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A RESULT ON PRIME RINGS WITH GENERALIZED DERIVATIONS

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Abstract

In this paper we investigate the following result. Let R be a prime ring, Q its symmetric Martindale quotient ring, C its extended centroid, I a nonzero ideal of R. If F and G are the two generalized derivation of R such that $(\mathbf{F}(\mathbf{xy}) + \mathbf{G}(\mathbf{yx}))^n - (\mathbf{xy} \mp \mathbf{yx})^n = 0$, for all $x, y \in I$, then either R is commutative or $F(x) = x$, $G(x) = \mp x$ for all $x \in R$ and $n = 1$.

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1. Introduction

Throughout this paper R represents a prime ring with center $Z(R)$, U stands for Utumi quotient ring with extended centroid C and Q appear for the symmetric Martindale quotient ring. For detailed conceptual knowledge about U, Q, C, one refer to [5].

An additive mapping $d : R \to R$ will be called a derivation on R if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. Let $q \in R$ be a fixed element. A map $d : R \to R$

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defined by $d(x) = [q, x] = qx - xq$, $x \in R$, is a derivation on R, which is called inner derivation defined by q. An additive map $F : R \to R$ is said to be a generalized derivation if there exists a derivation d of R such that, for all $x, y \in R$, $F(xy) = F(x)y + xd(y)$. Basic examples of generalized derivations are the usual derivations on R and left R-module mappings from R into itself. An important example is a map of the form $F(x) = ax + xb$, for some $a, b \in R$; such generalized derivations are called inner. In [12], Lee proved that every generalized derivation can be uniquely extended to a generalized derivation of Q and thus all generalized derivations of R implicitly assumed to be defined on the whole of Q. In particular, Lee proved the following: Let R be a semiprime ring. Then every generalized derivation F on a dense right ideal of R can be uniquely extended to Q and assumes the form $F(x) = ax + d(x)$ for some $a \in Q$ and a derivation d on Q.

In [6], Daif and Bell proved that if R is a semiprime ring with a nonzero ideal I and d is a derivation of R such that $d([x, y]) = [x, y]$ for all $x, y \in I$, then $I \subseteq Z(R)$. In particular, if R is prime ring, then R must be commutative. Authors [14] observe that: Let R be a prime ring, I a nonzero ideal of R and n a fixed positive integer. If R admits a generalized derivation F associated with a derivation d such that $(F([x, y]))^n = [x, y]$ for all $x, y \in I$, then either R is commutative or $n = 1$, $d = 0$ and F is the identity map on R.

Recently in [9], Huang and Davvaz consider the situation $(F([x, y]))^m = [x, y]^n$ for all $x, y \in R$. More precisely, they proved the following. Let R be a prime ring and m, n fixed positive integers. If R admits a generalized derivation F associated with a nonzero derivation d such that $(F([x, y]))^m = [x, y]^n$ for all $x, y \in R$, then R is commutative.

Very recently authors in [8] proved that: Let R be a non commutative prime ring, I a nonzero ideal of R, F a generalized derivation of R, $n \geq 1$ a fixed integer. If $0 \neq p$ such that $p(F(x)F(y) - xy)^n = 0$ for all $x, y \in I$, then there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$ with $\lambda^{2n} = 1$.

Carry on with the current the investigation we proved the following. Let R be a prime ring, I be a nonzero ideal of R, C represents the extended centroid of R and $n \geq 1$ is a fixed integer. If F and G are the two generalized derivation of R such that $(\mathbf{F}(\mathbf{xy}) + \mathbf{G}(\mathbf{yx}))^n - (\mathbf{xy} \mp \mathbf{yx})^n = 0$, for all $x, y \in I$, then either R is commutative and $F(x) = x$, $G(x) = \mp x$ for all $x \in R$ and $n = 1$.

2. Main results

We begin with the following lemmas as it's plays key role in our theorem.

Lemma 2.1. *Let $R = M_k(F)$ be a ring and $k \geq 2$ and $a, b, p, q \in R$. Suppose that $(axy + byx + [p, xy] + [q, yx])^n - (xy \pm yx)^n = 0$ for all $x, y \in R$, where $n \geq 1$ is a fixed integer. Then $a, b, p, q \in F \cdot I_k$.*

Proof. Let $a = (a_{ij})_{k\times k}$, $b = (b_{ij})_{k\times k}$, $p = (p_{ij})_{k\times k}$, $q = (q_{ij})_{k\times k}$, where $a_{ij}, b_{ij}, p_{ij}, q_{ij}$ in F. Denote e_{ij} the usual matrix with unit 1 in $(i,j)^{th}$ entry and zero elsewhere. We have $(ae_{12} + be_{12}e_{11} + [p, e_{12}] + [q, e_{12}])^n - (e_{12} \mp e_{12}e_{11})^n = 0$ and $(ae_{12} + b + pe_{12} - e_{12}p + qe_{12} - e_{12}q)^n - (e_{12} \mp e_{12})^n = 0$. That is, $\big((a+p)e_{12} - e_{12}(p+q) + qe_{12}\big)^n - (e_{12})^n = 0$. Multiply the above equation from right side by e_{12}, we get $\big(e_{12}(p+q)e_{12}\big)^n = 0$.

Next case. We have $\big((a+p)xy + (b+q)yx - xyp - yxq\big)^n - (xy \mp yx)^n = 0$. Choose $x = e_{11}$ and $y = e_{12}$, we obtain $\big((a+p)e_{11}e_{12} + (b+q)e_{12}e_{11} - e_{11}e_{12}p - e_{12}e_{11}q\big)^n - (e_{11}xe_{12} \mp e_{12}e_{11})^n = 0$. Multiplying right side by e_{12}, we find that $(-e_{12}pe_{12})^n = 0$ or $(-1)^n(e_{12}pe_{12})^n = 0$ or $(e_{12}pe_{12})^n = 0$, which implies that $a_{21} = 0$. Similarly $a_{12} = 0$. Hence $p = (p_{ij})$ is a diagonal matrix and $a_{ii} = a_{jj}$, where $i \neq j$. Hence p is a scalar matrix. Therefore, $p \in F \cdot I_k$. So, our identity reduces to $\big(axy + (b+q)yx - yxq\big)^n - (xy \mp yx)^n = 0$. Choose $x = e_{11}$ and $y = e_{12}$, we obtain $\big(ae_{11}e_{12} + (b+q)e_{12}e_{11} - e_{12}e_{11}q\big)^n - (e_{11}e_{12} \mp e_{12}e_{11})^n = 0$. Multiplying right side by e_{12}, we find that $(-e_{12}qe_{12})^n = 0$ or $(-1)^n(e_{12}qe_{12})^n = 0$ or $(e_{12}qe_{12})^n = 0$, which implies that $q_{12} = 0$. Similarly $q_{21} = 0$. We can get q is a diagonal matrix and hence q is a scalar matrix. Therefore, $q \in F \cdot I_k$. Hence, our identity reduces to $\big(axy + byx\big)^n - (xy \mp yx)^n = 0$. Choose $x = e_{11}$ and $y = e_{12}$, we get $\big(ae_{11}e_{12} + be_{12}e_{11}\big)^n - (e_{11}e_{12} \mp e_{12}e_{11})^n = 0$. Which implies that $\big(ae_{12} - (e_{12})^n = 0$. Left multiplying by e_{12}, we arrive at $(e_{12}ae_{12})^n = 0$. Which implies that $e_{12} = 0$ and $e_{21} = 0$. Use similar arguments, we find that $b \in F \cdot I_k$. ■

Lemma 2.2. *Let R be a prime ring, I be a nonzero ideal of R, C represents the extended centroid of R and $n \geq 1$ is a fixed integer. Suppose that for some $a, b, p, q \in R$, and $(axy + byx + [p, xy] + [q, yx])^n - (xy \mp yx)^n = 0$, for all $x, y \in I$, then $a, b, p, q \in C$.*

Proof. Since I satisfies the generalized polynomial identity

$$(2.1)\quad f(x,y) = (axy + byx + [p, xy] + [q, yx])^n - (xy \mp yx)^n \text{ for all } x, y \in R.$$

Hence U also satisfied the above GPI and $f(x,y) = 0$ for all $x, y \in U$ by [2].

We now consider that U does not satisfy any non-trivial GPI. By equation (2.1), we can say

$$(2.2)\quad ((a+p)xy + (b+q)yx - xyp - yxq)^n - (xy \mp yx)^n = 0 \text{ for all } x, y \in R.$$

Since x and y is given by $T = U *_C C\{x, y\}$, the free product of U and $C\{x, y\}$. If $p \notin C$, then $\{1, p\}$ is linearly independent over C. But if $q \notin span_C\{1, p\}$, then $\{1, p, q\}$ will be linearly independent over C. Therefore, we get a contradiction by equation (2.2). If $q \in span_C\{1, p\}$, then q can be written in the form for some scalars $\alpha, \gamma \in C$, $q = \alpha + \gamma p$. In this case, we will also arrive at contradiction by

(2.2). This clearly implies that $p \in C$. By using similar approach as above we can get $q, a+p, b+q \in C$ and hence a, b, p, q must be in C. Further we assume that (2.1) is a non trivial GPI for U. In such case, if C is infinite, we have $f(x, y) = 0$ for all $x, y \in U \bigotimes_C \bar{C}$, where $\bar{C}$ represents the algebraic closure of C. We can replace R by U or $U \bigotimes_C \bar{C}$ as C is finite or infinite respectively following the fact that both U and $U \bigotimes_C \bar{C}$ are centrally closed prime algebras [10]. Also, we may assume that $C = Z(R)$ and R is centrally closed C-algebra. By the theorem of Martindale [15], R is a primitive ring with nonzero socle $soc(R)$ and C as the associated division ring. Therefore, R is isomorphic to a dense ring of linear transformations of a vector space V over C from the theorem of Jacobson [7].

Let $dim_c V = k$, then $R \cong M_k(C)$ for $k \geq 1$. If $k = 1$, then R will be commutative and $a, b, p, q \in C$. If $k \geq 2$, then conclusion follows from Lemma 2.1.

If V is finite dimensional over C, then for any $e^2 = e \in sco(R)$, we have $eRe \cong M_l(C)$, where $l = dim_c Ve$. If $a, b, p, q \in C$, there is nothing to do. So, we consider all $a, b, p, q \notin C$. In this case at least one of a, b, p, q does not centralize the nonzero ideal $soc(R)$. Hence there exists $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in soc(R)$ such that either $[a, \alpha_1] = 0$ or $[b, \alpha_2] = 0$ or $[p, \alpha_3] = 0$ or $[q, \alpha_4] = 0$. An application of Litoff's theorem [1] enable us to take as idempotent $e \in soc(R)$ such that $a\alpha_1, \alpha_1 a, b\alpha_2, \alpha_2 b, p\alpha_3, \alpha_3 p, q\alpha_4, \alpha_4 q, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in eRe$. Therefore we can have $eRe \cong M_k(C)$ with $k = dim_c Ve$.

Replacing x by e and y by $ex(1-e)$ in (2.2) to get

$$((a+p)ex(1-e) - ex(1-e)p)^n - (ex(1-e))^n = 0 \quad \text{for all } x \in R. \tag{2.3}$$

The multiplication of equation (2.3) with $(1-e)$ from left yields that

$$(1-e)((a+p)ex(1-e))^n = 0 \quad \text{for all } x \in R. \tag{2.4}$$

A simple manipulation of equation (2.4) gives that $\{(1-e)(a+p)ex\}^{n+1} = 0$ for all $x \in R$. By Levitzki's [4], we can find $(1-e)(a+p)eR = 0$. This implies that $(1-e)(a+p)e = 0$. In the same way we can show that $(1-e)(b+q)e = 0$. Hence we have that

$$(a+p)e = e(a+p)e \quad \text{and} \quad (b+q)e = e(b+q)e.$$

Since R satisfies for all $x, y \in R$

$$e\{((a+p)exeye + (b+q)eyexe - exeyep - eyexeq)^n - (exeye \mp eyexe)^n\}e = 0. \tag{2.5}$$

and eRe satisfies

$$(e(a+p)exy + e(b+q)eyx - xyepe - yxeqe)^n - (xy \mp yx)^n = 0 \tag{2.6}$$

for all $x, y \in R$.

We have all eae, ebe, epe, eqe are central elements of eRe by above finite dimensional case. Which leads to a contradiction. This gives the assertion of lemma. ∎

Theorem 2.1. *Let R be a prime ring, I be a nonzero ideal of R, C represents the extended centroid of R and $n \geq 1$ is a fixed integer. If F and G are the two generalized derivation of R such that $(\mathbf{F}(\mathbf{xy}) + \mathbf{G}(\mathbf{yx}))^n - (\mathbf{xy} \mp \mathbf{yx})^n = 0$, for all $x, y \in I$, then either R is commutative or $F(x) = x$, $G(x) = \mp x$ for all $x \in R$ and $n = 1$.*

Proof. By our hypothesis, it is given that

$$(\mathbf{F}(\mathbf{xy}) + \mathbf{G}(\mathbf{yx}))^n - (\mathbf{xy} \mp \mathbf{yx})^n = 0 \quad \text{for all } x, y \in U. \tag{2.7}$$

Following [12], we can find $a, b \in U$ such that $F(x) = ax + \delta(x)$ and $G(x) = bx + \eta(x)$, where η, δ are derivations on U. Since I, R, U satisfy the same generalized polynomial identity and same differential identity by [2] and [11] respectively, we obtain

$$(axy + \delta(xy) + byx + \eta(yx))^n - (xy \mp yx)^n = 0 \quad \text{for all } x, y \in U. \tag{2.8}$$

This also implies that

$$(axy + \delta(x)y + x\delta(y) + byx + \eta(y)x + y\eta(x))^n - (xy \mp yx)^n = 0 \tag{2.9}$$

for all $x, y \in U$.

At this step the two case arises as below.

Case 1. Let us suppose that δ and η are two inner derivations of U, define as $\delta(x) = [p, x]$ and $\eta(x) = [q, x]$ for all $x \in U$, for some p, q belongs to U. Hence U satisfies

$$(axy + [p, xy] + byx + [q, yx])^n - (xy \mp yx)^n = 0 \quad \text{for all } x, y \in U. \tag{2.10}$$

With the help of Lemma 2.2, as all $a, b, p, q \in C$, then U satisfies

$$(axy + byx)^n - (xy \mp yx)^n = 0 \quad \text{for all } x, y \in U. \tag{2.11}$$

Equation (2.11) is a polynomial identity for U. Then by [3], there will be a field $\mathcal{F}$ such that $U \subseteq M_k(\mathcal{F})$, where $M_k(\mathcal{F})$ is the ring of $k \times k$ matrices of F. Also U and $M_k(\mathcal{F})$ satisfy the same polynomial identity. If $k = 1$, then U and R will obviously be commutative. Now investigate the case for $k \geq 2$ and putting $x = e_{ij}$ and $y = e_{jj}$ for $i \neq j$, then we get $(ae_{ij})^n - e_{ij} = 0$. For $n \geq 2, e_{ij} = 0$, a contradiction. this imply that $n = 1$ and $(a - 1)xy + (b \mp 1)yx = 0$ for all x, y in $M_k(\mathcal{F})$. If we put e_{ii} and e_{ij} in place of x and y respectively, then we get $(a - 1)e_{ij} = 0$, and hence $a = 1$. Again for $i \neq j$, put e_{ii} and e_{ij} in place of y and x respectively, then we get $(b \mp 1)e_{ij} = 0$, and hence $b = \mp 1$. With these values of $a = 1$ and $b = \mp 1$, we have $F(x) = x$ and $G(x) = \mp x$ for all $x \in U$.

Case 2. Let us assume that δ and η are not both inner derivations of U and also suppose that δ and η are linearly C-dependent modulo U_{int}. So, have $\sigma, \tau \in C$ such that $\sigma\delta + \tau\eta = a\delta_{q_1}$, and $a\delta_{q_1} = [q_1, x]$ for some $q_1 \in U$ and for all $x \in U$.

If $\sigma \neq 0$, then $\delta(x) = \lambda\eta(x) + [f, x]$ for all $x \in U$, where $\lambda = -\tau\sigma^{-1}$ and $f = \sigma^{-1}q_1$. Therefore, η can not be inner derivation of U. By equation (2.8), we find for all $x, y \in U$

$$(axy+\lambda\eta(x)y+\eta x\eta(y)+[f,xy]+byx+\eta(y)x+y\eta(x))^n-(xy\mp yx)^n=0. \tag{2.12}$$

From the theorem of Kharchenko [13], U satisfies the following

$$(axy+\lambda sy+\lambda xt+[f,xy]+byx+tx+ys)^n-(xy\mp yx)^n=0 \tag{2.13}$$

for all $x, y \in U$.

If R is commutative, then we have done. If R is non-commutative, then there exists $q \in U$ such that $q \neq U$. Substituting $[q, x]$ for x and $[q, y]$ for t in (2.13) to get

$$(axy+\lambda[q,x]y+\lambda x[q,y]+[f,xy]+byx+[q,y]x+y[q,x])^n-(xy\mp yx)^n=0 \tag{2.14}$$

for all $x, y \in U$.

Since U satisfies (2.14) we have

$$(axy+[\lambda q+f,xy]+byx+[q,yx])^n-(xy\mp yx)^n=0 \quad \text{for all } x,y\in U. \tag{2.15}$$

This observed that $q \in C$, which is a contradiction by Lemma 2.2.

Next consider $\sigma = 0$, then we have $\tau \neq 0$ and $f' = q_1\tau^{-1}$ such that $\eta(x) = [f', x]$ for all x in U. By equation (2.8), we can write

$$(axy+\delta(x)y+x\delta(y)+byx+[f',yx])^n-(xy\mp yx)^n=0 \quad \text{for all } x,y\in U. \tag{2.16}$$

Again using [13], U satisfies

$$(axy+sy+xt+byx+[f',yx])^n-(xy\mp yx)^n=0 \quad \text{for all } x,y\in U. \tag{2.17}$$

If we take $y = 0$ in above equation (2.17), then U satisfies $(xt)^n = 0$, for all $x, t \in U$. Hence by using the same arguments as above, R will be commutative. ■

Example 2.1. The following example justify that the theorem does not hold for arbitrary ring.

Let $R = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} | a, b \in \mathbb{Z} \right\}$ be a ring and $I = \left\{ \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} | b \in \mathbb{Z} \right\}$ be a non zero ideal of R. Define mappings $F, G, d, g : R \to R$ by $F\left(\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$, $G\left(\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} a & -b \\ 0 & 0 \end{bmatrix}$, $d\left(\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & -b \\ 0 & 0 \end{bmatrix}$, $g\left(\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & -2b \\ 0 & 0 \end{bmatrix}$.

Then F, G are generalized derivations with respective associated derivations d, g. We observe that $(\mathbf{F}(\mathbf{xy}) + \mathbf{G}(\mathbf{yx}))^n - (\mathbf{xy} \mp \mathbf{yx})^n = 0$ for all $x, y \in I$. But R is not commutative and $F(x) \neq x$ and $G(x) \neq \mp x$.

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