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ON nd-$K^*(n, r)$-FULL HYPERSUBSTITUTIONS

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Abstract

Based on the notion of $K^*(n, r)$-full terms defined by the authors, nd-$K^*(n, r)$-full hypersubstitutions are defined. It turns out that the extension of an nd-$K^*(n, r)$-full hypersubstitution is an endomorphism of the algebra of tree languages of nd-$K^*(n, r)$-full terms.

Keywords: $K^*(n, r)$-full term, nd-$K^*(n, r)$-full hypersubstitution, full term, nd-full hypersubstitution.

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1. Introduction and preliminaries

The concept of a term is one of the fundamental concepts of universal algebra. It plays plays a major role for classifying algebras into subclasses. Sets of terms of type τ are called tree languages.

In 2008, Denecke and others [3] considered the notion of the power set of the set of all terms of type τ. In case of a finite type of algebras, this study can be applied in the theoretical computer science. Moreover, on the power set of the set of all terms, an $(n+1)$-ary superposition operation is defined and then a heterogeneous algebra, which is called the *power clone of type* τ, is constructed. It turns out that such power clone satisfies the well-known clone axioms (C1), (C2), (C3), where (C1) is the superassociative law (see [3]).

It is widely accepted that *tree transducers* are algebraic machines which generalize automata, i.e., transducers transform terms of one fixed type into terms of a second type. To study such tool, tree transformations are essential concepts. It follows that many algebraists are interested to study *non-deterministic hypersubstitutions*, mappings which take operation symbols to sets of terms of the corresponding arity. In [3] it was proved that the set of all non-deterministic hypersubstitutions together with one suitable associative operation and an identity mapping forms a monoid.

After that, Denecke and Sarasit [2] defined binary associative operations on tree languages and examined the properties of this semigroup. In 2011, the properties of semigroups constructed from tree languages and tree language product were studied (see [1]).

Recently, the monoid of non-deterministic full hypersubstitutions was studied by Lekkoksung in 2019. Many properties of the superposition operation on the power set of the set of all full terms of type τ_n were investigated. For more details, we refer to [4].

In the same year, in [5], the authors introduced the concept of $K^*(n,r)$-full terms which extend the notion of full terms by the following. Let n be a fixed positive integer. Let $\tau_n = (n_i)_{i\in I}$ be an n-ary type of algebras with operation symbols f_i indexed by some set I, each f_i has arity $n_i = n$. Let $X_n = \{x_1, \ldots, x_n\}$ be an n-element alphabet of variables. The full transformation semigroup $T(\{1,\ldots,n\})$ is the set of all maps $\alpha : \{1,\ldots,n\} \to \{1,\ldots,n\}$ together the usual composition of mappings. Indeed, $T(\{1,\ldots,n\})$ is the monoid, an identity map 1_n acts as its identity. For a fixed integer $0 < r \leq n$, it is well-known that the set

$$K(n,r) = \{\alpha \in T(\{1,\ldots,n\}) \mid |ran(\alpha)| \leq r\}$$

of all restricted range transformations is a subsemigroup of $T(\{1,\ldots,n\})$. Then,

$$K^*(n,r) = K(n,r) \cup \{1_n\}$$

is a submonoid of $T(\{1,\ldots,n\})$. Define (see [5]):

(i) $f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$ is an *n-ary $K^*(n,r)$-full term of type* τ_n if f_i is an n-ary operation symbol and $\alpha \in K^*(n,r)$.

(ii) $f_i(t_1, \ldots, t_n)$ is an *n-ary $K^*(n,r)$-full term of type* τ_n if f_i is an n-ary operation symbol, and $t_1, \ldots, t_n$ are n-ary $K^*(n,r)$-full terms of type τ_n.

(iii) The set of all n-ary $K^*(n,r)$-full terms of type τ_n closed under finite applications of (ii) is denoted by $W_{\tau_n}^{K^*(n,r)}(X_n)$.

Example 1. Consider type τ_3 with a singleton set I, i.e., $\tau_3 = (3)$. Then $f(x_1,x_1,x_1), f(x_2,x_2,x_2), f(x_3,x_3,x_3), f(x_3,x_2,x_2), f(x_2,x_1,x_1), f(x_2,x_3,x_3)$, $f(x_1,x_2,x_3), f(f(x_1,x_2,x_3), f(x_1,x_3,x_3), f(x_3,x_2,x_3)) \in W_{\tau_3}^{K^*(3,2)}(X_3)$.
But, $x_1, x_2, x_3, f(x_2,x_3,x_1), f(x_3,x_2,x_1) \notin W_{\tau_3}^{K^*(3,2)}(X_3)$.

The paper is motivated by several recent studies in this research direction, we present tree languages over the set of all $K^*(n,r)$-full terms of type τ_n. The appropriate superposition operation on the power set of $K^*(n,r)$-full terms of type τ_n is defined and several algebraic properties of such operation are provided. Moreover, this operation satisfies also the superassociative law. In Section 3, the concept of non-deterministic $K^*(n,r)$-full hypersubstitutions of type τ_n is proposed. Furthermore, we define an associative binary operation on the set of all non-deterministic $K^*(n,r)$-full hypersubstitutions of type τ_n and determine an element acting as an identity.

2. SUPERPOSITION OPERATIONS OF SET OF $K^*(n,r)$-FULL TERMS

Let $P^*(W_{\tau_n}^{K^*(n,r)}(X_n))$ be the set of all non-empty subsets of $W_{\tau_n}^{K^*(n,r)}(X_n)$. Define the superposition operation

$$S_{nd}^n : P^*(W_{\tau_n}^{K^*(n,r)}(X_n))^{n+1} \to P^*(W_{\tau_n}^{K^*(n,r)}(X_n))$$

by, for $T, T_1, \ldots, T_n$ in $P^*(W_{\tau_n}^{K^*(n,r)}(X_n))$,

(i) if $T = \{f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})\}$ where $\alpha \in K^*(n,r)$, then
$S_{nd}^n(T, T_1, \ldots, T_n) = \{f_i(t_{\alpha(1)}, \ldots, t_{\alpha(n)}) \mid t_{\alpha(1)} \in T_{\alpha(1)}, \ldots, t_{\alpha(n)} \in T_{\alpha(n)}\}$;

(ii) if $T = \{f_i(t_1, \ldots, t_n)\}$, then
$S_{nd}^n(T, T_1, \ldots, T_n) = \{f_i(r_1, \ldots, r_n) \mid r_k \in S_{nd}^n(\{t_k\}, T_1, \ldots, T_n), 1 \le k \le n\}$;

(iii) if T is an arbitrary non-empty subset of $W_{\tau_n}^{K^*(n,r)}(X_n)$, then
$S_{nd}^n(T, T_1, \ldots, T_n) = \bigcup_{t \in T} S_{nd}^n(\{t\}, T_1, \ldots, T_n)$.

Example 2. Let us consider $T = \{f(x_3,x_2,x_2)\}, T_1 = \{f(x_1,x_2,x_1), f(x_3,x_2,x_2), f(x_3,x_2,x_3)\}, T_2 = \{f(x_2,x_2,x_3)\}$ and $T_3 = \{f(x_1,x_3,x_3)\}$ in $P^*(W_{(3)}^{K^*(3,2)}(X_3))$. Then

$$\begin{aligned} S_{nd}^3(T, T_1, T_2, T_3) &= S_{nd}^3(\{f(x_3,x_2,x_2)\}, T_1, T_2, T_3) \\ &= \{f(t_3, t_2, t_2) \mid t_2 \in T_2, t_3 \in T_3\} \\ &= \{f(f(x_1,x_3,x_3), f(x_2,x_2,x_3), f(x_2,x_2,x_3))\}. \end{aligned}$$

Example 3. Consider $T = \{f(x_2,x_2,x_1), f(x_3,x_2,x_3), f(x_1,x_2,x_2)\}$, $T_1 = \{f(x_1, x_1, x_2)\}$, $T_2 = \{f(x_2,x_3,x_3)\}$ and $T_3 = \{f(x_1,x_2,x_1)\}$ in $P^*(W_{(3)}^{K^*(3,2)}(X_3))$. We have

$$\begin{aligned}
S^3_{nd}(\{f(x_2,x_2,x_1)\},T_1,T_2,T_3) &= \{f(t_2,t_2,t_1) \mid t_1 \in T_1, t_2 \in T_2\}\\
&= \{f(f(x_2,x_3,x_3), f(x_2,x_3,x_3), f(x_1,x_1,x_2))\},\\
S^3_{nd}(\{f(x_3,x_2,x_3)\},T_1,T_2,T_3) &= \{f(t_3,t_2,t_3) \mid t_2 \in T_2, t_3 \in T_3\}\\
&= \{f(f(x_1,x_2,x_1), f(x_2,x_3,x_3), f(x_1,x_2,x_1))\},\\
S^3_{nd}(\{f(x_1,x_2,x_2),T_1,T_2,T_3\}) &= \{f(t_1,t_2,t_2) \mid t_1 \in T_1, t_2 \in T_2\}\\
&= \{f(f(x_1,x_1,x_2), f(x_2,x_3,x_3), f(x_2,x_3,x_3))\}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&S^3_{nd}(T,T_1,T_2,T_3)\\
&= S^3_{nd}(\{f(x_2,x_2,x_1)\},T_1,T_2,T_3) \cup S^3_{nd}(\{f(x_3,x_2,x_3)\},T_1,T_2,T_3)\\
&\quad \cup S^3_{nd}(\{f(x_1,x_2,x_2),T_1,T_2,T_3\})\\
&= \{f(f(x_2,x_3,x_3), f(x_2,x_3,x_3), f(x_1,x_1,x_2))\} \cup \{f(f(x_1,x_2,x_1), f(x_2,x_3,x_3),\\
&\quad f(x_1,x_2,x_1))\} \cup \{f(f(x_1,x_1,x_2), f(x_2,x_3,x_3), f(x_2,x_3,x_3))\}\\
&= \{f(f(x_2,x_3,x_3), f(x_2,x_3,x_3), f(x_1,x_1,x_2)), f(f(x_1,x_2,x_1), f(x_2,x_3,x_3),\\
&f(x_1,x_2,x_1)), f(f(x_1,x_1,x_2), f(x_2,x_3,x_3), f(x_2,x_3,x_3))\}.
\end{aligned}$$

If $t = f_i(t_1,\ldots,t_n) \in W_{\tau_n}^{K^*(n,r)}(X_n)$ and $\alpha \in K^*(n,r)$, then $t_\alpha = f_i\big(t_{\alpha(1)},\ldots, t_{\alpha(n)}\big)$. Let $T \in P^*\left(W_{\tau_n}^{K^*(n,r)}(X_n)\right)$. For $\alpha \in K^*(n,r)$, define

$$T_\alpha = \{t_\alpha \mid t \in T\}.$$

Example 4. Consider $T = \{f(x_2,x_2,x_3), f(x_1,x_3,x_1), f(x_1,x_2,x_2)\}$ in $P^*\left(W_{(3)}^{K^*(3,2)}(X_3)\right)$ and $\alpha = \begin{pmatrix} 1 & 2 & 3\\ 1 & 2 & 1\end{pmatrix}$ in $K^*(3,2)$. Then

$$\begin{aligned}
T_\alpha &= \{f(x_2,x_2,x_3)_\alpha, f(x_1,x_3,x_1)_\alpha, f(x_1,x_2,x_2)_\alpha\}\\
&= \big\{f(x_{\alpha(2)},x_{\alpha(2)},x_{\alpha(3)}), f(x_{\alpha(1)},x_{\alpha(3)},x_{\alpha(1)}), f(x_{\alpha(1)},x_{\alpha(2)},x_{\alpha(2)})\big\}\\
&= \{f(x_2,x_2,x_1), f(x_1,x_1,x_1), f(x_1,x_2,x_2)\}.
\end{aligned}$$

We then prove the following propositions.

Proposition 5. *Let n be a fixed positive integer. If $T, T_1, \ldots, T_n \in P^*\left(W_{\tau_n}^{K^*(n,r)}(X_n)\right)$ and $\alpha \in K^*(n,r)$, then*

$$S^n_{nd}(T_\alpha, T_1, \ldots, T_n) = S^n_{nd}\left(T, T_{\alpha(1)}, \ldots, T_{\alpha(n)}\right).$$

Proof. Let $T = \{f_i(x_{\beta(1)},\dots,x_{\beta(n)})\}$ where $\beta \in K^*(n,r)$.
Then $T_\alpha = \{f_i(x_{\alpha(\beta(1))},\dots,x_{\alpha(\beta(n))})\}$. Thus

$$\begin{aligned} S^n_{nd}(T_\alpha,T_1,\dots,T_n) &= S^n_{nd}(\{f_i(x_{\alpha(\beta(1))},\dots,x_{\alpha(\beta(n))})\},T_1,\dots,T_n) \\ &= \{f_i(t_{\alpha(\beta(1))},\dots,t_{\alpha(\beta(n))}) \mid t_{\alpha(\beta(k))} \in T_{\alpha(\beta(k))}, 1 \le k \le n\} \\ &= S^n_{nd}(\{f_i(x_{\beta(1)},\dots,x_{\beta(n)})\},T_{\alpha(1)},\dots,T_{\alpha(n)}) \\ &= S^n_{nd}(T,T_{\alpha(1)},\dots,T_{\alpha(n)}). \end{aligned}$$

Let $T = \{f_i(r_1,\dots,r_n)\}$ and assume that, for all $1 \le k \le n$,

$$S^n_{nd}(\{r_k\}_\alpha,T_1,\dots,T_n) = S^n_{nd}\left(\{r_k\},T_{\alpha(1)},\dots,T_{\alpha(n)}\right).$$

Then $T_\alpha = \{f_i(r_{\alpha(1)},\dots,r_{\alpha(n)})\}$. Thus

$$\begin{aligned} &S^n_{nd}(T_\alpha,T_1,\dots,T_n) \\ &= S^n_{nd}(\{f_i(r_{\alpha(1)},\dots,r_{\alpha(n)})\},T_1,\dots,T_n) \\ &= \{f_i(s_{\alpha(1)},\dots,s_{\alpha(n)}) \mid s_{\alpha(k)} \in S^n_{nd}(\{r_{\alpha(k)}\},T_1,\dots,T_n), 1 \le k \le n\} \\ &= \{f_i(s_{\alpha(1)},\dots,s_{\alpha(n)}) \mid s_{\alpha(k)} \in S^n_{nd}(\{r_k\}_\alpha,T_1,\dots,T_n), 1 \le k \le n\} \\ &= \{f_i(s_{\alpha(1)},\dots,s_{\alpha(n)}) \mid s_{\alpha(k)} \in S^n_{nd}(\{r_k\},T_{\alpha(1)},\dots,T_{\alpha(n)})\} \\ &= S^n_{nd}(\{f_i(r_1,\dots,r_n)\},T_{\alpha(1)},\dots,T_{\alpha(n)}) \\ &= S^n_{nd}(T,T_{\alpha(1)},\dots,T_{\alpha(n)}). \end{aligned}$$

If T is an arbitrary non-empty subset of $W^{K^*(n,r)}_{\tau_n}(X_n)$, then

$$\begin{aligned} S^n_{nd}(T_\alpha,T_1,\dots,T_n) &= \bigcup_{t\in T} S^n_{nd}(\{t_\alpha\},T_1,\dots,T_n) \\ &= \bigcup_{t\in T} S^n_{nd}(\{t\}_\alpha,T_1,\dots,T_n) \\ &= \bigcup_{t\in T} S^n_{nd}(\{t\},T_{\alpha(1)},\dots,T_{\alpha(n)}) \\ &= S^n_{nd}(T,T_{\alpha(1)},\dots,T_{\alpha(n)}). \end{aligned}$$

■

Proposition 6. *Let n be a fixed positive integer. If $T,T_1,\dots,T_n \in P^*\left(W^{K^*(n,r)}_{\tau_n}(X_n)\right)$ and $\alpha \in K^*(n,r)$, then*

$$S^n_{nd}(T_\alpha,T_1,\dots,T_n) = \left(S^n_{nd}(T,T_1,\dots,T_n)\right)_\alpha.$$

Proof. Let $T = \{f_i(x_{\beta(1)}, \ldots, x_{\beta(n)})\}$ where $\beta \in K^*(n,r)$. Then

$$\begin{aligned}
S^n_{nd}(T_\alpha, T_1, \ldots, T_n) &= S^n_{nd}((\{f_i(x_{\beta(1)}, \ldots, x_{\beta(n)})\})_\alpha, T_1, \ldots, T_n) \\
&= S^n_{nd}(\{f_i(x_{\alpha(\beta(1))}, \ldots, x_{\alpha(\beta(n))})\}, T_1, \ldots, T_n) \\
&= \{f_i(t_{\alpha(\beta(1))}, \ldots, t_{\alpha(\beta(n))}) \mid t_{\alpha(\beta(k))} \in T_{\alpha(\beta(k))}, 1 \le k \le n\} \\
&= \{f_i(t_{\beta(1)}, \ldots, t_{\beta(n)})_\alpha \mid t_{\beta(k)} \in T_{\beta(k)}, 1 \le k \le n\} \\
&= (\{f_i(t_{\beta(1)}, \ldots, t_{\beta(n)}) \mid t_{\beta(k)} \in T_{\beta(k)}, 1 \le k \le n\})_\alpha \\
&= (S^n_{nd}(\{f_i(x_{\beta(1)}, \ldots, x_{\beta(n)})\}, T_1, \ldots, T_n))_\alpha \\
&= (S^n_{nd}(T, T_1, \ldots, T_n))_\alpha.
\end{aligned}$$

Let $T = \{f_i(r_1, \ldots, r_n)\}$ and assume that, for all $1 \le k \le n$,

$$S^n_{nd}(\{r_k\}_\alpha, T_1, \ldots, T_n) = (S^n_{nd}(\{r_k\}, T_1, \ldots, T_n))_\alpha.$$

Then

$$\begin{aligned}
&S^n_{nd}(T_\alpha, T_1, \ldots, T_n) \\
&= S^n_{nd}((\{f_i(r_1, \ldots, r_n)\})_\alpha, T_1, \ldots, T_n) \\
&= S^n_{nd}(\{f_i(r_{\alpha(1)}, \ldots, r_{\alpha(n)})\}, T_1, \ldots, T_n) \\
&= \{f_i(s_{\alpha(1)}, \ldots, s_{\alpha(n)}) \mid s_{\alpha(k)} \in S^n_{nd}(\{r_{\alpha(k)}\}, T_1, \ldots, T_n), 1 \le k \le n\} \\
&= \{f_i(s_{\alpha(1)}, \ldots, s_{\alpha(n)}) \mid s_{\alpha(k)} \in S^n_{nd}(\{r_k\}_\alpha, T_1, \ldots, T_n), 1 \le k \le n\} \\
&= \{f_i(s_{\alpha(1)}, \ldots, s_{\alpha(n)}) \mid s_{\alpha(k)} \in (S^n_{nd}(\{r_k\}, T_1, \ldots, T_n))_\alpha, 1 \le k \le n\} \\
&= \{f_i(s_1, \ldots, s_n)_\alpha \mid (s_k)_\alpha \in (S^n_{nd}(\{r_k\}, T_1, \ldots, T_n))_\alpha, 1 \le k \le n\} \\
&= (\{f_i(s_1, \ldots, s_n) \mid s_k \in S^n_{nd}(\{r_k\}, T_1, \ldots, T_n)\})_\alpha, 1 \le k \le n \\
&= (S^n_{nd}(\{f_i(r_1, \ldots, r_n)\}, T_1, \ldots, T_n))_\alpha \\
&= (S^n_{nd}(T, T_1, \ldots, T_n))_\alpha.
\end{aligned}$$

If T is an arbitrary non-empty subset of $W^{K^*(n,r)}_{\tau_n}(X_n)$, then

$$\begin{aligned}
S^n_{nd}(T_\alpha, T_1, \ldots, T_n) &= \bigcup_{t \in T} S^n_{nd}(\{t_\alpha\}, T_1, \ldots, T_n) \\
&= \bigcup_{t \in T} S^n_{nd}(\{t\}_\alpha, T_1, \ldots, T_n) \\
&= \bigcup_{t \in T} (S^n_{nd}(\{t\}, T_1, \ldots, T_n))_\alpha \\
&= \left(\bigcup_{t \in T} S^n_{nd}(\{t\}, T_1, \ldots, T_n)\right)_\alpha \\
&= (S^n_{nd}(T, T_1, \ldots, T_n))_\alpha.
\end{aligned}$$

■

By Proposition 5 and Proposition 6, we have the following.

Corollary 7. *Let n be a fixed positive integer.*
If $T, T_1, \ldots, T_n \in P^*(W_{\tau_n}^{K^*(n,r)}(X_n))$ and $\alpha \in K^*(n,r)$, then

$$S_{nd}^n\left(T, T_{\alpha(1)}, \ldots, T_{\alpha(n)}\right) = (S_{nd}^n(T, T_1, \ldots, T_n))_\alpha.$$

Now we can state and prove the following result.

Theorem 8. *Let n be a fixed positive integer. If $T, T_1, \ldots, T_n, S_1, \ldots, S_n \in P^*(W_{\tau_n}^{K^*(n,r)}(X_n))$, then*

$$\begin{aligned}&S_{nd}^n(S_{nd}^n(T, S_1, \ldots, S_n), T_1, \ldots, T_n)\\ &= S_{nd}^n(T, S_{nd}^n(S_1, T_1, \ldots, T_n), \ldots, S_{nd}^n(S_n, T_1, \ldots, T_n)).\end{aligned}$$

Proof. Let $T = \{f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})\}$ where $\alpha \in K^*(n,r)$. Then

$$\begin{aligned}&S_{nd}^n(S_{nd}^n(T, S_1, \ldots, S_n), T_1, \ldots, T_n)\\ &= S_{nd}^n(S_{nd}^n(\{f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})\}, S_1, \ldots, S_n), T_1, \ldots, T_n)\\ &= S_{nd}^n(\{f_i(s_{\alpha(1)}, \ldots, s_{\alpha(n)}) \mid s_{\alpha(k)} \in S_{\alpha(k)}, 1 \le k \le n\}, T_1, \ldots, T_n)\\ &= \left\{f_i(r_{\alpha(1)}, \ldots, r_{\alpha(n)}) \mid r_{\alpha(k)} \in S_{nd}^n(\{s_{\alpha(k)} \mid s_{\alpha(k)} \in S_{\alpha(k)}, 1 \le k \le n\}, T_1, \ldots, T_n)\right\}\\ &= \left\{f_i(r_{\alpha(1)}, \ldots, r_{\alpha(n)}) \mid r_{\alpha(k)} \in S_{nd}^n(S_{\alpha(k)}, T_1, \ldots, T_n), 1 \le k \le n\right\}\\ &= S_{nd}^n(\{f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})\}, S_{nd}^n(S_1, T_1, \ldots, T_n), \ldots, S_{nd}^n(S_n, T_1, \ldots, T_n))\\ &= S_{nd}^n(T, S_{nd}^n(S_1, T_1, \ldots, T_n), \ldots, S_{nd}^n(S_n, T_1, \ldots, T_n)).\end{aligned}$$

Let $T = \{f_i(r_1, \ldots, r_n)\}$ where $r_1, \ldots, r_n \in W_{\tau_n}^{K^*(n,r)}(X_n)$ and assume that, for all $1 \le k \le n$,

$$\begin{aligned}&S_{nd}^n(S_{nd}^n(\{r_k\}, S_1, \ldots, S_n), T_1, \ldots, T_n)\\ &= S_{nd}^n(\{r_k\}, S_{nd}^n(S_1, T_1, \ldots, T_n), \ldots, S_{nd}^n(S_n, T_1, \ldots, T_n)).\end{aligned}$$

Then

$$\begin{aligned}&S_{nd}^n(S_{nd}^n(T, S_1, \ldots, S_n), T_1, \ldots, T_n)\\ &= S_{nd}^n(S_{nd}^n(\{f_i(r_1, \ldots, r_n)\}, S_1, \ldots, S_n), T_1, \ldots, T_n)\\ &= S_{nd}^n(\{f_i(s_1, \ldots, s_n) \mid s_k \in S_{nd}^n(\{r_k\}, S_1, \ldots, S_n), 1 \le k \le n\}, T_1, \ldots, T_n)\\ &= \{f_i(p_1, \ldots, p_n) \mid p_k \in S_{nd}^n(\{s_k \mid s_k \in S_{nd}^n(\{r_k\}, S_1, \ldots, S_n)\}, T_1, \ldots, T_n),\\ &\quad 1 \le k \le n\}\\ &= \{f_i(p_1, \ldots, p_n) \mid p_k \in S_{nd}^n(S_{nd}^n(\{r_k\}, S_1, \ldots, S_n), T_1, \ldots, T_n), 1 \le k \le n\}\\ &= \{f_i(p_1, \ldots, p_n) \mid p_k \in S_{nd}^n(\{r_k\}, S_{nd}^n(S_1, T_1, \ldots, T_n), \ldots, S_{nd}^n(S_n, T_1, \ldots, T_n))\}\\ &= S_{nd}^n(\{f_i(r_1, \ldots, r_n)\}, S_{nd}^n(S_1, T_1, \ldots, T_n), \ldots, S_{nd}^n(S_n, T_1, \ldots, T_n))\\ &= S_{nd}^n(T, S_{nd}^n(S_1, T_1, \ldots, T_n), \ldots, S_{nd}^n(S_n, T_1, \ldots, T_n)).\end{aligned}$$

If T is an arbitrary non-empty subset of $W_{\tau_n}^{K^*(n,r)}(X_n)$, then

$$\begin{aligned}
&S^n_{nd}(S^n_{nd}(T, S_1, \ldots, S_n), T_1, \ldots, T_n)\\
&= S^n_{nd}\Bigg(\bigcup_{t\in T} S^n_{nd}(\{t\}, S_1, \ldots, S_n), T_1, \ldots, T_n\Bigg)\\
&= \bigcup_{t\in T} S^n_{nd}(S^n_{nd}(\{t\}, S_1, \ldots, S_n), T_1, \ldots, T_n)\\
&= \bigcup_{t\in T} S^n_{nd}(\{t\}, S^n_{nd}(S_1, T_1, \ldots, T_n), \ldots, S^n_{nd}(S_n, T_1, \ldots, T_n))\\
&= S^n_{nd}(T, S^n_{nd}(S_1, T_1, \ldots, T_n), \ldots, S^n_{nd}(S_n, T_1, \ldots, T_n)).
\end{aligned}$$

■

Using the definition of $W_{\tau_n}^{K^*(n,r)}(X_n)$, if we set $r = n$, then $W_{\tau_n}^{K^*(n,r)}(X_n) = W_{\tau_n}^{F}(X_n)$, the set of all n-ary full terms of type τ_n. According to Theorem 8, we have the following corollary.

Corollary 9 ([4], Theorem 8)**.** *Let n be a fixed positive integer. If $T, T_1, \ldots, T_n$, $S_1, \ldots, S_n \subseteq W_{\tau_n}^{F}(X_n)$, then*

$$\begin{aligned}
&S^n_{nd}(S^n_{nd}(T, S_1, \ldots, S_n), T_1, \ldots, T_n)\\
&= S^n_{nd}(T, S^n_{nd}(S_1, T_1, \ldots, T_n), \ldots, S^n_{nd}(S_n, T_1, \ldots, T_n)).
\end{aligned}$$

3. Non-deterministic $K^*(n,r)$-Full Hypersubstitutions

In this section, we first introduce the concept of a mapping which takes the set of all operation symbols of type τ_n to the collection of all non-empty subsets of $W_{\tau_n}^{K^*(n,r)}(X_n)$. We then define a binary associative operation for such mappings and study some algebraic properties.

Definition. A *non-deterministic $K^*(n,r)$-full hypersubstitution* or *nd-$K^*(n,r)$-full hypersubstitution* of type τ_n is a map $\sigma^{nd} : \{f_i \mid i \in I\} \to P^*(W_{\tau_n}^{K^*(n,r)}(X_n))$. The set of all $nd - K^*(n,r)$-full hypersubstitutions of type τ_n will be denoted by $nd - Hyp^{K^*(n,r)}(\tau_n)$.

Any nd-$K^*(n,r)$-full hypersubstitution of type τ_n can be extended to a mapping

$$\hat{\sigma}^{nd} : P^*(W_{\tau_n}^{K^*(n,r)}(X_n)) \to P^*(W_{\tau_n}^{K^*(n,r)}(X_n))$$

as follows:

(i) $\hat{\sigma}^{nd}[\{f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})\}] = (\sigma^{nd}(f_i))_\alpha$ where $\alpha \in K^*(n,r)$;

(ii) $\hat{\sigma}^{nd}[\{f_i(t_1, \ldots, t_n)\}] = S^n_{nd}(\sigma^{nd}(f_i), \hat{\sigma}^{nd}[\{t_1\}], \ldots, \hat{\sigma}^{nd}[\{t_n\}])$ and assume that $\hat{\sigma}^{nd}[\{t_1\}], \ldots, \hat{\sigma}^{nd}[\{t_n\}]$ are already defined;

(iii) $\hat{\sigma}^{nd}[T] = \bigcup_{t\in T} \hat{\sigma}^{nd}[\{t\}]$ where T is an arbitrary non-empty subset of $W_{\tau_n}^{K^*(n,r)}(X_n)$.

Example 10. Consider a type $\tau_4 = (4,4)$ and $\alpha, \beta \in K^*(4,3)$ defined by

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 4 & 2 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 3 & 4 \end{pmatrix}.$$

Let $T = \{f(g(x_{\beta(1)}, x_{\beta(2)}, x_{\beta(3)}, x_{\beta(4)}), f(x_{\alpha(1)}, x_{\alpha(2)}, x_{\alpha(3)}, x_{\alpha(4)}), f(x_{\alpha(1)}, x_{\alpha(2)}, x_{\alpha(3)}, x_{\alpha(4)}), g(x_{\beta(1)}, x_{\beta(2)}, x_{\beta(3)}, x_{\beta(4)}))\}$ and let

$$\sigma^{nd} : \{f, g\} \to P^*(W^{K^*(4,3)}(X_4))$$

be defined by

$$\sigma^{nd}(f) = \{f(x_{\alpha(1)}, x_{\alpha(2)}, x_{\alpha(3)}, x_{\alpha(4)})\},$$
$$\sigma^{nd}(g) = \{g(x_{\beta(1)}, x_{\beta(2)}, x_{\beta(3)}, x_{\beta(4)})\}.$$

Then

$$\begin{aligned}
\hat{\sigma}^{nd}[T] &= \hat{\sigma}^{nd}[\{f(g(x_{\beta(1)}, x_{\beta(2)}, x_{\beta(3)}, x_{\beta(4)}), f(x_{\alpha(1)}, x_{\alpha(2)}, x_{\alpha(3)}, x_{\alpha(4)}),\\
&\quad f(x_{\alpha(1)}, x_{\alpha(2)}, x_{\alpha(3)}, x_{\alpha}(4)), g(x_{\beta(1)}, x_{\beta(2)}, x_{\beta(3)}, x_{\beta(4)}))\}]\\
&= S^4_{nd}(\sigma^{nd}(f), \hat{\sigma}^{nd}[\{g(x_{\beta(1)}, x_{\beta(2)}, x_{\beta(3)}, x_{\beta(4)})\}],\\
&\quad \hat{\sigma}^{nd}[\{f(x_{\alpha(1)}, x_{\alpha(2)}, x_{\alpha(3)}, x_{\alpha(4)})\}], \hat{\sigma}^{nd}[\{f(x_{\alpha(1)}, x_{\alpha(2)}, x_{\alpha(3)}, x_{\alpha(4)})\}],\\
&\quad \hat{\sigma}^{nd}[\{g(x_{\beta(1)}, x_{\beta(2)}, x_{\beta(3)}, x_{\beta(4)})\}])\\
&= S^4_{nd}(\sigma^{nd}(f), (\sigma^{nd}(g))_\beta, (\sigma^{nd}(f))_\alpha, (\sigma^{nd}(f))_\alpha, (\sigma^{nd}(g))_\beta).\\
&= S^4_{nd}(\sigma^{nd}(f), (\{g(x_{\beta(1)}, x_{\beta(2)}, x_{\beta(3)}, x_{\beta(4)})\})_\beta,\\
&\quad (\{f(x_{\alpha(1)}, x_{\alpha(2)}, x_{\alpha(3)}, x_{\alpha(4)})\})_\alpha, (\{f(x_{\alpha(1)}, x_{\alpha(2)}, x_{\alpha(3)}, x_{\alpha(4)})\})_\alpha,\\
&\quad (\{g(x_{\beta(1)}, x_{\beta(2)}, x_{\beta(3)}, x_{\beta(4)})\})_\beta)\\
&= S^4_{nd}(\sigma^{nd}(f), (\{g(x_2, x_2, x_3, x_4)\})_\beta, (\{f(x_1, x_4, x_4, x_2)\})_\alpha,\\
&\quad (\{f(x_1, x_4, x_4, x_2)\})_\alpha, (\{g(x_2, x_2, x_3, x_4)\})_\beta)\\
&= S^4_{nd}(\sigma^{nd}(f), \{g(x_{\beta(2)}, x_{\beta(2)}, x_{\beta(3)}, x_{(\beta(4)})\}, \{f(x_{\alpha(1)}, x_{\alpha(4)}, x_{\alpha(4)}, x_{\alpha(2)})\},\\
&\quad \{f(x_{\alpha(1)}, x_{\alpha(4)}, x_{\alpha(4)}, x_{\alpha(2)})\}, \{g(x_{\beta(2)}, x_{\beta(2)}, x_{\beta(3)}, x_{(\beta(4)})\})\\
&= S^4_{nd}(\{f(x_{\alpha(1)}, x_{\alpha(2)}, x_{\alpha(3)}, x_{\alpha(4)})\}, \{g(x_2, x_2, x_3, x_4)\}, \{f(x_1, x_2, x_2, x_4)\},\\
&\quad \{f(x_1, x_2, x_2, x_4)\}, \{g(x_2, x_2, x_3, x_4)\})\\
&= S^4_{nd}(\{f(x_1, x_4, x_4, x_2)\}, \{g(x_2, x_2, x_3, x_4)\}, \{f(x_1, x_2, x_2, x_4)\},\\
&\quad \{f(x_1, x_2, x_2, x_4)\}, \{g(x_2, x_2, x_3, x_4)\})\\
&= \{f(r_1, r_4, r_4, r_2) \mid r_1 \in \{g(x_2, x_2, x_3, x_4)\}, r_2 \in \{f(x_1, x_2, x_2, x_4)\},\\
&\quad r_4 \in \{g(x_2, x_2, x_3, x_4)\}\}\\
&= \{f(g(x_2, x_2, x_3, x_4), g(x_2, x_2, x_3, x_4), g(x_2, x_2, x_3, x_4), f(x_1, x_2, x_2, x_4))\}.
\end{aligned}$$

Using the definition of $\hat{\sigma}^{nd}$ and T_α, we have the following lemma.

Lemma 11. *Let* $T \in P^*\left(W_{\tau_n}^{K^*(n,r)}(X_n)\right)$ *and* $\alpha \in K^*(n,r)$*. Then*

$$\hat{\sigma}^{nd}[T_\alpha] = (\hat{\sigma}^{nd}[T])_\alpha.$$

Proof. Let $T = \{f_i(x_{\beta(1)}, \ldots, x_{\beta(n)})\}$ where $\beta \in K^*(n,r)$. Then

$$\begin{aligned}
\hat{\sigma}^{nd}[T_\alpha] &= \hat{\sigma}^{nd}[(\{f_i(x_{\beta(1)}, \ldots, x_{\beta(n)})\})_\alpha] \\
&= \hat{\sigma}^{nd}[\{f_i(x_{\alpha(\beta(1))}, \ldots, x_{\alpha(\beta(n))})\}] \\
&= \hat{\sigma}^{nd}[\{f_i(x_{(\alpha\circ\beta)(1)}, \ldots, x_{(\alpha\circ\beta)(n)})\}] \\
&= (\sigma^{nd}(f_i))_{(\alpha\circ\beta)} = ((\sigma^{nd}(f_i))_\beta)_\alpha \\
&= (\hat{\sigma}^{nd}[\{f_i(x_{\beta(1)}, \ldots, x_{\beta(n)})\}])_\alpha = (\hat{\sigma}^{nd}[T])_\alpha.
\end{aligned}$$

Let $T = \{f_i(r_1, \ldots, r_n)\}$ where $r_1, \ldots, r_n \in W_{\tau_n}^{K^*(n,r)}(X_n)$ and assume that $\hat{\sigma}^{nd}[\{r_k\}_\alpha] = (\hat{\sigma}^{nd}[\{r_k\}])_\alpha, 1 \leq k \leq n$. Then

$$\begin{aligned}
\hat{\sigma}^{nd}[T_\alpha] &= \hat{\sigma}^{nd}[(\{f_i(r_1, \ldots, r_n)\})_\alpha] = \hat{\sigma}^{nd}[\{f_i(r_{\alpha(1)}, \ldots, r_{\alpha(n)}\}] \\
&= S_{nd}^n(\sigma^{nd}(f_i), \hat{\sigma}^{nd}[\{r_{\alpha(1)}\}], \ldots, \hat{\sigma}^{nd}[\{r_{\alpha(n)}\}]) \\
&= S_{nd}^n(\sigma^{nd}(f_i), \hat{\sigma}^{nd}[\{r_1\}_\alpha], \ldots, \hat{\sigma}^{nd}[\{r_n\}_\alpha]) \\
&= S_{nd}^n(\sigma^{nd}(f_i), (\hat{\sigma}^{nd}[\{r_1\}])_\alpha, \ldots, (\hat{\sigma}^{nd}[\{r_n\}])_\alpha) \\
&= (S_{nd}^n(\sigma^{nd}(f_i), \hat{\sigma}^{nd}[\{r_1\}], \ldots, \hat{\sigma}^{nd}[\{r_n\}])_\alpha \\
&= (\hat{\sigma}^{nd}[\{f_i(r_1, \ldots, r_n)\}])_\alpha = (\hat{\sigma}^{nd}[T])_\alpha.
\end{aligned}$$

If T is an arbitrary non-empty subset of $W_{\tau_n}^{K^*(n,r)}(X_n)$, then

$$\begin{aligned}
\hat{\sigma}^{nd}[T_\alpha] &= \bigcup_{t\in T} \hat{\sigma}^{nd}[\{t_\alpha\}] \\
&= \bigcup_{t\in T} (\hat{\sigma}^{nd}[\{t\}])_\alpha \\
&= \left(\bigcup_{t\in T} (\hat{\sigma}^{nd}[\{t\}])\right)_\alpha \\
&= (\hat{\sigma}^{nd}[T])_\alpha.
\end{aligned}$$

■

Consider the algebra $(P^*(W_{\tau_n}^{K^*(n,r)}(X_n)); S_{nd}^n)$ of type $(n+1)$, we have the following.

Theorem 12. *For $\sigma \in nd - Hyp^{K^*(n,r)}(\tau_n)$, the extension*

$$\hat{\sigma}^{nd} : P^*\left(W_{\tau_n}^{K^*(n,r)}(X_n)\right) \to P^*\left(W_{\tau_n}^{K^*(n,r)}(X_n)\right)$$

is an endomorphism on the algebra $\left(P^*\big(W_{\tau_n}^{K^*(n,r)}(X_n)\big); S_{nd}^n\right)$.

Proof. Let $T, T_1, \ldots, T_n \in P^*(W_{\tau_n}^{K^*(n,r)}(X_n))$. We will show by induction on the complexity of T that $\hat{\sigma}^{nd}[S_{nd}^n(T, T_1, \ldots, T_n)] = S_{nd}^n(\hat{\sigma}^{nd}[T], \hat{\sigma}^{nd}[T_1], \ldots, \hat{\sigma}^{nd}[T_n])$. Let $T = \{f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})\}$. Then

$$\begin{aligned}
&\hat{\sigma}^{nd}[S_{nd}^n(T, T_1, \ldots, T_n)]\\
&= \hat{\sigma}^{nd}[S_{nd}^n(\{f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})\}, T_1, \ldots, T_n)]\\
&= \hat{\sigma}^{nd}[\{f_i(t_{\alpha(1)}, \ldots, t_{\alpha(n)}) \mid t_{\alpha(1)} \in T_{\alpha(1)}, \ldots, t_{\alpha(n)} \in T_{\alpha(n)}\}]\\
&= S_{nd}^n(\sigma^{nd}(f_i), \hat{\sigma}^{nd}[\{t_{\alpha(1)} \mid t_{\alpha(1)} \in T_{\alpha(1)}\}], \ldots, \hat{\sigma}^{nd}[\{t_{\alpha(n)} \mid t_{\alpha(n)} \in T_{\alpha(n)}\}])\\
&= S_{nd}^n(\sigma^{nd}(f_i), \hat{\sigma}^{nd}[T_{\alpha(1)}], \ldots, \hat{\sigma}^{nd}[T_{\alpha(n)}])\\
&= S_{nd}^n(\sigma^{nd}(f_i), \hat{\sigma}^{nd}[T_1]_\alpha, \ldots, \hat{\sigma}^{nd}[T_n]_\alpha)\\
&= S_{nd}^n((\sigma^{nd}(f_i))_\alpha, \hat{\sigma}^{nd}[T_1], \ldots, \hat{\sigma}^{nd}[T_n])\\
&= S_{nd}^n(\hat{\sigma}^{nd}[\{f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})\}], \hat{\sigma}^{nd}[T_1], \ldots, \hat{\sigma}^{nd}[T_n])\\
&= S_{nd}^n(\hat{\sigma}^{nd}[T], \hat{\sigma}^{nd}[T_1], \ldots, \hat{\sigma}^{nd}[T_n]).
\end{aligned}$$

Let $T = \{f_i(r_1, \ldots, r_n)\}$ where $r_1, \ldots, r_n \in W_{\tau_n}^{K^*(n,r)}(X_n)$ and assume that

$$\hat{\sigma}^{nd}[S_{nd}^n(\{r_k\}, T_1, \ldots, T_n)] = S_{nd}^n\left(\hat{\sigma}^{nd}[\{r_k\}], \hat{\sigma}^{nd}[T_1], \ldots, \hat{\sigma}^{nd}[T_n]\right)$$

for all $1 \leq k \leq n$. Then

$$\begin{aligned}
&\hat{\sigma}^{nd}[S_{nd}^n(T, T_1, \ldots, T_n)]\\
&= \hat{\sigma}^{nd}[S_{nd}^n(\{f_i(r_1, \ldots, r_n)\}, T_1, \ldots, T_n)]\\
&= \hat{\sigma}^{nd}[\{f_i(s_1, \ldots, s_n) \mid s_1 \in S_{nd}^n(\{r_1\}, T_1, \ldots, T_n), \ldots, s_n \in S_{nd}^n(\{r_n\}, T_1, \ldots, T_n)\}]\\
&= S_{nd}^n(\sigma^{nd}(f_i), \hat{\sigma}^{nd}[\{s_1 \mid s_1 \in S_{nd}^n(\{r_1\}, T_1, \ldots, T_n)\}], \ldots,\\
&\quad \hat{\sigma}^{nd}[\{s_n \mid s_n \in S_{nd}^n(\{r_n\}, T_1, \ldots, T_n)\}])\\
&= S_{nd}^n(\sigma^{nd}(f_i), \hat{\sigma}^{nd}[S_{nd}^n(\{r_1\}, T_1, \ldots, T_n)], \ldots, \hat{\sigma}^{nd}[S_{nd}^n(\{r_1\}, T_1, \ldots, T_n)])\\
&= S_{nd}^n(\sigma^{nd}(f_i), S_{nd}^n(\hat{\sigma}^{nd}[\{r_1\}], \hat{\sigma}^{nd}[T_1], \ldots, \hat{\sigma}^{nd}[T_n]), \ldots, S_{nd}^n(\hat{\sigma}^{nd}[\{r_n\}],\\
&\quad \hat{\sigma}^{nd}[T_1], \ldots, \hat{\sigma}^{nd}[T_n]))\\
&= S_{nd}^n(S_{nd}^n(\sigma^{nd}(f_i), \hat{\sigma}^{nd}[\{r_1\}], \ldots, \hat{\sigma}^{nd}[\{r_n\}]), \hat{\sigma}^{nd}[T_1], \ldots, \hat{\sigma}^{nd}[T_n])\\
&= S_{nd}^n(\hat{\sigma}^{nd}[\{f_i(r_1, \ldots, r_n)\}], \hat{\sigma}^{nd}[T_1], \ldots, \hat{\sigma}^{nd}[T_n])\\
&= S_{nd}^n(\hat{\sigma}^{nd}[T], \hat{\sigma}^{nd}[T_1], \ldots, \hat{\sigma}^{nd}[T_n]).
\end{aligned}$$

If T is an arbitrary non-empty subset of $W_{\tau_n}^{K^*(n,r)}(X_n)$, then

$$\begin{aligned}
\hat{\sigma}^{nd}[S^n_{nd}(T,T_1,\ldots,T_n)] &= \hat{\sigma}^{nd}\left[\bigcup_{t\in T} S^n_{nd}(\{t\},T_1,\ldots,T_n)\right] \\
&= \bigcup_{t\in T} \hat{\sigma}^{nd}[S^n_{nd}(\{t\},T_1,\ldots,T_n)] \\
&= \bigcup_{t\in T} S^n_{nd}\left(\hat{\sigma}^{nd}[\{t\}],\hat{\sigma}^{nd}[T_1],\ldots,\hat{\sigma}^{nd}[T_n]\right) \\
&= S^n_{nd}\left(\bigcup_{t\in T}\hat{\sigma}^{nd}[\{t\}],\hat{\sigma}^{nd}[T_1],\ldots,\hat{\sigma}^{nd}[T_n]\right) \\
&= S^n_{nd}\left(\hat{\sigma}^{nd}[T],\hat{\sigma}^{nd}[T_1],\ldots,\hat{\sigma}^{nd}[T_n]\right).
\end{aligned}$$

■

The following corollary is a special case of Theorem 12.

Corollary 13 ([4], Theorem 11)**.** *A mapping* $\hat{\sigma}^{nd} : P\left(W^F_{\tau_n}(X_n)\right) \to P\left(W^F_{\tau_n}(X_n)\right)$ *is an endomorphism of* $\left(P\left(W^F_{\tau_n}(X_n)\right); S^n_{nd}\right)$.

For $\sigma_1^{nd},\sigma_2^{nd} \in$ nd-$Hyp^{K^*(n,r)}(\tau_n)$, define

$$\sigma_1^{nd} \circ_{nd} \sigma_2^{nd} = \hat{\sigma}_1^{nd} \circ \sigma_2^{nd}$$

where $\circ$ is the usual composition of mappings.

The property of $\hat{\sigma}^{nd}$ has shown as follows.

Lemma 14. *Let* $\sigma_1^{nd},\sigma_2^{nd} \in$ *nd-*$Hyp^{K^*(n,r)}(\tau_n)$. *Then*

$$\left(\sigma_1^{nd} \circ_{nd} \sigma_2^{nd}\right)^{\hat{}}[T] = (\hat{\sigma}_1^{nd} \circ \hat{\sigma}_2^{nd})[T].$$

Proof. Let $T = \left\{f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)})\right\}$ where $\alpha \in K^*(n,r)$. Then

$$\begin{aligned}
\left(\sigma_1^{nd} \circ_{nd} \sigma_2^{nd}\right)^{\hat{}}[T] &= \left(\sigma_1^{nd} \circ_{nd} \sigma_2^{nd}\right)^{\hat{}}[\{f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)})\}] \\
&= \left((\sigma_1^{nd} \circ_{nd} \sigma_2^{nd})(f_i)\right)_\alpha = \left((\hat{\sigma}_1^{nd} \circ \sigma_2^{nd})(f_i)\right)_\alpha \\
&= \left(\hat{\sigma}_1^{nd}[\sigma_2^{nd}(f_i)]\right)_\alpha = \hat{\sigma}_1^{nd}\left[(\sigma_2^{nd}(f_i))_\alpha\right] \\
&= \hat{\sigma}_1^{nd}\left[\hat{\sigma}_2^{nd}\left[\{f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)})\}\right]\right] \\
&= \hat{\sigma}_1^{nd}\left[\hat{\sigma}_2^{nd}[T]\right] = \left(\hat{\sigma}_1^{nd} \circ \hat{\sigma}_2^{nd}\right)[T].
\end{aligned}$$

Let $T = \{f_i(r_1,\ldots,r_n)\}$ where $r_1,\ldots,r_n \in W_{\tau_n}^{K^*(n,r)}(X_n)$ and assume that

$$\left(\sigma_1^{nd} \circ_{nd} \sigma_2^{nd}\right)^{\hat{}}[\{r_k\}] = (\hat{\sigma}_1^{nd} \circ \hat{\sigma}_2^{nd})[\{r_k\}]$$

for all $1 \leq k \leq n, n \in \mathbb{N}$. Then

$$\begin{aligned}
&\left(\sigma_1^{nd} \circ_{nd} \sigma_2^{nd}\right)^{\hat{}}[T]\\
&= \left(\sigma_1^{nd} \circ_{nd} \sigma_2^{nd}\right)^{\hat{}}[\{f_i(r_1, \ldots, r_n)\}]\\
&= S_{nd}^n\big(\left(\sigma_1^{nd} \circ_{nd} \sigma_2^{nd}\right)(f_i), \left(\sigma_1^{nd} \circ_{nd} \sigma_2^{nd}\right)^{\hat{}}[\{r_1\}], \ldots, \left(\sigma_1^{nd} \circ_{nd} \sigma_2^{nd}\right)^{\hat{}}[\{r_n\}]\big)\\
&= S_{nd}^n\big(\left(\hat{\sigma}_1^{nd} \circ \sigma_2^{nd}\right)(f_i), \left(\hat{\sigma}_1^{nd} \circ \hat{\sigma}_2^{nd}\right)[\{r_1\}], \ldots, \left(\hat{\sigma}_1^{nd} \circ \hat{\sigma}_2^{nd}\right)[\{r_n\}]\big)\\
&= S_{nd}^n\big(\hat{\sigma}_1^{nd}\left[\sigma_2^{nd}(f_i)\right], \hat{\sigma}_1^{nd}\left[\hat{\sigma}_2^{nd}[\{r_1\}]\right], \ldots, \hat{\sigma}_1^{nd}\left[\hat{\sigma}_2^{nd}[\{r_n\}]\right]\big)\\
&= \hat{\sigma}_1^{nd}\left[S_{nd}^n\big(\sigma_2^{nd}(f_i), \hat{\sigma}_2^{nd}[\{r_1\}], \ldots, \hat{\sigma}_2^{nd}[\{r_n\}]\big)\right]\\
&= \hat{\sigma}_1^{nd}\left[\hat{\sigma}_2^{nd}\left[\{f_i(r_1, \ldots, r_n)\}\right]\right] = \hat{\sigma}_1^{nd}\left[\hat{\sigma}_2^{nd}[T]\right] = \left(\hat{\sigma}_1^{nd} \circ \hat{\sigma}_2^{nd}\right)[T].
\end{aligned}$$

If T is an arbitrary non-empty subset of $W_{\tau_n}^{K^*(n,r)}(X_n)$, then

$$\begin{aligned}
\left(\sigma_1^{nd} \circ_{nd} \sigma_2^{nd}\right)^{\hat{}}[T] &= \bigcup_{t \in T} \left(\sigma_1^{nd} \circ_{nd} \sigma_2^{nd}\right)^{\hat{}}[\{t\}] = \bigcup_{t \in T} \left(\hat{\sigma}_1^{nd} \circ \hat{\sigma}_2^{nd}\right)[\{t\}]\\
&= \bigcup_{t \in T} \hat{\sigma}_1^{nd}\left[\hat{\sigma}_2^{nd}[\{t\}]\right] = \hat{\sigma}_1^{nd}\left[\bigcup_{t \in T} \hat{\sigma}_2^{nd}[\{t\}]\right]\\
&= \hat{\sigma}_1^{nd}\left[\hat{\sigma}_2^{nd}[T]\right] = \left(\hat{\sigma}_1^{nd} \circ \hat{\sigma}_2^{nd}\right)[T].
\end{aligned}$$

■

Lemma 15. *The binary operation $\circ_{nd}$ is associative.*

Proof. Let $\sigma_1^{nd}, \sigma_2^{nd}, \sigma_3^{nd} \in$ nd-$Hyp^{K^*(n,r)}(\tau_n)$. Then

$$\begin{aligned}
\sigma_1^{nd} \circ_{nd} \left(\sigma_2^{nd} \circ_{nd} \sigma_3^{nd}\right) &= \hat{\sigma}_1^{nd} \circ \left(\sigma_2^{nd} \circ_{nd} \sigma_3^{nd}\right)\\
&= \hat{\sigma}_1^{nd} \circ \left(\hat{\sigma}_2^{nd} \circ \sigma_3^{nd}\right)\\
&= \left(\hat{\sigma}_1^{nd} \circ \hat{\sigma}_2^{nd}\right) \circ \sigma_3^{nd}\\
&= \left(\sigma_1^{nd} \circ_{nd} \sigma_2^{nd}\right)^{\hat{}} \circ \sigma_3^{nd}\\
&= \left(\sigma_1^{nd} \circ_{nd} \sigma_2^{nd}\right) \circ_{nd} \sigma_3^{nd}.
\end{aligned}$$

■

Let $\sigma_{id}^{nd} \in$ nd-$Hyp^{K^*(n,r)}(\tau_n)$ be such that $\sigma_{id}^{nd}(f_i) = \{f_i(x_1, \ldots, x_n)\}$ for all $i \in I$.

Now we have the following lemma.

Lemma 16. *Let $T \subseteq W_{\tau_n}^{K^*(n,r)}(X_n)$. Then $\hat{\sigma}_{id}^{nd}[T] = T$.*

Proof. Let $T = \left\{f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})\right\}$ where $\alpha \in K^*(n,r)$. Then

$$\begin{aligned}
\hat{\sigma}_{id}^{nd}[T] &= \hat{\sigma}_{id}^{nd}\left[\{f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})\}\right]\\
&= \left(\sigma_{id}^{nd}(f_i)\right)_{\alpha} = \left(\{f_i(x_1, \ldots, x_n)\}\right)_{\alpha}\\
&= \left\{f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})\right\} = T.
\end{aligned}$$

Let $T = \{f_i(r_1, \ldots, r_n)\}$ where $r_1, \ldots, r_n \in W_{\tau_n}^{K^*(n,r)}(X_n)$ and assume that

$$\hat{\sigma}_{id}^{nd}[\{r_k\}] = \{r_k\}$$

for all $1 \leq k \leq n, n \in \mathbb{N}$. Then

$$\begin{aligned}
\hat{\sigma}_{id}^{nd}[T] &= \hat{\sigma}_{id}^{nd}[\{f_i(r_1, \ldots, r_n)\}] \\
&= S_{nd}^{n}\big(\sigma_{id}^{nd}(f_i), \hat{\sigma}_{id}^{nd}[\{r_1\}], \ldots, \hat{\sigma}_{id}^{nd}[\{r_n\}]\big) \\
&= S_{nd}^{n}(\{f_i(x_1, \ldots, x_n)\}, \{r_1\}, \ldots, \{r_n\}) \\
&= \{f_i(s_1, \ldots, s_n) \mid s_k \in \{r_k\}, 1 \leq k \leq n\} \\
&= \{f_i(r_1, \ldots, r_n)\} = T.
\end{aligned}$$

If T is an arbitrary non-empty subset of $W_{\tau_n}^{K^*(n,r)}(X_n)$, then

$$\hat{\sigma}_{id}^{nd}[T] = \bigcup_{t \in T} \hat{\sigma}_{id}^{nd}[\{t\}] = \bigcup_{t \in T} \{t\} = T.$$

■

Hence we have the following theorem.

Theorem 17. $\big(\text{nd-}Hyp^{K^*(n,r)}(\tau_n); \circ_{nd}, \sigma_{id}^{nd}\big)$ *is a monoid.*

Proof. The fact that $Hyp^{K^*(n,r)}(\tau_n)$ is a semigroup with respect to a binary operation $\circ_{nd}$ follows from Lemma 15. To prove that σ_{id}^{nd} is an identity element, let $\sigma^{nd} \in \text{nd-}Hyp^{K^*(n,r)}(\tau_n)$ and f_i be an operation symbol. We show that $\big(\sigma^{nd} \circ_{nd} \sigma_{id}^{nd}\big)(f_i) = \sigma^{nd}(f_i) = \big(\sigma_{id}^{nd} \circ_{nd} \sigma^{nd}\big)(f_i)$. From Lemma 16, it follows that

$$\begin{aligned}
(\sigma^{nd} \circ_{nd} \sigma_{id}^{nd})(f_i) &= \hat{\sigma}^{nd}[\sigma_{id}^{nd}(f_i)] \\
&= \hat{\sigma}^{nd}[\{f_i(x_1, \ldots, x_n)\}] \\
&= \sigma^{nd}(f_i) \\
&= \hat{\sigma}_{id}^{nd}[\sigma^{nd}(f_i)] \\
&= (\sigma_{id}^{nd} \circ_{nd} \sigma^{nd})(f_i).
\end{aligned}$$

■

Finally, we recall that nd-$Hyp^{F}(\tau_n)$ is a set of all mappings

$$\sigma^{nd} : \{f_i \mid i \in I\} \to P(W_{\tau_n}^{F}(X_n)).$$

The following corollary is the special case of the main result presented above:

Corollary 18 ([4], Theorem 16)**.** *The structure* $\big(\text{nd-}Hyp^{F}(\tau_n); \circ_{nd}, \sigma_{id}^{nd}\big)$ *is a monoid.*

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