# ON ORDER PRIME DIVISOR GRAPHS OF FINITE GROUPS 

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#### Abstract

The order prime divisor graph $\mathscr{P} \mathscr{D}(G)$ of a finite group $G$ is a simple graph whose vertex set is $G$ and two vertices $a, b \in G$ are adjacent if and only if either $a b=e$ or $o(a b)$ is some prime number, where $e$ is the identity element of the group $G$ and $o(x)$ denotes the order of an element $x \in G$. In this paper, we establish the necessary and sufficient condition for the completeness of order prime divisor graph $\mathscr{P} \mathscr{D}(G)$ of a group $G$. Concentrating on the graph $\mathscr{P} \mathscr{D}\left(D_{n}\right)$, we investigate several properties like degrees, girth, regularity, Eulerianity, Hamiltonicity, planarity etc. We characterize some graph theoretic properties of $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{n}\right), \mathscr{P} \mathscr{D}\left(S_{n}\right), \mathscr{P} \mathscr{D}\left(A_{n}\right)$.


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## 1. Introduction

Defining graphs over groups help us for studying the interplay between algebraic properties and graph-theoretic properties and structures. For a finite group $G$, one can associate a certain type of graph, order prime divisor graph $\mathscr{P} \mathscr{D}(G)$,

[^0]and investigate the interplay between the group-theoretic properties of $G$ and the graph-theoretic properties of order prime divisor graph $\mathscr{P} \mathscr{D}(G)$.

There are a number of constructions of graphs from groups or semigroups in the literature. Here we begin by introducing some well-known graphs associated with semigroups or groups. In 1964, Bosák [1] studied certain graph over semigroups. Then Csákány and Pollák [2] defined intersection graphs of nontrivial proper subgroups of groups. In [9], Zelinka studied intersection graphs of nontrivial subgroups of finite Abelian groups. Later on, the intersection graph of ideals of rings was studied by Chakrabarty, Ghosh, Mukherjee and Sen [3].

In [6], Kelarev and Quinn introduced the notion of the (directed) power graph $P(G)$ of a group $G$ and described the structure of the (directed) power graphs of all finite abelian groups. According to them the (directed) power graph $P(G)$ of a group $G$ is a directed graph with the set $G$ of vertices, and with all edges $(u, v)$ such that $u \neq v$ and $v$ is the power of $u$. Later on, Chakrabarty, Ghosh and Sen [4] defined the undirected power graph $\mathscr{G}(S)$ of a semigroup $S$ as the undirected graph with vertex set $S$ and distinct vertices $a$ and $b$ are adjacent if $a^{m}=b$ or $b^{m}=a$ for some positive integer $m$.

In 2009, the authors [7] defined order prime graph and studied its properties. According to them the order prime graph $O P(\Gamma)$ of a finite group $\Gamma$ is a graph with vertex set $\Gamma$ and two vertices are adjacent in $O P(\Gamma)$ if and only if their orders are relatively prime in $\Gamma$.

In this paper, a new type of graph, called order prime divisor graph, is defined and studied its properties. For a finite group $G$, the order prime divisor graph of $G$, denoted by $\mathscr{P} \mathscr{D}(G)$, is a simple graph with vertex set $G$ and two vertices $a, b$ are adjacent in $\mathscr{P} \mathscr{D}(G)$ if and only if either $o(a b)=1$ or $o(a b)=p$ for some prime $p$, i.e., either $a$ and $b$ are inverse to each other or $a b$ is an element of prime order. Thus in $\mathscr{P} \mathscr{D}(G)$ two vertices $a, b$ are adjacent if and only if $o(a b)$ divides $p$ for some prime number $p$ and that's why we have named this graph as order prime divisor graph. We note that if $G$ is a group, then $o(a b)=o(b a)$ for any two elements $a, b \in G$. Clearly, by definition, order prime divisor graph $\mathscr{P} \mathscr{D}(G)$ of a finite group $G$ contains no isolated vertices. Following examples show that cyclic groups may have disconnected order prime divisor graphs where as non cyclic (even non-commutative) groups have connected order prime divisor graphs.


Figure 1. $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{4}\right)$
Figure 2. $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{2} \oplus \mathbb{Z}_{2}\right)$


Figure 3. $\mathscr{P} \mathscr{D}\left(D_{4}\right)$


Figure 4. $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{9}\right)$

In this paper, $\mathbb{Z}_{n}=\{\overline{0}, \overline{1}, \ldots, \overline{n-1}\}$ is the additive group of integers modulo $n, U(n)=\left\{\bar{x} \in \mathbb{Z}_{n}: \operatorname{gcd}(x, n)=1\right\}$ denotes the group of units of the ring $\left(\mathbb{Z}_{n},+, \cdot\right)$ of order $\varphi(n), D_{n}$ is the dihedral group of order $2 n, S_{n}$ is the permutation group on $n$ symbols, $A_{n}$ is the alternating group on $n$ symbols, $G L_{n}\left(F_{q}\right)$ denotes the general linear group over a finite field $F_{q}, S L_{n}\left(F_{q}\right)$ denotes the special linear group over a finite field $F_{q}, T$ denotes a non-commutative group containing 12 elements and is defined by $T=\left\{\langle a, b\rangle: o(a)=6, b^{2}=a^{3}, b a=a^{-1} b\right\}, K_{n}$ denotes the complete graph with $n$ vertices, $K_{r, s}$ denotes the complete bipartite graph or biclique where partite sets have sizes $r$ and $s, C_{n}$ denotes a cycle with $n$ vertices, $\operatorname{deg}(v)$ denotes the degree of a vertex $v, a \leftrightarrow b$ denotes that vertices $a, b$ are adjacent. For usual algebraic terms, we refer to [5], and we refer to [8] for graph-theoretic terms, definitions and notations.

## 2. Some properties of $\mathscr{P} \mathscr{D}(G)$

In this section, we study some interesting properties of $\mathscr{P} \mathscr{D}(G)$. We also establish here the necessary and sufficient condition for the order prime divisor graph $\mathscr{P} \mathscr{D}(G)$ of a finite group to be complete.

Theorem 2.1. For a finite group $G, \mathscr{P} \mathscr{D}(G)$ is complete if and only if each non identity element of $G$ is of prime order.

Proof. Let $G$ be a finite group with order of each non identity element is prime. Let $a, b$ be any two distinct elements of $G$. If $a b=e$, then $o(a b)=1$ and thus $a$ and $b$ are adjacent in $\mathscr{P} \mathscr{D}(G)$. Suppose $a b \neq e$. Then by the given hypothesis it follows that $o(a b)$ is some prime and hence these two elements $a$ and $b$ are adjacent in $\mathscr{P} \mathscr{D}(G)$. Therefore, between any two distinct vertices in $\mathscr{P} \mathscr{D}(G)$, there is an edge and hence $\mathscr{P} \mathscr{D}(G)$ is complete.

Conversely, let $G$ be finite group for which $\mathscr{P} \mathscr{D}(G)$ is complete. Let $a(\neq e)$ be any element of $G$. Since $\mathscr{P} \mathscr{D}(G)$ is complete, we must have an edge between the vertices $a$ and $e$. Therefore $o(a)=o(a e)$ must be prime. Consequently, order of any non identity element of $G$ is prime.

Corollary 2.2. Let $G$ be a group of prime order. Then $\mathscr{P} \mathscr{D}(G)$ is complete.
Corollary 2.3. $S_{3}, A_{4}, A_{5}$ are non-commutative groups whose order prime divisor graphs $\mathscr{P} \mathscr{D}\left(S_{3}\right), \mathscr{P} \mathscr{D}\left(A_{4}\right), \mathscr{P} \mathscr{D}\left(A_{5}\right)$ are complete.

Corollary 2.4. Let $G$ be a finite commutative group. Then $\mathscr{P} \mathscr{D}(G)$ is complete if and only if $G \cong \underbrace{\mathbb{Z}_{p} \oplus \mathbb{Z}_{p} \oplus \cdots \oplus \mathbb{Z}_{p}}_{n-\text { fold }}$ for some positive integer $n$.
Proof. First suppose that $G \cong \underbrace{\mathbb{Z}_{p} \oplus \mathbb{Z}_{p} \oplus \cdots \oplus \mathbb{Z}_{p}}_{n \text {-fold }}$ for some positive integer $n$. Then order of any non identity element of $G$ is $p$ and hence by Theorem 2.1, it follows that $\mathscr{P} \mathscr{D}(G)$ is complete.

Conversely, let $G$ be a finite commutative group such that $\mathscr{P} \mathscr{D}(G)$ is complete. Then by Theorem 2.1, it follows that every non identity element of $G$ is of prime order. Now by the structure theorem of finite commutative group, we have $G \cong \mathbb{Z}_{p_{1}}{ }^{n_{1}} \oplus \mathbb{Z}_{p_{2}}{ }^{n_{2}} \oplus \cdots \oplus \mathbb{Z}_{p_{k}}{ }^{{ }^{n}}$, , where $p_{1}, p_{2}, \ldots, p_{k}$ are primes (not necessarily distinct) and $n_{1}, n_{2}, \ldots, n_{k}$ are positive integers. First we show that $n_{1}=$ $n_{2}=\cdots=n_{k}=1$. On the contrary, suppose $n_{i}>1$, for some $i \in\{1,2, \ldots, k\}$. Then $G$ contains elements of composite order $p_{i}{ }^{n_{i}}$, a contradiction. Therefore, $n_{1}=n_{2}=\cdots=n_{k}=1$ and thus $G \cong \mathbb{Z}_{p_{1}} \oplus \mathbb{Z}_{p_{2}} \oplus \cdots \oplus \mathbb{Z}_{p_{k}}$. We now show that $p_{1}=p_{2}=\cdots=p_{k}$. Suppose $p_{i} \neq p_{j}$, for some $1 \leq i, j \leq k$. Then $G$ contains elements of composite order $p_{i} p_{j}$, which is again a contradiction. Therefore, $p_{1}=p_{2}=\cdots=p_{k}=p$ (say) and thus $G \cong \underbrace{\mathbb{Z}_{p} \oplus \mathbb{Z}_{p} \oplus \cdots \oplus \mathbb{Z}_{p}}_{k-\text { fold }}$.

Theorem 2.5. If a finite group $G$ has a subgroup of order $p$ (where $p$ is prime), then $\mathscr{P} \mathscr{D}(G)$ contains a clique with $p$ vertices.

Proof. Let $H$ be a subgroup of a finite group $G$ such that $|H|=p$ for some prime number $p$. Then every non identity element in $H$ is of order $p$. Therefore, $\mathscr{P} \mathscr{D}(H)$ is a complete graph containing $p$ vertices. Since $\mathscr{P} \mathscr{D}(H)$ is a subgraph of $\mathscr{P} \mathscr{D}(G)$, it follows that $\mathscr{P} \mathscr{D}(G)$ has a clique $\mathscr{P} \mathscr{D}(H)$ with $p$ vertices.

Remark 2.6. (i) The vertices of every clique in $\mathscr{P} \mathscr{D}(G)$ may not form a subgroup of $G$. For example, in the order prime divisor graph $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{4} \oplus \mathbb{Z}_{4}\right)$, $\{(\overline{0}, \overline{1}),(\overline{0}, \overline{3}),(\overline{2}, \overline{1}),(\overline{2}, \overline{3})\},\{(\overline{1}, \overline{0}),(\overline{1}, \overline{2}),(\overline{3}, \overline{0}),(\overline{3}, \overline{2})\},\{(\overline{1}, \overline{1}),(\overline{1}, \overline{3}),(\overline{3}, \overline{1}),(\overline{3}, \overline{3})\}$ form three different clique but none of them forms a subgroup of $\mathbb{Z}_{4} \oplus \mathbb{Z}_{4}$.
(ii) If the vertices of a clique in $\mathscr{P} \mathscr{D}(G)$ form a subgroup $H$ of $G$ then order of $H$ may not be prime. For example, in the order prime divisor graph $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{4} \oplus \mathbb{Z}_{4}\right)$, $\{(\overline{0}, \overline{0}),(\overline{0}, \overline{2}),(\overline{2}, \overline{0}),(\overline{2}, \overline{2})\}$ form a subgroup of $\mathbb{Z}_{4} \oplus \mathbb{Z}_{4}$, which is of order 4 .
Theorem 2.7. Let $G$ be a finite group and $p(\geq 5)$ be a prime such that $p$ divides $|G|$. Then $\mathscr{P} \mathscr{D}(G)$ is a non planar graph.

Proof. Let $G$ be a group of order $n$ and $p(\geq 5)$ be a prime such that $p$ divides $n$. Then by Cauchy's Theorem, $G$ has an element of order $p$ and hence a subgroup $H$ of order $p$. Now by Theorem 2.5, $\mathscr{P} \mathscr{D}(H)=K_{p}$ is a subgraph of $\mathscr{P} \mathscr{D}(G)$. Since $p \geq 5$, it follows that $\mathscr{P} \mathscr{D}(G)$ contains $K_{5}$ as a subgraph. Hence by Kuratowski's Theorem, it follows that $\mathscr{P} \mathscr{D}(G)$ is a non planar graph.

Remark 2.8. Converse of Theorem 2.7 is not true in general. From Corollary 2.3, we have $\mathscr{P} \mathscr{D}\left(S_{3}\right)$ and $\mathscr{P} \mathscr{D}\left(A_{4}\right)$ are complete graphs and hence they are non planar as well as non outerplanar, though there are no prime greater or equal to 5 dividing the order of the groups.

Corollary 2.9. If a finite group $G$ has a centre of prime order $p$, then $\mathscr{P} \mathscr{D}(G)$ contains a clique with $p$ vertices.

Corollary 2.10. If $m$ is a positive integer such that $2^{m}-1$ is prime (called Mersenne prime), then $\mathscr{P} \mathscr{D}\left(G L_{n}\left(F_{2^{m}}\right)\right)$ has a clique with $2^{m}-1$ vertices.

Proof. Now $Z\left(G L_{n}\left(F_{2^{m}}\right)\right)=\left\{A=\left(a_{i j}\right)_{n \times n}: a_{i j}=0,1 \leq i \neq j \leq n ; a_{i i}=a \in\right.$ $\left.F_{2^{m}} \backslash\{0\}\right\}(n>1)$ implies $\left|Z\left(G L_{n}\left(F_{2^{m}}\right)\right)\right|=2^{m}-1$. Hence by Corollary 2.9, we have $\mathscr{P} \mathscr{D}\left(G L_{n}\left(F_{2^{m}}\right)\right)$ has a clique with $2^{m}-1$ vertices.

Corollary 2.11. Let $F_{q}$ be a finite field with $q$ elements and $n$ be a positive integers such that $\operatorname{gcd}(n, q-1)$ is prime. Then $\mathscr{P} \mathscr{D}\left(S L_{n}\left(F_{q}\right)\right)$ has a clique with $\operatorname{gcd}(n, q-1)$ vertices.

Proof. Let $n$ be a positive integer such that $\operatorname{gcd}(n, q-1)$ is prime. For $n(>1)$, we have $Z\left(S L_{n}\left(F_{q}\right)\right)=\left\{A=\left(a_{i j}\right)_{n \times n}: a_{i j}=0, i \neq j ; a_{i i}=a \in F_{q} \backslash\{0\}\right.$; $\left.a^{n}=1\right\}$ implies $\left|Z\left(S L_{n}\left(F_{q}\right)\right)\right|=\operatorname{gcd}(n, q-1)$. Hence by Corollary 2.9, we have $\mathscr{P} \mathscr{D}\left(S L_{n}\left(F_{q}\right)\right)$ has a clique with $\operatorname{gcd}(n, q-1)$ vertices.

Recall that the girth of a graph with at least one cycle is the length of its shortest cycle. A graph with no cycle has infinite girth. A graph with no cycle is said to be an acyclic graph. A forest is an acyclic graph whereas a tree is a connected acyclic graph.

Theorem 2.12. Let $G$ be a finite group and $p$ is an odd prime such that $p$ divides $|G|$. Then $\mathscr{P} \mathscr{D}(G)$ is neither bipartite nor a tree and $\operatorname{girth}(\mathscr{P} \mathscr{D}(G))=3$.

Proof. Let $G$ be a group of order $n$ and $p$ be an odd prime such that $p$ divides $n$. Then $G$ contains an element $x$ of order $p$. Since $p$ is an odd prime, we have $x \neq x^{-1}$ and $o\left(x^{-1}\right)=p$. This implies $x \leftrightarrow e \leftrightarrow x^{-1} \leftrightarrow x$ forms a cycle in $\mathscr{P} \mathscr{D}(G)$ of length 3. Since $\mathscr{P} \mathscr{D}(G)$ is a simple graph, we must have $\operatorname{girth}(\mathscr{P} \mathscr{D}(G))=3$. Also $\mathscr{P} \mathscr{D}(G)$ contains an odd cycle implies $\mathscr{P} \mathscr{D}(G)$ is neither bipartite nor a tree.

Theorem 2.13. If $G_{1}, G_{2}$ are two finite groups such that $G_{1} \cong G_{2}$, then $\mathscr{P} \mathscr{D}\left(G_{1}\right)$ $\cong \mathscr{P} \mathscr{D}\left(G_{2}\right)$.

Proof. Let $G_{1}$ and $G_{2}$ be two finite groups and $\psi: G_{1} \longrightarrow G_{2}$ be an isomorphism. Let $a, b \in G_{1}$ be adjacent in $\mathscr{P} \mathscr{D}\left(G_{1}\right)$. Then either $o(a b)=1$ or $o(a b)=p$ for some prime $p$. Since $\psi$ is an isomorphism, we have $o(\psi(a) \psi(b))=o(\psi(a b))=$ $o(a b)$. If $o(a b)=1$, then $o(\psi(a) \psi(b))=1$ and hence $\psi(a)$ and $\psi(b)$ are adjacent in $\mathscr{P} \mathscr{D}\left(G_{2}\right)$. On the other hand, if $o(a b)=p$, then $o(\psi(a) \psi(b))=p$ and hence $\psi(a)$ is adjacent to $\psi(b)$ in $\mathscr{P} \mathscr{D}\left(G_{2}\right)$. Conversely, if $\psi(a)$ and $\psi(b)$ are adjacent in $\mathscr{P} \mathscr{D}\left(G_{2}\right)$, one can easily check that $a$ and $b$ are adjacent in $\mathscr{P} \mathscr{D}\left(G_{1}\right)$. Hence $\mathscr{P} \mathscr{D}\left(G_{1}\right) \cong \mathscr{P} \mathscr{D}\left(G_{2}\right)$.

Remark 2.14. The converse of Theorem 2.13 is not true in general. For example we consider two non-isomorphic groups $\mathbb{Z}_{27} \oplus \mathbb{Z}_{3}$ and $\mathbb{Z}_{9} \oplus \mathbb{Z}_{9}$. Then both the graphs $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{27} \oplus \mathbb{Z}_{3}\right)$ and $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{9} \oplus \mathbb{Z}_{9}\right)$ have five components. Out of these five components one component is a 8-regular graph with 9 vertices and each of remaining four components is a 9-regular graph with 18 vertices. Therefore, $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{27} \oplus \mathbb{Z}_{3}\right) \cong \mathscr{P} \mathscr{D}\left(\mathbb{Z}_{9} \oplus \mathbb{Z}_{9}\right)$ though $\mathbb{Z}_{27} \oplus \mathbb{Z}_{3} \nsubseteq \mathbb{Z}_{9} \oplus \mathbb{Z}_{9}$.

Corollary 2.15. For a finite group $G, \operatorname{Aut}(G) \subseteq \operatorname{Aut}(\mathscr{P} \mathscr{D}(G))$.
Proof. Follows from Theorem 2.13.
Theorem 2.16. Let $G$ be a finite commutative group. Then $\mathscr{P} \mathscr{D}(G)$ has at least two pendant vertices if and only if $G \cong \mathbb{Z}_{2^{r}}$, for some positive integer $r$.

Proof. Let $G \cong \mathbb{Z}_{2^{r}}$, for some positive integer $r$. Since $\overline{2^{r-1}} \in \mathbb{Z}_{2^{r}}$ is the unique element of order 2 , we have $\operatorname{deg}(\overline{0})=1=\operatorname{deg}\left(\overline{2^{r-1}}\right)$ in $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{2^{r}}\right)$. Hence $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{2^{r}}\right)$ has at least two pendant vertices.

Conversely, let $G$ be a finite commutative group of order $n$ such that $\mathscr{P} \mathscr{D}(G)$ contains at least two pendant vertices. First we prove that $n$ has no odd prime divisor. For this, let $p$ be any odd prime divisor of $n$. Then $G$ contains at least two elements $x$ and $y$ of order $p$. Now let $a \in G$ be any element. Then $a \leftrightarrow a^{-1} x$ and $a \leftrightarrow a^{-1} y$ in $\mathscr{P} \mathscr{D}(G)$ and thus $\operatorname{deg}(a) \geq 2$ in $\mathscr{P} \mathscr{D}(G)$. Hence $\mathscr{P} \mathscr{D}(G)$ contains no pendant vertices, a contradiction. Therefore, 2 is the only one divisor of $n$ and hence $G \cong \mathbb{Z}_{2^{n_{1}}} \oplus \mathbb{Z}_{2^{n_{2}}} \oplus \cdots \oplus \mathbb{Z}_{2^{n_{k}}}$ for some positive integers $n_{1}, n_{2}, \ldots, n_{k}$. We now show that $k=1$. On the contrary, let $k>1$. Then $G$ contains at least three elements of order 2 and hence degree of every vertex in $\mathscr{P} \mathscr{D}(G)$ is at least 3. Thus $\mathscr{P} \mathscr{D}(G)$ contains no pendant vertices, which is again a contradiction. Therefore, $k=1$ and consequently $G \cong \mathbb{Z}_{2^{r}}$, for some positive integer $r$.

Corollary 2.17. For a finite commutative group $G, \mathscr{P} \mathscr{D}(G)$ is a forest if and only if either $G \cong \mathbb{Z}_{2}$ or $G \cong \mathbb{Z}_{4}$.

Proof. First we assume that $G$ is isomorphic to either $\mathbb{Z}_{2}$ or $\mathbb{Z}_{4}$. Then clearly $\mathscr{P} \mathscr{D}(G)$ is a forest.

Conversely, let $G$ be a finite commutative group such that $\mathscr{P} \mathscr{D}(G)$ is a forest. We know that every tree with at least two vertices has at least two pendant vertices and every component of a forest is a tree. Hence by Theorem 2.16, $G$ is of the form $\mathbb{Z}_{2^{r}}$, for some positive integer $r$. We now show that $r$ must be equal to 1 or 2 . On the contrary, let $r \geq 3$. Then $\overline{1} \leftrightarrow \overline{2^{r-1}-1} \leftrightarrow \overline{2^{r-1}+1} \leftrightarrow \overline{2^{r}-1} \leftrightarrow \overline{1}$ forms a cycle in $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{2^{r}}\right)$. Hence $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{2^{r}}\right)$ is not a forest, a contradiction. This contradiction ensures that $r \leq 2$. Consequently, $G$ is isomorphic to either $\mathbb{Z}_{2}$ or $\mathbb{Z}_{4}$.

We need a result that follows from [5, Chapter 9, Section 9.5, Corollary 20].
Theorem 2.18. Let $n \geq 2$ be an integer with factorization $n=p_{1}{ }^{k_{1}} p_{2}{ }^{k_{2}} \cdots p_{r}{ }^{k_{r}}$ in $\mathbb{Z}$, where $p_{1}, p_{2}, \ldots, p_{r}$ are distinct primes. Then
(i) $U(n) \cong U\left(p_{1}^{k_{1}}\right) \times U\left(p_{2}^{k_{2}}\right) \times \cdots \times U\left(p_{r}^{k_{r}}\right)$,
(ii) $U\left(2^{\alpha}\right) \cong \mathbb{Z}_{2} \times \mathbb{Z}_{2^{\alpha-2}}$, for all $\alpha \geq 2$,
(iii) $U\left(p^{\alpha}\right) \cong \mathbb{Z}_{p^{\alpha-1}(p-1)}$ for any odd prime $p$.

Theorem 2.19. For any integer $n \geq 3$, the order prime divisor graph $\mathscr{P} \mathscr{D}\left(U\left(2^{n}\right)\right)$ has no pendant vertices.
Proof. For any $n \geq 3, U\left(2^{n}\right) \cong \mathbb{Z}_{2} \oplus \mathbb{Z}_{2^{n-2}}$ and hence by Theorem 2.16 , it follows that $\mathscr{P} \mathscr{D}\left(U\left(2^{n}\right)\right)$ has no pendant vertices.

Corollary 2.20. For any odd prime $p$ and for any positive integer $n$, the order prime divisor graph $\mathscr{P} \mathscr{D}\left(U\left(p^{n}\right)\right)$ has no pendant vertices.

Proof. For any odd prime $p, U\left(p^{n}\right) \cong \mathbb{Z}_{p^{n-1}(p-1)}$. Therefore, by Theorem 2.16, it follows that for any odd prime $p$ and for positive integer $n$, the order prime divisor graph $\mathscr{P} \mathscr{D}\left(U\left(p^{n}\right)\right)$ has no pendant vertices.

## 3. ORdER PRIME DIVISOR GRAPH OF THE GROUP $\mathbb{Z}_{n}$

A graph $G$ is said to be a $k$-regular graph ( $k$ is a non negative integer) if the degree of each vertex of $G$ is $k$. In this section we study $k$-regular order prime divisor graph of the group $\left(\mathbb{Z}_{n},+\right)$. From Corollary 2.2, it follows that order prime divisor graph of any group of prime order $p$ is $(p-1)$-regular. In this section we study order prime divisor graph of any cyclic group of order $2 p$, where $p$ is prime.

Theorem 3.1. For any cyclic group $G$ of order $2 p$, where $p$ is an odd prime, $\mathscr{P} \mathscr{D}(G)$ is a p-regular graph.

Proof. Let $G$ be a cyclic group of order $2 p$. Then $G$ contains exactly one element of order $1,(p-1)$ elements of order $p$, one element of order 2 and $(p-1)$ elements of order $2 p$. Hence $G$ has exactly $(p-1)+1=p$ elements of prime order. If $x$ is any element of $G$, then $o(x)=1$ or 2 or $p$ or $2 p$. We consider the following cases.

Case 1. Suppose $o(x)=1$, then $x=e$. Now $\operatorname{deg}(e)$ in $\mathscr{P} \mathscr{D}(G)$ is exactly equal to the number of prime order elements in $G$. Therefore, $\operatorname{deg}(e)=p$ in $\mathscr{P} \mathscr{D}(G)$.

Case 2. Assume that $o(x)=2$. Since $G$ is cyclic so $G$ has only one element of order 2 . Clearly $x$ is adjacent to $e$. Also for any other element $y \in G$ with $o(y)=p$, we see that $x$ is adjacent to $x^{-1} y$. Thus the number of such $y$ is $(p-1)$. Therefore the total number of adjacent vertices of $x$ in $\mathscr{P} \mathscr{D}(G)$ is $1+(p-1)=p$.

Case 3. Suppose $o(x)=p$. There are $(p-1)$ elements of order $p$. In this case $e$ and $x^{-1}$ are both adjacent to $x$. Moreover, for any other prime order element $z\left(\neq x, x^{2}\right) \in G$, we see that $x$ is adjacent to $x^{-1} z$. The number of such $z$ is $(p-2)[(p-3)$ elements of order $p$ and one element of order 2$]$. Therefore the total number of adjacent vertices of $x$ in $\mathscr{P} \mathscr{D}(G)$ is $1+1+(p-2)=p$.

Case 4. Suppose that $o(x)=2 p$. Then obviously $o\left(x^{2}\right)=p$. Clearly, $x^{-1}$ is adjacent to $x$. Moreover, for any other prime order element $u\left(\neq x^{2}\right) \in G$, we see that $x$ is adjacent to $x^{-1} u$. The number of such $u$ is $(p-1)[(p-2)$ elements of order $p$ and one element of order 2]. Hence the total number of adjacent vertices of $x$ in $\mathscr{P} \mathscr{D}(G)$ is $1+(p-1)=p$.

Therefore, considering all the cases we have $\operatorname{deg}(x)=p$ in $\mathscr{P} \mathscr{D}(G)$. Since $x \in G$ is arbitrary, we must have $\mathscr{P} \mathscr{D}(G)$ is a $p$-regular graph.

Corollary 3.2. For any odd prime $p, \mathscr{P} \mathscr{D}\left(\mathbb{Z}_{2 p}\right)$ is a p-regular graph.
Remark 3.3. Figure 1 shows that $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{4}\right)$ is 1 -regular and thus Theorem 3.1 is not true when $p$ is even prime.
Lemma 3.4. $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{2^{n}}\right)$ is disconnected for $n \geq 2$ and non regular for $n \geq 3$.
Proof. For $n \geq 2$, the set $\left\{\overline{0}, \overline{2^{n-1}}\right\}$ forms a connected component of $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{2^{n}}\right)$ and thus $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{2^{n}}\right)$ is a disconnected graph.

Now, we consider the graph $\mathscr{P} \mathscr{D}\left(\underline{\mathbb{Z}_{2^{n}}}\right)$ for $n \geq 3$. In this graph $\operatorname{deg}(\overline{1})=2$ because $\overline{1}$ is adjacent to $\overline{2^{n}-1}$ and $\overline{2^{n-1}-1}$, whereas $\operatorname{deg}\left(\overline{2^{n-1}}\right)=1$, since $o\left(\overline{2^{n-1}}\right)=2$ in the group $\mathbb{Z}_{2^{n}}$. Hence $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{2^{n}}\right)$ is non regular.

Lemma 3.5. For any odd prime $p$ and any integer $n \geq 2, \mathscr{P} \mathscr{D}\left(\mathbb{Z}_{p^{n}}\right)$ is a non regular graph.

Proof. Since $\mathbb{Z}_{p^{n}}$ is a cyclic group, the number of elements of order $p$ is $\varphi(p)=$ $p-1$. This implies $\operatorname{deg}(\overline{0})=p-1$ in $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{p^{n}}\right)$. Again, $\overline{1}$ is adjacent to
$\overline{p^{n-1}-1}, \overline{2 p^{n-1}-1}, \overline{3 p^{n-1}-1}, \ldots, \overline{(p-1) p^{n-1}-1}$ and $\overline{p^{n}-1}$ in $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{p^{n}}\right)$. Therefore, $\operatorname{deg}(\overline{1})=p$ in $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{p^{n}}\right)$. Hence $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{p^{n}}\right)$ is not a regular graph.

Lemma 3.6. If $n \neq p, 2 p$ for some prime $p$ and if $n=p_{1}^{r_{1}} p_{2}^{r_{2}} \cdots p_{m}^{r_{m}}$, where $p_{1}, p_{2}, \ldots, p_{m}$ are distinct primes and $r_{1}, r_{2}, \ldots, r_{m}$ are positive integers, then $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{n}\right)$ is not a regular graph.

Proof. Similar to the proof of Lemma 3.5, we can prove that $\operatorname{deg}(\overline{0})=\varphi\left(p_{1}\right)+$ $\varphi\left(p_{2}\right)+\cdots+\varphi\left(p_{m}\right)$ and $\operatorname{deg}(\overline{1})=\varphi\left(p_{1}\right)+\varphi\left(p_{2}\right)+\cdots+\varphi\left(p_{m}\right)+1$ in $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{n}\right)$. Hence $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{n}\right)$ is not a regular graph.

We are now in a position to characterize all finite cyclic groups $\mathbb{Z}_{n}$ for which $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{n}\right)$ is regular.

Theorem 3.7. $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{n}\right)$ is a regular graph if and only if $n=p$ or $2 p$ for some prime $p$.

Proof. First we assume that $n=p$ for some prime $p$. Then by Theorem 2.1, it follows that $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{p}\right)$ is a complete graph with $p$ vertices and hence $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{p}\right)$ is a $(p-1)$-regular graph. On the other hand if $n=2 p$ for some odd prime $p$, then by Corollary 3.2, we have $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{2 p}\right)$ is a $p$-regular graph. Moreover, from Figure 1, we see that $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{4}\right)$ is 1-regular.

Converse part follows from Lemma 3.6.
Corollary 3.8. For any positive integer $n(\neq 4)$, if $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{n}\right)$ is regular then it is connected.

Proof. From Theorem 3.7, we have $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{n}\right)$ is regular if and only if $n=p$ or $2 p$ for some prime $p$. Now for any prime $p, \mathscr{P} \mathscr{D}\left(\mathbb{Z}_{p}\right)$ is complete and hence it is connected. If $p$ is an odd prime, then by Theorem 3.7, we have $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{2 p}\right)$ is a $p$-regular graph with $2 p$ vertices and hence it is connected.

Remark 3.9. The converse of Corollary 3.8 is not true in general. The following graph shows that $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{15}\right)$ is connected but not regular.

Definition 3.10 [8]. The line graph of a graph $G$, written $L(G)$, is the graph whose vertices are the edges of $G$, with $e f \in E(L(G))$ when $e=u v$ and $f=v w$ in $G$.

Theorem 3.11. For any odd prime $p$,
(i) the line graph $L\left(\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{p}\right)\right)$ of $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{p}\right)$ is $(2 p-4)$-regular,
(ii) the line graph $L\left(\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{2 p}\right)\right)$ of $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{2 p}\right)$ is $(2 p-2)$-regular.


Figure 5. $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{15}\right)$
Proof. (i) Let $p$ be an odd prime. Then $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{p}\right)$ is $(p-1)$-regular. Let $e$ be any edge of $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{p}\right)$. Then $e$ represents a vertex $e_{v}$ in $L\left(\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{p}\right)\right)$. Let $a_{e}$ and $b_{e}$ be the end vertices of the edge $e$ in $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{p}\right)$. Again $\operatorname{deg}\left(a_{e}\right)=\operatorname{deg}\left(b_{e}\right)=(p-1)$ in $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{p}\right)$. Since $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{p}\right)$ is simple, the edge $e$ has the vertex $a_{e}$ in common with $(p-2)$ edges and the vertex $b_{e}$ in common with $(p-2)$ edges in $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{p}\right)$. Hence $\operatorname{deg}\left(e_{v}\right)=(p-2)+(p-2)=(2 p-4)$ in $L\left(\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{p}\right)\right)$. Therefore $L\left(\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{p}\right)\right)$ is $(2 p-4)$-regular graph.
(ii) Since $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{2 p}\right)$, where $p$ is any odd prime, is $p$-regular, so by the similar argument as in (i) of this theorem, we have the line graph $L\left(\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{2 p}\right)\right)$ of $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{2 p}\right)$ is $(2 p-2)$-regular.

Theorem 3.12 (Dirac [8]). Let $G$ be a simple graph with $n(>2)$ vertices. If $\operatorname{deg}(v) \geq \frac{n}{2}$ for every vertex $v$ of $G$, then $G$ is Hamiltonian.

Theorem 3.13. If $n=p$ or $2 p$, where $p$ an odd prime, then $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{n}\right)$ is a Hamiltonian graph.

Proof. Let $p$ be an odd prime. From Theorem 2.1, it follows that $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{p}\right)$ is complete and hence it is Hamiltonian. On the other hand from Theorem 3.7, we have $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{2 p}\right)$ is a $p$-regular graph and by Theorem 3.12, it follows that $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{2 p}\right)$ is a Hamiltonian graph.

Theorem 3.14. For a finite commutative group $G, \mathscr{P} \mathscr{D}(G)$ is a bipartite graph if and only if $G \cong \mathbb{Z}_{2^{r}}$, for some positive integer $r$.

Proof. Let $G \cong \mathbb{Z}_{2^{r}}$, for some positive integer $r$. If $G$ is isomorphic to either $\mathbb{Z}_{2}$ or $\mathbb{Z}_{2^{2}}$ then clearly $\mathscr{P} \mathscr{D}(G)$ is a bipartite graph. We now consider $r \geq 3$. Then in the order prime divisor graph $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{2^{r}}\right), \operatorname{deg}(\overline{0})=\operatorname{deg}\left(\overline{2^{r-1}}\right)=\operatorname{deg}\left(\overline{2^{r-2}}\right)=$ $\operatorname{deg}\left(\overline{3 \cdot 2^{r-2}}\right)=1$ and also $\left\{\overline{0}, \overline{2^{r-1}}\right\},\left\{\overline{2^{r-2}}, \overline{3 \cdot 2^{r-2}}\right\}$ form two components. Let $\bar{x} \in \mathbb{Z}_{2^{r}}$ such that $\bar{x} \notin\left\{\overline{0}, \overline{2^{r-1}}, \overline{2^{r-2}}, \overline{3 \cdot 2^{r-2}}\right\}$. Then $\left\{\bar{x}, \overline{2^{r-1}-x}, \overline{2^{r-1}+x}\right.$, $\left.\overline{2^{r}-x}\right\}$ forms a component which is isomorphic to $C_{4}$. Hence every component of $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{2^{r}}\right)$ is isomorphic to either $K_{2}$ or $C_{4}$. Hence $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{2^{r}}\right)$ has no odd cycle and consequently, $\mathscr{P} \mathscr{D}(G)$ is a bipartite graph.

Conversely, let $G$ be a finite commutative group of order $n$ such that $\mathscr{P} \mathscr{D}(G)$ is a bipartite graph. We now show that $n$ is not divisible by any odd prime $p$. If any odd prime $p$ divides $n$, then $G$ contains an element $x$ of order $p$. Since $p$ is an odd prime, we have $x \neq x^{-1}$ and $o\left(x^{-1}\right)=p$. This implies $x \leftrightarrow e \leftrightarrow x^{-1} \leftrightarrow x$ forms a cycle in $\mathscr{P} \mathscr{D}(G)$ of length 3 . Therefore $\mathscr{P} \mathscr{D}(G)$ contains an odd cycle, which contradicts that $\mathscr{P} \mathscr{D}(G)$ is a bipartite graph. Hence $n$ is of the form $2^{r}$, for some positive integer $r$. Therefore $G \cong \mathbb{Z}_{2^{n_{1}}} \oplus \mathbb{Z}_{2^{n_{2}}} \oplus \cdots \oplus \mathbb{Z}_{2^{n_{k}}}$ for some positive integers $n_{1}, n_{2}, \ldots, n_{k}$. We now show that $k=1$. On the contrary, let $k>1$. Then $\left.\left\{\left(\frac{2^{n_{1}-1}}{}, \overline{0}, \ldots, \overline{0}\right),\left(\overline{0}, \overline{2^{n_{2}-1}}, \ldots, \overline{0}\right), \overline{2^{n_{1}-1}}, \frac{2^{n_{2}-1}}{}, \ldots, \overline{0}\right)\right\}$ forms a triangle. Thus $\mathscr{P} \mathscr{D}(G)$ is not bipartite graph, which is again a contradiction. Therefore, $k=1$ and thus $G \cong \mathbb{Z}_{2^{r}}$, for some positive integer $r$.

Theorem 3.15. Let $p$ be an odd prime and $n \geq 2$ is an integer. Then $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{p^{n}}\right)$ have one $(p-1)$-regular component and $\frac{p^{n-1}-1}{2}$ components each of them is $p$ regular.

Proof. In $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{p^{n}}\right),\left\{\overline{0}, \overline{p^{n-1}}, \overline{2 p^{n-1}}, \ldots, \overline{(p-1) p^{n-1}}\right\}$ forms a component with $p$ vertices and this component is of $(p-1)$-regular.

Let $A_{i}=\left\{\bar{i}, \overline{p^{n-1}-i}, \overline{2 p^{n-1}-i}, \ldots, \overline{p^{n}-i}, \overline{p^{n-1}+i}, \overline{2 p^{n-1}+i}, \ldots\right.$,
$\left.\overline{(p-1) p^{n-1}+i}\right\}=U_{i} \cup V_{i}$, where $U_{i}=\left\{\overline{p^{n-1}-i}, \overline{2 p^{n-1}-i}, \ldots, \overline{p^{n}-i}\right\}$ and $V_{i}=\left\{\bar{i}, \overline{p^{n-1}+i}, \overline{2 p^{n-1}+i}, \ldots, \overline{(p-1) p^{n-1}+i}\right\}$, for all $i=1,2, \ldots, \frac{p^{n-1}-1}{2}$.

We now show that every $A_{i}$, for all $i=1,2, \ldots, \frac{p^{n-1}-1}{2}$, contains $2 p$ elements. Let $\overline{u_{1}}, \overline{u_{2}} \in U_{i}$ be any two elements of $U_{i}$. Let $\overline{u_{1}}=\overline{r p^{n-1}-i}, \overline{u_{2}}=\overline{s p^{n-1}-i}$ for some $r, s \in\{1,2, \ldots, p\}$. If possible let $\overline{u_{1}}=\overline{u_{2}}$, i.e., $r p^{n-1}-i+t_{1} p^{n}=$ $s p^{n-1}-i+t_{2} p^{n}$, i.e., $(r-s)=\left(t_{2}-t_{1}\right) p$ which is possible only when $r=s$ and hence $U_{i}$ contains $p$ elements. Similarly, we can show that $V_{i}$ contains $p$ elements. Now we need to show that $U_{i} \cap V_{i}=\emptyset$. If possible let $\bar{u} \in U_{i} \cap V_{i}$. Then $u=c p^{n-1}-i+t_{3} p^{n}=d p^{n-1}+i+t_{4} p^{n}$, where $c \in\{1,2, \ldots, p\}$ and $d \in\{0,1, \ldots,(p-1)\}$, i.e., $(c-d) p^{n-1}+\left(t_{3}-t_{4}\right) p^{n}=2 i$, i.e., $2 i$ is divisible by $p^{n-1}$, a contradiction since $2 i \leq p^{n-1}-1<p^{n-1}$. Therefore, $U_{i} \cap V_{i}=\emptyset$ and hence $A_{i}$ contains $2 p$ elements.

Let $\overline{r p^{n-1}-i} \in U_{i}$ for some $r \in\{1,2, \ldots, p\}$, and $\overline{s p^{n-1}+i} \in V_{i}$ for some $s \in\{0,1, \ldots,(p-1)\}$, be any two elements. In $\mathbb{Z}_{p^{n}}, o\left(\overline{r p^{n-1}-i}+\overline{s p^{n-1}+i}\right)=p$
and thus every element of $U_{i}$ is adjacent to each element of $V_{i}$. But for any two elements $\overline{r p^{n-1}-i}, \overline{t p^{n-1}-i} \in U_{i}(r, t \in\{1,2, \ldots, p\})$, we have $o\left(\overline{r_{1} p^{n-1}-i}+\right.$ $\left.r_{2} p^{n-1}-i\right) \neq p$ and thus no two elements of $U_{i}$ are adjacent. Similarly we can show that no two elements in $V_{i}$ are adjacent. Hence degree of every vertex in $A_{i}$, for all $i=1,2, \ldots, \frac{p^{n-1}-1}{2}$, is $p$ in $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{p^{n}}\right)$.

We now show that every $A_{i}$, for all $i=1,2, \ldots, \frac{p^{n-1}-1}{2}$, forms a component of $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{p^{n}}\right)$. Let $x \in A_{i}$. Then $x$ is adjacent to either $\bar{i}$ or $\overline{p^{n-1}-i}$ in $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{p^{n}}\right)$. Therefore every element of $A_{i}$ is connected to each other by a path of length at most 2. It is easy to verify that there is no edge between a vertex in $A_{i}$ and a vertex in $A_{j}$ for $i \neq j$ and $i, j \in\left\{1,2, \ldots, \frac{p^{n-1}-1}{2}\right\}$. Therefore, every $A_{i}$, for all $i=1,2, \ldots, \frac{p^{n-1}-1}{2}$, forms a $p$-regular component with $2 p$ vertices. Hence the theorem.

## 4. ORDER PRIME DIVISOR GRAPH OF THE GROUP $D_{n}$

For each positive integer $n \geq 3$, the dihedral group of degree $n$, denoted by $D_{n}$, is a non-commutative group containing $2 n$ elements and is defined by $D_{n}=$ $\left\{\langle a, b\rangle: o(a)=n, o(b)=2, b a=a^{-1} b\right\}$. The study of dihedral group $D_{n}$ helps us to characterize non commutative groups. In this section, we establish some graph-theoretic properties of $\mathscr{P} \mathscr{D}\left(D_{n}\right)$.

Before going to our results, we first consider the order prime divisor graph $\mathscr{P} \mathscr{D}\left(D_{8}\right)$.

## Example 4.1.



Figure 6. $\mathscr{P} \mathscr{D}\left(D_{8}\right)$

From Figure 6, we have $\operatorname{deg}(a)=\operatorname{deg}\left(a^{3}\right)=\operatorname{deg}\left(a^{5}\right)=\operatorname{deg}\left(a^{7}\right)=10$ and $\operatorname{deg}(e)=\operatorname{deg}\left(a^{2}\right)=\operatorname{deg}\left(a^{4}\right)=\operatorname{deg}\left(a^{6}\right)=\operatorname{deg}(b)=\operatorname{deg}(a b)=\operatorname{deg}\left(a^{2} b\right)=$ $\operatorname{deg}\left(a^{3} b\right)=\operatorname{deg}\left(a^{4} b\right)=\operatorname{deg}\left(a^{5} b\right)=\operatorname{deg}\left(a^{6} b\right)=\operatorname{deg}\left(a^{7} b\right)=9$ in order prime divisor graph $\mathscr{P} \mathscr{D}\left(D_{8}\right)$. Here $8=2^{3}$ and note that degree of any vertex in $\mathscr{P} \mathscr{D}\left(D_{8}\right)$ is either $\varphi(2)+8$ or $\varphi(2)+8+1$.

Theorem 4.2. Let $n(\geq 3)$ be a number and $n=p_{1}^{r_{1}} p_{2}^{r_{2}} \cdots p_{m}^{r_{m}}$ be the factorization of $n$ as product of distinct primes and their positive powers. Then the degree of a vertex of $\mathscr{P} \mathscr{D}\left(D_{n}\right)$ is either $\varphi\left(p_{1}\right)+\varphi\left(p_{2}\right)+\cdots+\varphi\left(p_{m}\right)+n$ or $\varphi\left(p_{1}\right)+\varphi\left(p_{2}\right)+$ $\cdots+\varphi\left(p_{m}\right)+n+1$.

Proof. Now $D_{n}=\left\{\langle a, b\rangle: o(a)=n, o(b)=2, b a=a^{-1} b\right\}$. Let $H=\langle a\rangle$ and $K=H b$. Then $D_{n}=H \cup K$ and all the $n$ elements of $K$ are of order 2. Also the subgroup $H$ contains $\varphi\left(p_{1}\right)$ elements of order $p_{1}, \varphi\left(p_{2}\right)$ elements of order $p_{2}$, and so on, $\varphi\left(p_{m}\right)$ elements of order $p_{m}$. Now $\operatorname{deg}(e)$ in $\mathscr{P} \mathscr{D}\left(D_{n}\right)$ is exactly equal to the number of prime order elements in $D_{n}$. Therefore, $\operatorname{deg}(e)=$ $\varphi\left(p_{1}\right)+\varphi\left(p_{2}\right)+\cdots+\varphi\left(p_{m}\right)+n$ in $\mathscr{P} \mathscr{D}\left(D_{n}\right)$. Let $x(\neq e) \in D_{n}$ be any element.

If $o(x)=2$, then $e$ is adjacent to $x$. Also for any other element $y(\neq x) \in G$ with prime order, we see that $x$ is adjacent to $x^{-1} y$. Thus the total number of adjacent vertices of $x$ in $\mathscr{P} \mathscr{D}\left(D_{n}\right)$ is $\varphi\left(p_{1}\right)+\varphi\left(p_{2}\right)+\cdots+\varphi\left(p_{m}\right)+n$.

Suppose $o(x)=p$, where $p$ is an odd prime. Then $x^{2}$ is again an element of order $p$. In this case $e$ and $x^{-1}$ are adjacent to $x$. Moreover, for any other element $z\left(\neq x, x^{2}\right) \in G$ with prime order, we see that $x$ is adjacent to $x^{-1} z$. Thus the total number of adjacent vertices of $x$ in $\mathscr{P} \mathscr{D}\left(D_{n}\right)$ is $\varphi\left(p_{1}\right)+\varphi\left(p_{2}\right)+\cdots+\varphi\left(p_{m}\right)+n$.

Finally, suppose that $o(x)$ is composite. Then in this case $x^{-1}$ is adjacent to $x$. If $x^{2}$ is an element of prime order, then for any other element $u\left(\neq x^{2}\right) \in G$ with prime order, we see that $x$ is adjacent to $x^{-1} u$. Thus, the total number of adjacent vertices of $x$ in $\mathscr{P} \mathscr{D}\left(D_{n}\right)$ is $\varphi\left(p_{1}\right)+\varphi\left(p_{2}\right)+\cdots+\varphi\left(p_{m}\right)+n$. On the other hand, if $o\left(x^{2}\right)$ is composite, then for any other element $u \in G$ with prime order, we see that $x$ is adjacent to $x^{-1} u$. Hence the total number of adjacent vertices of $x$ in $\mathscr{P} \mathscr{D}\left(D_{n}\right)$ is $\varphi\left(p_{1}\right)+\varphi\left(p_{2}\right)+\cdots+\varphi\left(p_{m}\right)+n+1$. Thus the proof is completed.

Theorem 4.3. $\mathscr{P} \mathscr{D}\left(D_{n}\right)(n \geq 3)$ is a regular graph if and only if $n=p$ or $2 p$ for some prime $p$.

Proof. Let $p(\geq 3)$ be a prime. Now $D_{p}$ is a non-commutative group of order $2 p$ such that every non identity element is of prime order. Hence by Theorem 2.1, it follows that $\mathscr{P} \mathscr{D}\left(D_{p}\right)$ is complete and thus it is $(2 p-1)$-regular.

For $p=2$, we have from Figure 3 that $\mathscr{P} \mathscr{D}\left(D_{2 p}\right)=\mathscr{P} \mathscr{D}\left(D_{4}\right)$ is 5-regular.
We now establish the regularity of the graph $\mathscr{P} \mathscr{D}\left(D_{2 p}\right)$, where $p$ is any odd prime. Note that $D_{2 p}=\left\{e, a, a^{2}, \ldots, a^{2 p-1}, b, a b, a^{2} b, \ldots, a^{2 p-1} b\right\}=H \cup K$, where $H=\langle a\rangle$ is the cyclic subgroup of $D_{2 p}$ generated by $a$ and $K=H b$ is a right
coset of $H$, different from $H$. Here every element of $K$ is of order 2. Moreover, $H$ contains exactly ( $p-1$ ) elements of order $p$ and a unique element of order 2. Let $a^{r} \in H$ and $a^{s} b \in K$. Then $a^{r} a^{s} b=a^{r+s(\bmod 2 p)} b$, for $r, s=1,2, \ldots, 2 p$. Thus $a^{r} a^{s} b \in K$ and hence $o\left(a^{r} a^{s} b\right)=2$. Therefore, every element of $H$ is adjacent to every element of $K$ in $\mathscr{P} \mathscr{D}\left(D_{2 p}\right)$. Let $a^{r}, a^{s} \in H$, then $a^{r} a^{s} \in H$, for $r, s=1,2, \ldots, 2 p$. Finally, for any two elements $a^{r} b, a^{s} b \in K$, we have $a^{r} b a^{s} b \in H$, for $r, s=1,2, \ldots, 2 p$. We now show that every vertex of $\mathscr{P} \mathscr{D}\left(D_{2 p}\right)$ is of degree $3 p$. For this let $x$ be any vertex of $\mathscr{P} \mathscr{D}\left(D_{2 p}\right)$.

Suppose $x \in H$. Then $x$ is adjacent to every element of $K$. Moreover, similar to the proof of Theorem 3.1, we can conclude that $x$ is adjacent to exactly $p$ elements of $H$. Hence, if $x \in H$, then $\operatorname{deg}(x)=3 p$. On the other hand, if $x \in K$, then all the $2 p$ elements of $H$ are adjacent to $x$. Since $x \in K$, we must have $x^{2}=e$ and $K=H x$. Also, $K x=(H x) x=H x^{2}=H$. Now $H$ contains total $p$ elements of prime order (exactly $p-1$ elements of order $p$ and 1 element of order 2 ). Therefore, $x$ is adjacent to exactly $p$ elements of $K$. Hence total number of adjacent vertices is $3 p$ and thus $\operatorname{deg}(x)=3 p$. Therefore, in either cases $\operatorname{deg}(x)=3 p$ in $\mathscr{P} \mathscr{D}\left(D_{2 p}\right)$. Consequently, $\mathscr{P} \mathscr{D}\left(D_{2 p}\right)$ is a $3 p$-regular graph.

Conversely, we assume that $\mathscr{P} \mathscr{D}\left(D_{n}\right)$ is regular. We show that $n=p$ or $2 p$ for some prime $p$. On the contrary we let $n \neq p, 2 p$ for any prime $p$. Let $n=p_{1}^{r_{1}} p_{2}^{r_{2}} \cdots p_{m}^{r_{m}}$, where $p_{1}, p_{2}, \ldots, p_{m}$ are distinct primes and $r_{1}, r_{2}, \ldots, r_{m}$ are positive integers. Then by Theorem 4.2, it follows that $\operatorname{deg}(e)=\varphi\left(p_{1}\right)+\varphi\left(p_{2}\right)+$ $\cdots+\varphi\left(p_{m}\right)+n$ whereas $\operatorname{deg}(a)=\varphi\left(p_{1}\right)+\varphi\left(p_{2}\right)+\cdots+\varphi\left(p_{m}\right)+n+1$ in $\mathscr{P} \mathscr{D}\left(D_{n}\right)$. This leads to $\mathscr{P} \mathscr{D}\left(D_{n}\right)$ is not regular, which is a contradiction. Consequently, $n=p$ or $2 p$ for some prime $p$.

From Corollary 3.2, Theorem 4.3, Figure 1 and Figure 2, we have the following result.

Theorem 4.4. If $G$ is a group of order $p$ or $2 p$, where $p$ is prime, then $\mathscr{P} \mathscr{D}(G)$ is regular.

Theorem 4.5. Let $p$ be an odd prime, then
(i) the line graph $L\left(\mathscr{P} \mathscr{D}\left(D_{p}\right)\right)$ of $\mathscr{P} \mathscr{D}\left(D_{p}\right)$ is $(4 p-4)$-regular.
(ii) the line graph $L\left(\mathscr{P} \mathscr{D}\left(D_{2 p}\right)\right)$ of $\mathscr{P} \mathscr{D}\left(D_{2 p}\right)$ is $(6 p-2)$-regular.

Proof. (i) For any prime $p \geq 3, \mathscr{P} \mathscr{D}\left(D_{p}\right)$ is ( $2 p-1$ )-regular graph. Thus by the similar argument of the proof of Theorem 3.11(i), it follows that the line graph $L\left(\mathscr{P} \mathscr{D}\left(D_{p}\right)\right)$ of $\mathscr{P} \mathscr{D}\left(D_{p}\right)$ is $(4 p-4)$-regular.
(ii) For any odd prime $p$, the graph $\mathscr{P} \mathscr{D}\left(D_{2 p}\right)$ is $3 p$-regular. So by the similar argument of the proof of Theorem 3.11(i), we have the line graph $L\left(\mathscr{P} \mathscr{D}\left(D_{2 p}\right)\right)$ of $\mathscr{P} \mathscr{D}\left(D_{2 p}\right)$ is $(6 p-2)$-regular.

Corollary 4.6. The line graph $L\left(\mathscr{P} \mathscr{D}\left(D_{4}\right)\right)$ of $\mathscr{P} \mathscr{D}\left(D_{4}\right)$ is 8 -regular graph.

Proof. Since $\mathscr{P} \mathscr{D}\left(D_{4}\right)$ is 5-regular graph, so by the similar argument of the proof of Theorem 3.11(i), it follows that the line graph $L\left(\mathscr{P} \mathscr{D}\left(D_{4}\right)\right)$ of $\mathscr{P} \mathscr{D}\left(D_{4}\right)$ is 8-regular.

Theorem 4.7. For any integer $n(\geq 3), \mathscr{P} \mathscr{D}\left(D_{n}\right)$ is connected as well as Hamiltonian.

Proof. Let $n(\geq 3)$ and $n=p_{1}^{r_{1}} p_{2}^{r_{2}} \cdots p_{m}^{r_{m}}$ be the factorization of $n$ as product of distinct primes and their positive powers. Then $\mathscr{P} \mathscr{D}\left(D_{n}\right)$ is a graph with $2 n$ vertices and by Theorem 4.2, we have degree of each vertex is either $\varphi\left(p_{1}\right)+$ $\varphi\left(p_{2}\right)+\cdots+\varphi\left(p_{m}\right)+n$ or $\varphi\left(p_{1}\right)+\varphi\left(p_{2}\right)+\cdots+\varphi\left(p_{m}\right)+n+1$. Thus $\operatorname{deg}(v) \geq n$ for every vertex in $\mathscr{P} \mathscr{D}\left(D_{n}\right)$. Therefore, by Theorem 3.12, it follows that $\mathscr{P} \mathscr{D}\left(D_{n}\right)$ is Hamiltonian and hence connected.

Theorem 4.8. The graph $\mathscr{P} \mathscr{D}\left(D_{n}\right)$ is not Eulerian for any positive integer $n(\geq 3)$.

Proof. Let $n(\geq 3)$ and $n=p_{1}^{r_{1}} p_{2}^{r_{2}} \cdots p_{m}^{r_{m}}$ be the factorization of $n$ as product of distinct primes and their positive powers. Then $\mathscr{P} \mathscr{D}\left(D_{n}\right)$ is a graph with $2 n$ vertices and by Theorem 4.2, we have degree of each vertex is either $\varphi\left(p_{1}\right)+\varphi\left(p_{2}\right)+$ $\cdots+\varphi\left(p_{m}\right)+n$ or $\varphi\left(p_{1}\right)+\varphi\left(p_{2}\right)+\cdots+\varphi\left(p_{m}\right)+n+1$. Now $\varphi\left(p_{1}\right)+\varphi\left(p_{2}\right)+\cdots+$ $\varphi\left(p_{m}\right)+n$ or $\varphi\left(p_{1}\right)+\varphi\left(p_{2}\right)+\cdots+\varphi\left(p_{m}\right)+n+1$ are two consecutive positive integers. So one of them must be even and another must be odd. Therefore, $\mathscr{P} \mathscr{D}\left(D_{n}\right)$ contains odd degree vertices. Consequently, $\mathscr{P} \mathscr{D}\left(D_{n}\right)$ is not Eulerian.

Remark 4.9. For any odd prime $p$, the graph $\mathscr{P} \mathscr{D}\left(D_{p}\right)$ is complete and hence $\operatorname{diam}\left(\mathscr{P} \mathscr{D}\left(D_{p}\right)\right)=1$.

Theorem 4.10. For any composite number $n(\geq 3), \operatorname{diam}\left(\mathscr{P} \mathscr{D}\left(D_{n}\right)\right)=2$.
Proof. Now $D_{n}=\left\{\langle a, b\rangle: o(a)=n, o(b)=2, b a=a^{-1} b\right\}$. Let $H=\langle a\rangle$, the cyclic subgroup generated by $a$ and $K=H b$ be the right coset of $H$ different from $H$. Then $D_{n}=H \cup K$. Here, every element of $K$ is of order 2. Let $a^{r} \in H$ and $a^{s} b \in K$. Then $a^{r} a^{s} b=a^{r+s(\bmod n)} b$, for $r, s=1,2, \ldots, n$. Hence $a^{r} a^{s} b \in K$ and thus $o\left(a^{r} a^{s} b\right)=2$. Therefore, every element of $H$ is adjacent to every element of $K$ in $\mathscr{P} \mathscr{D}\left(D_{n}\right)$. Let $x, y$ be any two vertices of $\mathscr{P} \mathscr{D}\left(D_{n}\right)$.

Case 1. If $H$ contains exactly one of $x$ or $y$ and $K$ contains the other, then $x \leftrightarrow y$ is a path in $\mathscr{P} \mathscr{D}\left(D_{n}\right)$ of length 1.

Case 2. We now consider the other possibility. Without loss of generality, we assume that $x, y \in H$. Then two sub-cases arise.

Subcase (a). Suppose $x y=e$ or $o(x y)=p$ for some prime $p$. Then $x \leftrightarrow y$ is a path in $\mathscr{P} \mathscr{D}\left(D_{n}\right)$ of length 1.

Subcase (b). If $x$ and $y$ are not adjacent in $\mathscr{P} \mathscr{D}\left(D_{n}\right)$. Then for any $z \in K$, we get a path $x \leftrightarrow z \leftrightarrow y$ in $\mathscr{P} \mathscr{D}\left(D_{n}\right)$ of length 2 .

Therefore, there is a path between any two vertices of $\mathscr{P} \mathscr{D}\left(D_{n}\right)$. Hence $\mathscr{P} \mathscr{D}\left(D_{n}\right)$ is connected and $\operatorname{diam}\left(\mathscr{P} \mathscr{D}\left(D_{n}\right)\right)=2$.

Theorem 4.11. For any $n(\geq 3)$, the graph $\mathscr{P} \mathscr{D}\left(D_{n}\right)$ is neither bipartite nor a tree. Moreover, $\operatorname{girth}\left(\mathscr{P} \mathscr{D}\left(D_{n}\right)\right)=3$.

Proof. For any $n \geq 3, D_{n}=\left\{\langle a, b\rangle: o(a)=n, o(b)=2, b a=a^{-1} b\right\}$. Let $p$ be a prime factor of $n$. Then $n=p q$ for some positive integer $q$. Now $a^{q} \in D_{n}$ such that $o\left(a^{q}\right)=p$. Then $a^{q} \leftrightarrow b \leftrightarrow e \leftrightarrow a^{q}$ forms a cycle of length 3. Since $\mathscr{P} \mathscr{D}\left(D_{n}\right)$ is a simple graph, it follows that $\operatorname{girth}\left(\mathscr{P} \mathscr{D}\left(D_{n}\right)\right)=3$. Since $\mathscr{P} \mathscr{D}\left(D_{n}\right)$ contains an odd cycle, it follows that $\mathscr{P} \mathscr{D}\left(D_{n}\right)$ is neither bipartite nor a tree.

Theorem 4.12. For any integer $(n \geq 3)$, $\mathscr{P} \mathscr{D}\left(D_{n}\right)$ is non planar.
Proof. We prove that $\mathscr{P} \mathscr{D}\left(D_{n}\right)$ is non planar for any integer $n(\geq 3)$. On the contrary, suppose $\mathscr{P} \mathscr{D}\left(D_{k}\right)$ is planar for some integer $k(\geq 3)$. First we claim that 2 is only one prime factor of $k$. If not, let $p$ be an odd prime factor of $k$ and thus $k=p^{m} q$ for some positive integers $m, q$ with $\operatorname{gcd}(p, q)=1$. Now $D_{k}=$ $\left\{\langle a, b\rangle: o(a)=k, o(b)=2, b a=a^{-1} b\right\}$. Let $x \in D_{k}$ such that $o(x)=p$. Then the induced subgraph of $\mathscr{P} \mathscr{D}\left(D_{k}\right)$ induced by the set of vertices $\left\{e, x, x^{-1}, b, x b\right\}$ forms the complete subgraph $K_{5}$. Hence by Kuratowski's Theorem, we conclude that $\mathscr{P} \mathscr{D}\left(D_{k}\right)$ is non planar, which contradicts our assumption that $\mathscr{P} \mathscr{D}\left(D_{k}\right)$ is planar. This contradiction ensures that 2 is the only one prime factor of $k$ and hence $k=2^{r}$ for some positive integer $r \geq 2$.

Now we show that $\mathscr{P} \mathscr{D}\left(D_{2^{r}}\right)$ is non planar. First we see from Fig. 3 that $\mathscr{P} \mathscr{D}\left(D_{4}\right)$ contains complete bipartite graph $K_{3,3}$ with bipartition $\left\{e, a, a^{3}\right\}$ and $\left\{b, a b, a^{3} b\right\}$ as a subgraph. Hence by Kuratowski's Theorem, we have $\mathscr{P} \mathscr{D}\left(D_{4}\right)$ is non planar. Moreover, for $r \geq 3$, we have $D_{2^{r}}$ has a subgroup isomorphic to $D_{4}$. Thus $\mathscr{P} \mathscr{D}\left(D_{2^{r}}\right)$ has a subgraph isomorphic to $\mathscr{P} \mathscr{D}\left(D_{4}\right)$. Since $\mathscr{P} \mathscr{D}\left(D_{4}\right)$ is non planar, it follows that $\mathscr{P} \mathscr{D}\left(D_{2^{r}}\right)=\mathscr{P} \mathscr{D}\left(D_{k}\right)$ is also non planar. Hence the theorem.

Theorem 4.13 (Brook's Theorem [8]). If $G$ is a connected graph other than a complete graph or an odd cycle, then $\chi(G) \leq \Delta(G)$, where $\chi(G)$ and $\Delta(G)$ are the chromatic number and the maximum vertex degree of $G$ respectively.

Remark 4.14. If $n$ is an odd prime, then $\mathscr{P} \mathscr{D}\left(D_{n}\right)$ is a complete graph with $2 n$ vertices and hence $\chi\left(\mathscr{P} \mathscr{D}\left(D_{n}\right)\right)=2 n$.

Theorem 4.15. Let $n(\geq 3)$ be a composite number and $n=p_{1}^{r_{1}} p_{2}^{r_{2}} \cdots p_{m}^{r_{m}}$ be the factorization of $n$, where $p_{1}, p_{2}, \ldots, p_{m}$ are distinct primes and $r_{1}, r_{2}, \ldots, r_{m}$ are positive integers. Then $\chi\left(\mathscr{P} \mathscr{D}\left(D_{n}\right)\right) \leq \varphi\left(p_{1}\right)+\varphi\left(p_{2}\right)+\cdots+\varphi\left(p_{m}\right)+n+1$.

Proof. If $n(\geq 3)$ and $n=p_{1}^{r_{1}} p_{2}^{r_{2}} \cdots p_{m}^{r_{m}}$ be the factorization of $n$, where $p_{1}, p_{2}$, $\ldots, p_{m}$ are distinct primes and $r_{1}, r_{2}, \ldots, r_{m}$ are positive integers. Then by Theorem 4.2, it follows that degree of every vertex in $\mathscr{P} \mathscr{D}\left(D_{n}\right)$ is either $\varphi\left(p_{1}\right)+$ $\varphi\left(p_{2}\right)+\cdots+\varphi\left(p_{m}\right)+n$ or $\varphi\left(p_{1}\right)+\varphi\left(p_{2}\right)+\cdots+\varphi\left(p_{m}\right)+n+1$. Therefore, $\Delta\left(\mathscr{P} \mathscr{D}\left(D_{n}\right)\right)=\varphi\left(p_{1}\right)+\varphi\left(p_{2}\right)+\cdots+\varphi\left(p_{m}\right)+n+1$. Moreover, $\mathscr{P} \mathscr{D}\left(D_{n}\right)$ is connected graph which is neither complete nor an odd cycle. Hence by Theorem 4.13, it follows that $\chi\left(\mathscr{P} \mathscr{D}\left(D_{n}\right)\right) \leq \varphi\left(p_{1}\right)+\varphi\left(p_{2}\right)+\cdots+\varphi\left(p_{m}\right)+n+1$.

## 5. Order prime divisor graphs of small finite groups

Here we discuss all possible order prime divisor graphs $\mathscr{P} \mathscr{D}(G)$, where $G$ is a group of order at most 15. For this purpose, we first exhibit the order prime divisor graph $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{12}\right)$.


Figure 7. $\mathscr{P} \mathscr{D}\left(\mathbb{Z}_{12}\right)$

| Order of Group $G$ | Group $G$ | Order Prime Divisor Graph $\mathscr{P} \mathscr{D}(G)$ |
| :---: | :---: | :---: |
| 2 | $\mathbb{Z}_{2}$ | $K_{2}$ |
| 3 | $\mathbb{Z}_{3}$ | $K_{3}$ |
| 4 | $\mathbb{Z}_{4}$ | $K_{2} \cup K_{2}$ (Figure 1) |
|  | $\mathbb{Z}_{2} \oplus \mathbb{Z}_{2}$ | $K_{4}$ (Figure 2) |
| 5 | $\mathbb{Z}_{5}$ | $K_{5}$ |
| 6 | $\mathbb{Z}_{6}$ | 3-regular connected graph |
|  | $S_{3}$ | $K_{6}$ |
| 7 | $\mathbb{Z}_{7}$ | $K_{7}$ |
|  | $\mathbb{Z}_{8}$ | $K_{2} \cup K_{2} \cup C_{4}$ |
|  | $\mathbb{Z}_{2} \oplus \mathbb{Z}_{4}$ | $K_{4} \cup K_{4}$ |
| 8 | $\mathbb{Z}_{2} \oplus \mathbb{Z}_{2} \oplus \mathbb{Z}_{2}$ | $K_{8}$ |
|  | $D_{4}$ | Figure 3 |
|  | $Q_{8}$ | $K_{2} \cup K_{2} \cup K_{2} \cup K_{2}$ |


| 9 | $\mathbb{Z}_{9}$ | Figure 4 |
| :---: | :---: | :---: |
|  | $\mathbb{Z}_{3} \oplus \mathbb{Z}_{3}$ | $K_{9}$ |
| 10 | $\mathbb{Z}_{10}$ | 5-regular connected graph |
|  | $D_{5}$ | $K_{10}$ |
| 11 | $\mathbb{Z}_{11}$ | $K_{11}$ |
|  | $\mathbb{Z}_{12}$ | Figure 7 |
|  | $\mathbb{Z}_{2} \oplus \mathbb{Z}_{2} \oplus \mathbb{Z}_{3}$ | 5-regular connected graph |
| 12 | $A_{4}$ | $K_{12}$ |
|  | $D_{6}$ | 9-regular connected graph |
|  | $T$ | Union of two 3-regular component |
| 13 | $\mathbb{Z}_{13}$ | $K_{13}$ |
| 14 | $\mathbb{Z}_{14}$ | 7-regular connected graph |
|  | $D_{7}$ | $K_{14}$ |
| 15 | $\mathbb{Z}_{15}$ | Figure 5 |

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