq m$. Then $f_i(t_1, \ldots, t_{n_i}) = f_i(t_{\alpha_i(1)}, \ldots, t_{\alpha_i(n_i)})$ for some $\alpha_i \in K_{n_i,n}$ and $i \in I_m^{\tau,*}$. Suppose that $|\alpha^{-1}(j)| = l_j$ for all $1 \leq j \leq m_0$. We observe that the term $f_i(x_{\alpha_i(1)}, \ldots, x_{\alpha_i(n_i)})$ is an m-ary k-term of type τ and for any $1 \leq j \leq m$,

$$\begin{aligned}
&\mathrm{vb}_j(S_m^{m_0}(f_i(x_{\alpha_i(1)}, \ldots, x_{\alpha_i(n_i)}), t_1, \ldots, t_{m_0}))\\
&= \sum_{p=1}^{m_0} \mathrm{vb}_p(f_i(x_{\alpha_i(1)}, \ldots, x_{\alpha_i(n_i)}))\mathrm{vb}_j(t_p)\\
&= \sum_{p=1}^{m_0} l_p \mathrm{vb}_j(t_p) = \sum_{l=1}^{n_i} \mathrm{vb}_j(t_l) \leq k.
\end{aligned}$$

Then

$$\begin{aligned}
&S_{m_0,m}^{(k)}\left(f_i\left(x_{\alpha_i(1)}, \ldots, x_{\alpha_i(n_i)}\right), t_1, \ldots, t_{m_0}\right)\\
&= S_m^{m_0}\left(f_i\left(x_{\alpha_i(1)}, \ldots, x_{\alpha_i(n_i)}\right), t_1, \ldots, t_{m_0}\right)\\
&= f_i\left(t_{\alpha_i(1)}, \ldots, t_{\alpha_i(n_i)}\right) = f_i\left(t_1, \ldots, t_{n_i}\right).
\end{aligned}$$

For the last case, if the cardinality of $\{t_1, \ldots, t_{n_i}\}$ is greater than m. Then we have $f_i(t_1, \ldots, t_{n_i}) = f_i\left(t_{\alpha_i(1)}, \ldots, t_{\alpha_i(n_i)}\right)$ for some $\alpha_i \in K_{n_i,n}^*$ and $i \in I_m^{\tau,*}$, so it is obvious that

$$S_{m,m}^{(k)}\left(f_i\left(t_{\alpha_i(1)}, \ldots, t_{\alpha_i(n_i)}\right), x_1, \ldots, x_m\right) = f_i(t_1, \ldots, t_{n_i}).$$

Therefore, we have that $f_i(t_1, \ldots, t_{n_i})$ is generated. ■

Proposition 14. *Let $k \in \mathbb{N}$ and $\tau = (n_i)_{i\in I}$ with $n_i = 1$ for all $i \in I$. Then* $\mathbf{clone}^{(k)}(\tau)$ *is free with respect to itself, and freely generated by* $\left(F_{\tau,n}^{(k)}\right)_{n\in\mathbb{N}}$.

Proof. Let $n \in \mathbb{N}$. Theorem 11 says that any mapping $\varphi' \colon F_{\tau,n}^{(k)} \to \mathrm{T}_\tau(X_n)$ can be extended to an endomorphism on $\mathrm{T}_\tau(X_n)$. Let $\varphi \colon F_{\tau,n}^{(k)} \to \mathrm{T}_\tau^{(k)}(X_n)$ be any mapping. By the above argument, φ can be also extended to an endomorphism on $\mathrm{T}_\tau^{(k)}(X_n)$. By Theorem 11 and Lemma 13, we obtain our proposition. ■

Now, we ask the following question. What is going on in the case that the operation symbols are not all unary? The following theorem addresses this question.

Theorem 15. *Let $\tau = (n_i)_{i\in I}$ be such that $n_j > 1$ for some $j \in I$. Then* $\mathbf{clone}^{(k)}(\tau)$ *is free with respect to itself, and freely generated by* $\big(F_{\tau,n}^{(k)}\big)_{n\in\mathbb{N}}$ *if and only if $k = 1$.*

Proof. ($\Leftarrow$) This direction follows the proof given in [3, Lemma 3.2].

($\Rightarrow$) Assume that $k \in \mathbb{N} \setminus \{1\}$. Suppose that $\mathbf{clone}^{(k)}(\tau)$ is free with respect to itself. For any $n > n_j$, we define a mapping $\varphi_n \colon F_{\tau,n}^{(k)} \to \mathrm{T}_\tau^{(k)}(X_n)$ by

$$\varphi_n(t) := \begin{cases} f_j(x_1, \ldots, x_1) & \text{if } k \geq n_j, \\ f_j(\underbrace{x_1, \ldots, x_1}_{k\text{-times}}, x_2, \ldots, x_{n_j-k+1}) & \text{if } k < n_j \end{cases}$$

for all $t \in F_{\tau,n}^{(k)}$.

If $k \geq n_j$, then we put $t = f_j(t', x_1, \ldots, x_1) \in \mathrm{T}_\tau^{(k)}(X_n)$, where $\mathrm{vb}_1(t') = \max\{pn_j - (p-1) : n_j^p \leq k \text{ and } p \in \mathbb{N}\}$. Since $\overline{\varphi}_n$ is a weak endomorphism,

$$\begin{aligned} \overline{\varphi}_n\left(f_j(x_1, \ldots, x_1)\right) &= \overline{\varphi}_n\big(S_{n,n}^{(k)}\big(f_j(x_1, \ldots, x_{n_j}), x_1, \ldots, x_1\big)\big) \\ &= S_{n,n}^{(k)}\big(\varphi_n\big(f_j(x_1, \ldots, x_{n_j})\big), \overline{\varphi}_n(x_1), \ldots, \overline{\varphi}_n(x_1)\big) \\ &= S_{n,n}^{(k)}\big(f_j(x_1, \ldots, x_1), x_1, \ldots, x_1\big) \\ &= S_n^n\big(f_j(x_1, \ldots, x_1), x_1, \ldots, x_1\big) \\ &= f_j(x_1, \ldots, x_1). \end{aligned}$$

This implies that $\mathrm{vb}_1(\overline{\varphi}_n(t')) = n_j^p$. We now observe that

$$\big(f_j\big(x_1, \ldots, x_{n_j}\big), t, x_2, \ldots, x_n\big) \in \mathrm{dom}(S_{n,n}^{(k)}),$$

but

$$\begin{aligned} &\big(\varphi_n\big(f_j(x_1, \ldots, x_{n_j})\big), \overline{\varphi}_n(t), \overline{\varphi}_n(x_2), \ldots, \overline{\varphi}_n(x_n)\big) \\ &= \big(f_j(x_1, \ldots, x_1), \overline{\varphi}_n(t), x_2, \ldots, x_n\big) \notin \mathrm{dom}(S_{n,n}^{(k)}) \end{aligned}$$

since $\overline{\varphi}_n(t)$ is not defined, indeed, $\mathrm{vb}_1(S_n^n(f_j(x_1, \ldots, x_1), \overline{\varphi}_n(t), x_2, \ldots, x_n)) = n_j^{p+1} > k$. This is a contradiction.

Now, if $k < n_j < n$, then we put $t = f_j(x_1, \ldots, x_{n_j}) \in \mathrm{T}_\tau^{(k)}(X_n)$ and $t' = f_j(\underbrace{x_1, \ldots, x_1}_{k\text{-times}}, x_2, \ldots, x_{n_j-k+1}) \in \mathrm{T}_\tau^{(k)}(X_n)$. Since $\overline{\varphi}_n$ is a weak endomorphism, we see that $\overline{\varphi}_n(t) = t'$. This implies that

$$\big(f_j(x_1, \ldots, x_{n_j}), t, x_2, \ldots, x_n\big) \in \mathrm{dom}(S_{n,n}^{(k)}),$$

but

$$\begin{aligned}&\big(\varphi_n\big(f_j(x_1,\ldots,x_{n_j})\big),\overline{\varphi}_n(t),\overline{\varphi}_n(x_2),\ldots,\overline{\varphi}_n(x_n)\big)\\&=\big(f_j(\underbrace{x_1,\ldots,x_1}_{k\text{-times}},x_2,\ldots,x_{n_j-k+1}),t',x_2,\ldots,x_n\big)\notin\mathrm{dom}(S^{(k)}_{n,n})\end{aligned}$$

since $\mathrm{vb}_1(S^n_n(f_j(\underbrace{x_1,\ldots,x_1}_{k\text{-times}},x_2,\ldots,x_{n_j-k+1}),t',x_2,\ldots,x_n))=k^2>k$. This is a contradiction. Altogether, we have that $\mathbf{clone}^{(k)}(\tau)$ is not free with respect to itself. Therefore, we obtain our theorem. ∎

3. Properties of $\mathbf{clone}^{(k)}(\tau)$

In the study of $\mathbf{clone}_{\mathrm{lin}}(\tau)$, K. Denecke showed that this many-sorted partial algebra satisfies (C1)–(C3) (see [3]). In this section we show that our many-sorted partial algebra $\mathbf{clone}^{(k)}(\tau)$ also satisfies these properties as weak identities.

We first recall the concept of weak identities. An equation $s\approx t$ of terms over the many-sorted partial algebra $\mathbf{A}$ is said to be a *weak identity* in $\mathbf{A}$ if after evaluation there holds: if the right hand side is defined, then the left hand side is defined or conversely, and both sides are equal (see [1, 3]).

Theorem 16. *For any $k\in\mathbb{N}$, the many-sorted partial algebra $\mathbf{clone}^{(k)}(\tau)$ satisfies* (C1)–(C3) *as weak identities.*

Proof. Let $k\in\mathbb{N}$.

(C1): We replace the variables by arbitrary $t_1,\ldots,t_p\in\mathrm{T}^{(k)}_\tau(X_n)$, $s_1,\ldots,s_n\in\mathrm{T}^{(k)}_\tau(X_m)$, $t\in\mathrm{T}^{(k)}_\tau(X_p)$, where $m,n,p\in\mathbb{N}$, and the operation symbol by the partial fundamental operation of $\mathbf{clone}^{(k)}(\tau)$. Then we have

$$\begin{aligned}&S^{(k)}_{p,m}\left(t,S^{(k)}_{n,m}(t_1,s_1,\ldots,s_n),\ldots,S^{(k)}_{n,m}(t_p,s_1,\ldots,s_n)\right)\\&\approx S^{(k)}_{n,m}\left(S^{(k)}_{p,n}(t,t_1,\ldots,t_p),s_1,\ldots,s_n\right).\end{aligned}\tag{1}$$

Assume that the right-hand-side of Equation (1) is defined. This means that

$$S^{(k)}_{n,m}\left(S^{(k)}_{p,n}(t,t_1,\ldots,t_p),s_1,\ldots,s_n\right)=S^n_m\big(S^p_n(t,t_1,\ldots,t_p),s_1,\ldots,s_n\big),$$

and for any $1\le i\le m$,

$$\mathrm{vb}_i\big(S^n_m\big(S^p_n(t,t_1,\ldots,t_p),s_1,\ldots,s_n\big)\big)\le k,$$

or,

$$\sum_{j=1}^{n} \mathrm{vb}_j\big(S_n^p(t, t_1, \ldots, t_p)\big)\mathrm{vb}_i(s_j) \leq k.$$

This yields

$$\sum_{j=1}^{n}\left(\sum_{l=1}^{p} \mathrm{vb}_l(t)\mathrm{vb}_j(t_l)\right)\mathrm{vb}_i(s_j) \leq k.$$

Since for any $1 \leq i \leq m$

$$\sum_{j=1}^{n}\left(\sum_{l=1}^{p} \mathrm{vb}_l(t)\mathrm{vb}_j(t_l)\right)\mathrm{vb}_i(s_j) = \sum_{j=1}^{p} \mathrm{vb}_j(t)\left(\sum_{l=1}^{n} \mathrm{vb}_l(t_j)\mathrm{vb}_i(s_l)\right),$$

any variable x_i occurring in

$$S_m^p\big(t, S_m^n(t_1, s_1, \ldots, s_n), \ldots, S_m^n(t_p, s_1, \ldots, s_n)\big)$$

is not exceed k. Therefore, the left-hand-side of Equation (1) is defined, indeed,

$$\begin{aligned}&S_{p,m}^{(k)}\Big(t, S_{n,m}^{(k)}(t_1, s_1, \ldots, s_n), \ldots, S_{n,m}^{(k)}(t_p, s_1, \ldots, s_n)\Big)\\ &= S_m^p\big(t, S_m^n(t_1, s_1, \ldots, s_n), \ldots, S_m^n(t_p, s_1, \ldots, s_n)\big).\end{aligned}$$

By (C1), we obtain our (1).

(C2): We replace the variable λ_i by $x_i \in X_n$, $\tilde{S}_m^n$ by $S_{n,m}^{(k)}$ and $\tilde{X}_1, \ldots, \tilde{X}_n$ by $t_1, \ldots, t_n \in \mathrm{T}_\tau^{(k)}(X_m)$. Then we have

$$S_{n,m}^{(k)}(x_i, t_1, \ldots, t_n) \approx t_i.$$

Since $t_1, \ldots, t_n \in \mathrm{T}_\tau^{(k)}(X_m)$, for any $1 \leq j \leq m$, $\mathrm{vb}_j(S_m^n(x_i, t_1, \ldots, t_n)) \leq k$. Thus, $S_{n,m}^{(k)}(x_i, t_1, \ldots, t_n) \approx S_m^n(x_i, t_1, \ldots, t_n) \approx t_i$.

(C3): This can be proved similarly to (C2) since $\mathrm{vb}_j(S_m^n(t, x_1, \ldots, x_n)) \leq k$ for any $1 \leq j \leq m$.

Altogether, we have that $\mathbf{clone}^{(k)}(\tau)$ satisfies (C1)–(C3) as weak identities. ∎

4. A mapping whose range is the set of k-terms

A mapping whose domain is the set of all operation symbols and whose range is the set of all terms preserving arities, is called a *hypersubstitution*. The set of all hypersubstitutions forms a monoid under a particular binary operation. This concept was first precisely given by Denecke, Lau, Pöschel and Schweigert (see

[4]). There are many papers contributing to this subject. For more information about hypersubstitutions and their applications can be found in [5] and [9].

Let us define the concept of hypersubstitutions in a formal way. A *hypersubstitution* of type τ is a mapping $\sigma\colon \{f_i : i \in I\} \to \mathrm{T}_\tau(X)$ such that $\sigma(f_i) \subseteq \mathrm{T}_\tau(X_{n_i})$ for any $i \in I$. The set of all hypersubstitutions of type τ is denoted by $\mathrm{Hyp}(\tau)$.

For any hypersubstitution σ can be extended to a mapping $\overline{\sigma}\colon \mathrm{T}_\tau(X) \to \mathrm{T}_\tau(X)$ defined as follows.

1. $\overline{\sigma}[x] := x$ for all $x \in X$.
2. If $f_i(s_1, \ldots, s_{n_i}) \in \mathrm{T}_\tau(X_n)$ and assume that $\overline{\sigma}[s_j]$ is already defined for all $1 \le j \le n_i$, then $\overline{\sigma}[f_i(s_1, \ldots, s_{n_i})] := S_n^{n_i}(\sigma(f_i), \overline{\sigma}[s_1], \ldots, \overline{\sigma}[s_{n_i}])$, where n is the maximum arity of the terms $\overline{\sigma}[s_1], \ldots, \overline{\sigma}[s_{n_i}]$.

It was proved that the structure $\mathbf{Hyp}(\tau) := \langle \mathrm{Hyp}(\tau); \circ_{\mathrm{h}}, \sigma_{\mathrm{id}}\rangle$ is a monoid, where $\circ_{\mathrm{h}}$ is a binary operation on $\mathrm{Hyp}(\tau)$ defined by $\sigma_1 \circ_{\mathrm{h}} \sigma_2 := \widehat{\sigma}_1 \circ \sigma_2$, and $\sigma_{\mathrm{id}}(f_i) := f_i(x_1, \ldots, x_{n_i})$ is a neutral element (see [9]).

In this section, we will consider the set of such mappings whose range is a restriction to the set of k-terms.

Definition. Let $k \in \mathbb{N}$. A hypersubstitution of type τ is a k-hypersubstitution of type τ if its range is the set of k-terms of the same type. Denoted by $\mathrm{Hyp}^{(k)}(\tau)$ the set of all k-hypersubstitution of type τ.

Any k-hypersubstitution of type τ is usually denoted by $\sigma^{(k)}$. When it is clear from the context, we always drop the superscript k.

It is not difficult to observe that $\mathrm{Hyp}^{(k)}(\tau) \subseteq \mathrm{Hyp}(\tau)$ for any $k \in \mathbb{N}$, but $\mathrm{Hyp}^{(k)}(\tau)$ is not closed under the binary operation $\circ_{\mathrm{h}}$. In other word, the set $\mathrm{Hyp}^{(k)}(\tau)$ together with $\circ_{\mathrm{h}}$ does not form a monoid as the following example shows.

Example 17. Let $\tau = (2)$ with a binary operation f, $k = 3$, $\sigma_1(f) = f(x_1, x_1)$ and $\sigma_2(f) = f(f(x_1, x_1), x_2)$. We see that $\sigma_1, \sigma_2 \in \mathrm{Hyp}^{(3)}(\tau)$. But

$$(\sigma_1 \circ_{\mathrm{h}} \sigma_2)(f) = \widehat{\sigma}_1[f(f(x_1, x_1), x_2)] = f(f(x_1, x_1), f(x_1, x_1)) \notin \mathrm{T}_\tau^{(3)}(X_2).$$

That is, $\sigma_1 \circ_{\mathrm{h}} \sigma_2 \notin \mathrm{Hyp}^{(3)}(\tau)$.

Any k-hypersubstitution σ of type τ can be extended to a partial mapping $\widehat{\sigma}\colon \mathrm{T}_\tau^{(k)}(X) \multimap \mathrm{T}_\tau^{(k)}(X)$ defined as follows. For any $n \in \mathbb{N}$, let $t \in \mathrm{T}_\tau^{(k)}(X_n)$,

1. $\widehat{\sigma}[x] := x$ for all $x \in X$.
2. $$\widehat{\sigma}[f_i(s_1, \ldots, s_{n_i})] := \begin{cases} \overline{\sigma}[f_i(s_1, \ldots, s_{n_i})] & \text{if } (\mathrm{K}_j) \text{ holds} \\ & \text{for all } 1 \le j \le m, \\ \text{not defined} & \text{otherwise,} \end{cases}$$

   for all $f_i(s_1, \ldots, s_{n_i}) \in \mathrm{T}_\tau^{(k)}(X_m)$.

We observe here that for any $\sigma \in \mathrm{Hyp}^{(k)}(\tau)$ and $t = f_i(s_1, \ldots, s_{n_i}) \in \mathrm{T}_\tau^{(k)}(X_n)$, $\widehat{\sigma}[t]$ is defined if and only if

$$\sum_{l=1}^{n_i} \mathrm{vb}_l(\sigma(f_i))\mathrm{vb}_j(\overline{\sigma}[s_l]) \leq k$$

for all $1 \leq j \leq n$.

Remark 18. For any $k \in \mathbb{N}$, let $\sigma \in \mathrm{Hyp}^{(k)}(\tau)$. We remark that the extension of σ is not a weak endomorphism. Let us consider the following example. We fix $k = 2$. Let $\tau = (2)$ with a binary operation symbol f. We can see easily by these settings:

1. $\sigma(f) = f(x_2, x_2)$,
2. $t = x_1$,
3. $t_1 = x_1$,
4. $t_2 = f(x_1, f(x_2, x_2))$,

that $\widehat{\sigma}[S_{2,2}^{(2)}(t, t_1, t_2)]$ is defined and it is equal to x_1, but $S_{2,2}^{(2)}(\widehat{\sigma}[t], \widehat{\sigma}[t_1], \widehat{\sigma}[t_2])$ is not define since $t_2 \notin \mathrm{dom}(\widehat{\sigma})$.

By the above discussion, we obtain the following proposition.

Proposition 19. *Let $k, m, n \in \mathbb{N}$ and $\sigma \in \mathrm{Hyp}^{(k)}(\tau)$. If $S_{n,m}^{(k)}(\widehat{\sigma}[t], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n])$ is defined, then $\widehat{\sigma}[S_{n,m}^{(k)}(t, t_1, \ldots, t_n)]$ is also defined. Moreover,*

$$S_{n,m}^{(k)}\big(\widehat{\sigma}[t], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n]\big) = \widehat{\sigma}\left[S_{n,m}^{(k)}(t, t_1, \ldots, t_n)\right]. \tag{2}$$

Proof. Assume that $S_{n,m}^{(k)}(\widehat{\sigma}[t], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n])$ is evaluable. That is, $\widehat{\sigma}[t] \in \mathrm{T}_\tau^{(k)}(X_n)$ and $\widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n] \in \mathrm{T}_\tau^{(k)}(X_m)$, and

$$\sum_{l=1}^{n} \mathrm{vb}_l(\widehat{\sigma}[t])\mathrm{vb}_j(\widehat{\sigma}[t_l]) \leq k,$$

for all $1 \leq j \leq m$. We show by induction on the complexity of the n-ary k-term t of type τ. If $t = x_i \in X_n$, then $S_{n,m}^{(k)}(\widehat{\sigma}[x_i], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n]) = S_m^n(\widehat{\sigma}[x_i], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n]) = S_m^n(x_i, \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n]) = \widehat{\sigma}[t_i] = \widehat{\sigma}[S_m^n(x_i, t_1, \ldots, t_n)] = \widehat{\sigma}[S_{n,m}^{(k)}(x_i, t_1, \ldots, t_n)]$. This shows that $\widehat{\sigma}[S_{n,m}^{(k)}(x_i, t_1, \ldots, t_n)]$ is defined and that (2) is valid. Now, let $t = f_i(s_1, \ldots, s_{n_i})$. Assume that, for all $1 \leq j \leq n_i$, if $S_{n,m}^{(k)}(\widehat{\sigma}[s_j], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n])$ is defined, then $\widehat{\sigma}[S_{n,m}^{(k)}(s_j, t_1, \ldots, t_n)]$ is defined and both are equal. Since

$$S_{n,m}^{(k)}\big(\widehat{\sigma}[f_i(s_1, \ldots, s_{n_i})], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n]\big)$$

is defined,

$$\begin{aligned}
&S^{(k)}_{n,m}\big(\widehat{\sigma}[f_i(s_1,\ldots,s_{n_i})],\widehat{\sigma}[t_1],\ldots,\widehat{\sigma}[t_n]\big)\\
&= S^{(k)}_{n,m}\big(S^{n_i}_n(\sigma(f_i),\overline{\sigma}[s_1],\ldots,\overline{\sigma}[s_{n_i}]),\widehat{\sigma}[t_1],\ldots,\widehat{\sigma}[t_n]\big)\\
&= S^{(k)}_{n,m}\big(S^{(k)}_{n_i,n}(\sigma(f_i),\widehat{\sigma}[s_1],\ldots,\widehat{\sigma}[s_{n_i}]),\widehat{\sigma}[t_1],\ldots,\widehat{\sigma}[t_n]\big)\\
&= S^{(k)}_{n_i,m}(\sigma(f_i),S^{(k)}_{n,m}(\widehat{\sigma}[s_1],\widehat{\sigma}[t_1],\ldots,\widehat{\sigma}[t_n]),\ldots,S^{(k)}_{n,m}(\widehat{\sigma}[s_{n_i}],\widehat{\sigma}[t_1],\ldots,\widehat{\sigma}[t_n]))\\
&\hfill \text{(by Theorem 16)}\\
&= S^{(k)}_{n_i,m}\big(\sigma(f_i),\widehat{\sigma}\big[S^{(k)}_{n,m}(s_1,t_1,\ldots,t_n)\big],\ldots,\widehat{\sigma}\big[S^{(k)}_{n,m}(s_{n_i},t_1,\ldots,t_n)\big]\big)\\
&\hfill \text{(by our presumption)}\\
&= S^{n_i}_m\big(\sigma(f_i),\overline{\sigma}\big[S^{(k)}_{n,m}(s_1,t_1,\ldots,t_n)\big],\ldots,\overline{\sigma}\big[S^{(k)}_{n,m}(s_{n_i},t_1,\ldots,t_n)\big]\big)\\
&= \overline{\sigma}\big[f_i(S^{(k)}_{n,m}(s_1,t_1,\ldots,t_n),\ldots,S^{(k)}_{n,m}(s_{n_i},t_1,\ldots,t_n))\big]\\
&= \widehat{\sigma}\big[f_i(S^{n}_{m}(s_1,t_1,\ldots,t_n),\ldots,S^{n}_{m}(s_{n_i},t_1,\ldots,t_n))\big]\\
&= \widehat{\sigma}\big[S^{n}_{m}(f_i(s_1,\ldots,s_{n_i}),t_1,\ldots,t_n)\big]\\
&= \widehat{\sigma}\big[S^{(k)}_{n,m}(f_i(s_1,\ldots,s_{n_i}),t_1,\ldots,t_n)\big].
\end{aligned}$$

This implies that $\widehat{\sigma}[S^{(k)}_{n,m}(f_i(s_1,\ldots,s_{n_i}),t_1,\ldots,t_n)]$ is defined and the equality holds. ∎

As a particular case of the above result, we obtain the following corollary.

Corollary 20. *Let $\sigma \in \mathrm{Hyp}^{(1)}(\tau)$. Then we have that $\widehat{\sigma}$ is a weak endomorphism on* $\mathbf{clone}^{(1)}(\tau)$.

Proof. Let $m,n \in \mathbb{N}$. Following from Proposition 19, we need to show only that $S^{(1)}_{n,m}(\widehat{\sigma}[t],\widehat{\sigma}[t_1],\ldots,\widehat{\sigma}[t_n])$ is defined if $\widehat{\sigma}[S^{(1)}_{n,m}(t,t_1,\ldots,t_n)]$ is defined. We assume that $\widehat{\sigma}[S^{(1)}_{n,m}(t,t_1,\ldots,t_n)]$ is evaluated. Since $\sigma(f_i) \in \mathrm{T}^{(1)}_\tau(X_{n_i})$ for all $i \in I$, it is clear that $\overline{\sigma}[t] \in \mathrm{T}^{(1)}_\tau(X_n)$ and $\overline{\sigma}[t_j] \in \mathrm{T}^{(1)}_\tau(X_m)$ for all $1 \le j \le n$. This means that $\overline{\sigma}[t] = \widehat{\sigma}[t]$ and $\overline{\sigma}[t_j] = \widehat{\sigma}[t_j]$ for all $1 \le j \le n$. Let us now consider

$$\begin{aligned}
\widehat{\sigma}\big[S^{(1)}_{n,m}(t,t_1,\ldots,t_n)\big] &= \overline{\sigma}\big[S^{(1)}_{n,m}(t,t_1,\ldots,t_n)\big] = \overline{\sigma}\big[S^{n}_{m}(t,t_1,\ldots,t_n)\big]\\
&= S^n_m\big(\overline{\sigma}[t],\overline{\sigma}[t_1],\ldots,\overline{\sigma}[t_n]\big) = S^n_m\big(\widehat{\sigma}[t],\widehat{\sigma}[t_1],\ldots,\widehat{\sigma}[t_n]\big).
\end{aligned}$$

Since

$$\mathrm{vb}_j\left(\mathrm{vb}_j\left(S^n_m(\widehat{\sigma}[t],\widehat{\sigma}[t_1],\ldots,\widehat{\sigma}[t_n])\right)\right) = \mathrm{vb}_j\left(\widehat{\sigma}\big[S^{(1)}_{n,m}(t,t_1,\ldots,t_n)\big]\right) \le 1$$

for all $1 \le j \le m$, we have that $\widehat{\sigma}\big[S^{(1)}_{n,m}(t,t_1,\ldots,t_n)\big] = S^n_m(\widehat{\sigma}[t],\widehat{\sigma}[t_1],\ldots,\widehat{\sigma}[t_n]) = S^{(1)}_{n,m}(\widehat{\sigma}[t],\widehat{\sigma}[t_1],\ldots,\widehat{\sigma}[t_n])$. This shows that $S^{(1)}_{n,m}(\widehat{\sigma}[t],\widehat{\sigma}[t_1],\ldots,\widehat{\sigma}[t_n])$ is defined. Furthermore, $S^{(1)}_{n,m}(\widehat{\sigma}[t],\widehat{\sigma}[t_1],\ldots,\widehat{\sigma}[t_n]) = \widehat{\sigma}[S^{(1)}_{n,m}(t,t_1,\ldots,t_n)]$. ∎

By Theorem 15, the following result is obtained.

Corollary 21. *Let $k \in \mathbb{N}$ and $\sigma \in \mathrm{Hyp}^{(k)}(\tau)$. Then $\widehat{\sigma}$ is a weak endomorphism on $\mathbf{clone}^{(k)}(\tau)$ if and only if $k = 1$.*

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