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ON ORDER PRIME DIVISOR GRAPHS OF FINITE GROUPS

Mridul K. Sen, Sunil K. Maity[1]

and

Sumanta Das

Department of Pure Mathematics
University of Calcutta
35, Ballygunge Circular Road, Kolkata-700019, India

e-mail: senmk6@gmail.com
skmpm@caluniv.ac.in
sumanta.das498@gmail.com

Abstract

The order prime divisor graph $\mathscr{PD}(G)$ of a finite group G is a simple graph whose vertex set is G and two vertices $a, b \in G$ are adjacent if and only if either $ab = e$ or $o(ab)$ is some prime number, where e is the identity element of the group G and $o(x)$ denotes the order of an element $x \in G$. In this paper, we establish the necessary and sufficient condition for the completeness of order prime divisor graph $\mathscr{PD}(G)$ of a group G. Concentrating on the graph $\mathscr{PD}(D_n)$, we investigate several properties like degrees, girth, regularity, Eulerianity, Hamiltonicity, planarity etc. We characterize some graph theoretic properties of $\mathscr{PD}(\mathbb{Z}_n)$, $\mathscr{PD}(S_n)$, $\mathscr{PD}(A_n)$.

Keywords: group, dihedral group, complete graph, Eulerian graph, regular graph, planar graph, order prime divisor graph.

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1. Introduction

Defining graphs over groups help us for studying the interplay between algebraic properties and graph-theoretic properties and structures. For a finite group G, one can associate a certain type of graph, order prime divisor graph $\mathscr{PD}(G)$,

[1]Corresponding author.

and investigate the interplay between the group-theoretic properties of G and the graph-theoretic properties of order prime divisor graph $\mathscr{PD}(G)$.

There are a number of constructions of graphs from groups or semigroups in the literature. Here we begin by introducing some well-known graphs associated with semigroups or groups. In 1964, Bosák [1] studied certain graph over semigroups. Then Csákány and Pollák [2] defined intersection graphs of nontrivial proper subgroups of groups. In [9], Zelinka studied intersection graphs of nontrivial subgroups of finite Abelian groups. Later on, the intersection graph of ideals of rings was studied by Chakrabarty, Ghosh, Mukherjee and Sen [3].

In [6], Kelarev and Quinn introduced the notion of the (directed) power graph $P(G)$ of a group G and described the structure of the (directed) power graphs of all finite abelian groups. According to them the (directed) power graph $P(G)$ of a group G is a directed graph with the set G of vertices, and with all edges (u, v) such that $u \neq v$ and v is the power of u. Later on, Chakrabarty, Ghosh and Sen [4] defined the undirected power graph $\mathscr{G}(S)$ of a semigroup S as the undirected graph with vertex set S and distinct vertices a and b are adjacent if $a^m = b$ or $b^m = a$ for some positive integer m.

In 2009, the authors [7] defined order prime graph and studied its properties. According to them the order prime graph $OP(\Gamma)$ of a finite group Γ is a graph with vertex set Γ and two vertices are adjacent in $OP(\Gamma)$ if and only if their orders are relatively prime in Γ.

In this paper, a new type of graph, called order prime divisor graph, is defined and studied its properties. For a finite group G, the order prime divisor graph of G, denoted by $\mathscr{PD}(G)$, is a simple graph with vertex set G and two vertices a, b are adjacent in $\mathscr{PD}(G)$ if and only if either $o(ab) = 1$ or $o(ab) = p$ for some prime p, i.e., either a and b are inverse to each other or ab is an element of prime order. Thus in $\mathscr{PD}(G)$ two vertices a, b are adjacent if and only if $o(ab)$ divides p for some prime number p and that's why we have named this graph as order prime divisor graph. We note that if G is a group, then $o(ab) = o(ba)$ for any two elements $a, b \in G$. Clearly, by definition, order prime divisor graph $\mathscr{PD}(G)$ of a finite group G contains no isolated vertices. Following examples show that cyclic groups may have disconnected order prime divisor graphs where as non cyclic (even non-commutative) groups have connected order prime divisor graphs.

Figure 1. $\mathscr{PD}(\mathbb{Z}_4)$

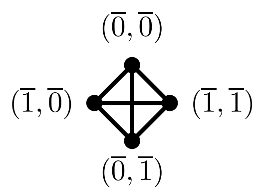

Figure 2. $\mathscr{PD}(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$

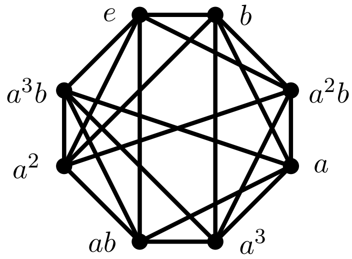

Figure 3. $\mathscr{PD}(D_4)$

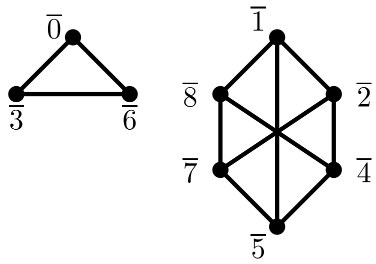

Figure 4. $\mathscr{PD}(\mathbb{Z}_9)$

In this paper, $\mathbb{Z}_n = \{\overline{0}, \overline{1}, \ldots, \overline{n-1}\}$ is the additive group of integers modulo n, $U(n) = \{\overline{x} \in \mathbb{Z}_n : \gcd(x,n) = 1\}$ denotes the group of units of the ring $(\mathbb{Z}_n, +, \cdot)$ of order $\varphi(n)$, D_n is the dihedral group of order $2n$, S_n is the permutation group on n symbols, A_n is the alternating group on n symbols, $GL_n(F_q)$ denotes the general linear group over a finite field F_q, $SL_n(F_q)$ denotes the special linear group over a finite field F_q, T denotes a non-commutative group containing 12 elements and is defined by $T = \{\langle a, b\rangle : o(a) = 6, b^2 = a^3, ba = a^{-1}b\}$, K_n denotes the complete graph with n vertices, $K_{r,s}$ denotes the complete bipartite graph or biclique where partite sets have sizes r and s, C_n denotes a cycle with n vertices, $\deg(v)$ denotes the degree of a vertex v, $a \leftrightarrow b$ denotes that vertices a, b are adjacent. For usual algebraic terms, we refer to [5], and we refer to [8] for graph-theoretic terms, definitions and notations.

2. Some properties of $\mathscr{PD}(G)$

In this section, we study some interesting properties of $\mathscr{PD}(G)$. We also establish here the necessary and sufficient condition for the order prime divisor graph $\mathscr{PD}(G)$ of a finite group to be complete.

Theorem 2.1. *For a finite group G, $\mathscr{PD}(G)$ is complete if and only if each non identity element of G is of prime order.*

Proof. Let G be a finite group with order of each non identity element is prime. Let a, b be any two distinct elements of G. If $ab = e$, then $o(ab) = 1$ and thus a and b are adjacent in $\mathscr{PD}(G)$. Suppose $ab \neq e$. Then by the given hypothesis it follows that $o(ab)$ is some prime and hence these two elements a and b are adjacent in $\mathscr{PD}(G)$. Therefore, between any two distinct vertices in $\mathscr{PD}(G)$, there is an edge and hence $\mathscr{PD}(G)$ is complete.

Conversely, let G be finite group for which $\mathscr{PD}(G)$ is complete. Let $a\,(\neq e)$ be any element of G. Since $\mathscr{PD}(G)$ is complete, we must have an edge between the vertices a and e. Therefore $o(a) = o(ae)$ must be prime. Consequently, order of any non identity element of G is prime. ■

Corollary 2.2. *Let G be a group of prime order. Then $\mathscr{PD}(G)$ is complete.*

Corollary 2.3. *S_3, A_4, A_5 are non-commutative groups whose order prime divisor graphs $\mathscr{PD}(S_3)$, $\mathscr{PD}(A_4)$, $\mathscr{PD}(A_5)$ are complete.*

Corollary 2.4. *Let G be a finite commutative group. Then $\mathscr{PD}(G)$ is complete if and only if $G \cong \underbrace{\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p}_{n-\text{fold}}$ for some positive integer n.*

Proof. First suppose that $G \cong \underbrace{\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p}_{n-\text{fold}}$ for some positive integer n. Then order of any non identity element of G is p and hence by Theorem 2.1, it follows that $\mathscr{PD}(G)$ is complete.

Conversely, let G be a finite commutative group such that $\mathscr{PD}(G)$ is complete. Then by Theorem 2.1, it follows that every non identity element of G is of prime order. Now by the structure theorem of finite commutative group, we have $G \cong \mathbb{Z}_{{p_1}^{n_1}} \oplus \mathbb{Z}_{{p_2}^{n_2}} \oplus \cdots \oplus \mathbb{Z}_{{p_k}^{n_k}}$, where $p_1, p_2, \ldots, p_k$ are primes (not necessarily distinct) and $n_1, n_2, \ldots, n_k$ are positive integers. First we show that $n_1 = n_2 = \cdots = n_k = 1$. On the contrary, suppose $n_i > 1$, for some $i \in \{1, 2, \ldots, k\}$. Then G contains elements of composite order ${p_i}^{n_i}$, a contradiction. Therefore, $n_1 = n_2 = \cdots = n_k = 1$ and thus $G \cong \mathbb{Z}_{p_1} \oplus \mathbb{Z}_{p_2} \oplus \cdots \oplus \mathbb{Z}_{p_k}$. We now show that $p_1 = p_2 = \cdots = p_k$. Suppose $p_i \neq p_j$, for some $1 \leq i, j \leq k$. Then G contains elements of composite order $p_i p_j$, which is again a contradiction. Therefore, $p_1 = p_2 = \cdots = p_k = p$ (say) and thus $G \cong \underbrace{\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p}_{k-\text{fold}}$. ■

Theorem 2.5. *If a finite group G has a subgroup of order p (where p is prime), then $\mathscr{PD}(G)$ contains a clique with p vertices.*

Proof. Let H be a subgroup of a finite group G such that $|H| = p$ for some prime number p. Then every non identity element in H is of order p. Therefore, $\mathscr{PD}(H)$ is a complete graph containing p vertices. Since $\mathscr{PD}(H)$ is a subgraph of $\mathscr{PD}(G)$, it follows that $\mathscr{PD}(G)$ has a clique $\mathscr{PD}(H)$ with p vertices. ■

Remark 2.6. (i) The vertices of every clique in $\mathscr{PD}(G)$ may not form a subgroup of G. For example, in the order prime divisor graph $\mathscr{PD}(\mathbb{Z}_4 \oplus \mathbb{Z}_4)$, $\{(\overline{0}, \overline{1}), (\overline{0}, \overline{3}), (\overline{2}, \overline{1}), (\overline{2}, \overline{3})\}$, $\{(\overline{1}, \overline{0}), (\overline{1}, \overline{2}), (\overline{3}, \overline{0}), (\overline{3}, \overline{2})\}$, $\{(\overline{1}, \overline{1}), (\overline{1}, \overline{3}), (\overline{3}, \overline{1}), (\overline{3}, \overline{3})\}$ form three different clique but none of them forms a subgroup of $\mathbb{Z}_4 \oplus \mathbb{Z}_4$.

(ii) If the vertices of a clique in $\mathscr{PD}(G)$ form a subgroup H of G then order of H may not be prime. For example, in the order prime divisor graph $\mathscr{PD}(\mathbb{Z}_4 \oplus \mathbb{Z}_4)$, $\{(\overline{0}, \overline{0}), (\overline{0}, \overline{2}), (\overline{2}, \overline{0}), (\overline{2}, \overline{2})\}$ form a subgroup of $\mathbb{Z}_4 \oplus \mathbb{Z}_4$, which is of order 4.

Theorem 2.7. *Let G be a finite group and $p\,(\geq 5)$ be a prime such that p divides $|G|$. Then $\mathscr{PD}(G)$ is a non planar graph.*

Proof. Let G be a group of order n and $p\,(\geq 5)$ be a prime such that p divides n. Then by Cauchy's Theorem, G has an element of order p and hence a subgroup H of order p. Now by Theorem 2.5, $\mathscr{P}\mathscr{D}(H) = K_p$ is a subgraph of $\mathscr{P}\mathscr{D}(G)$. Since $p \geq 5$, it follows that $\mathscr{P}\mathscr{D}(G)$ contains K_5 as a subgraph. Hence by Kuratowski's Theorem, it follows that $\mathscr{P}\mathscr{D}(G)$ is a non planar graph. ∎

Remark 2.8. Converse of Theorem 2.7 is not true in general. From Corollary 2.3, we have $\mathscr{P}\mathscr{D}(S_3)$ and $\mathscr{P}\mathscr{D}(A_4)$ are complete graphs and hence they are non planar as well as non outerplanar, though there are no prime greater or equal to 5 dividing the order of the groups.

Corollary 2.9. *If a finite group G has a centre of prime order p, then $\mathscr{P}\mathscr{D}(G)$ contains a clique with p vertices.*

Corollary 2.10. *If m is a positive integer such that $2^m - 1$ is prime (called Mersenne prime), then $\mathscr{P}\mathscr{D}(GL_n(F_{2^m}))$ has a clique with $2^m - 1$ vertices.*

Proof. Now $Z\big(GL_n(F_{2^m})\big) = \big\{A = (a_{ij})_{n\times n} : a_{ij} = 0, 1 \leq i \neq j \leq n; a_{ii} = a \in F_{2^m} \setminus \{0\}\big\}$ $(n > 1)$ implies $|Z(GL_n(F_{2^m}))| = 2^m - 1$. Hence by Corollary 2.9, we have $\mathscr{P}\mathscr{D}(GL_n(F_{2^m}))$ has a clique with $2^m - 1$ vertices. ∎

Corollary 2.11. *Let F_q be a finite field with q elements and n be a positive integers such that* $\gcd(n, q-1)$ *is prime. Then $\mathscr{P}\mathscr{D}(SL_n(F_q))$ has a clique with $gcd(n, q-1)$ vertices.*

Proof. Let n be a positive integer such that $\gcd(n, q-1)$ is prime. For $n\,(> 1)$, we have $Z(SL_n(F_q)) = \big\{A = (a_{ij})_{n\times n} : a_{ij} = 0,\ i \neq j; a_{ii} = a \in F_q \setminus \{0\}; a^n = 1\big\}$ implies $|Z(SL_n(F_q))| = \gcd(n, q-1)$. Hence by Corollary 2.9, we have $\mathscr{P}\mathscr{D}(SL_n(F_q))$ has a clique with $gcd(n, q-1)$ vertices. ∎

Recall that the girth of a graph with at least one cycle is the length of its shortest cycle. A graph with no cycle has infinite girth. A graph with no cycle is said to be an acyclic graph. A forest is an acyclic graph whereas a tree is a connected acyclic graph.

Theorem 2.12. *Let G be a finite group and p is an odd prime such that p divides $|G|$. Then $\mathscr{P}\mathscr{D}(G)$ is neither bipartite nor a tree and $girth(\mathscr{P}\mathscr{D}(G)) = 3$.*

Proof. Let G be a group of order n and p be an odd prime such that p divides n. Then G contains an element x of order p. Since p is an odd prime, we have $x \neq x^{-1}$ and $o(x^{-1}) = p$. This implies $x \leftrightarrow e \leftrightarrow x^{-1} \leftrightarrow x$ forms a cycle in $\mathscr{P}\mathscr{D}(G)$ of length 3. Since $\mathscr{P}\mathscr{D}(G)$ is a simple graph, we must have $girth(\mathscr{P}\mathscr{D}(G)) = 3$. Also $\mathscr{P}\mathscr{D}(G)$ contains an odd cycle implies $\mathscr{P}\mathscr{D}(G)$ is neither bipartite nor a tree. ∎

Theorem 2.13. *If G_1, G_2 are two finite groups such that $G_1 \cong G_2$, then $\mathscr{P}\mathscr{D}(G_1) \cong \mathscr{P}\mathscr{D}(G_2)$.*

Proof. Let G_1 and G_2 be two finite groups and $\psi : G_1 \longrightarrow G_2$ be an isomorphism. Let $a, b \in G_1$ be adjacent in $\mathscr{P}\mathscr{D}(G_1)$. Then either $o(ab) = 1$ or $o(ab) = p$ for some prime p. Since ψ is an isomorphism, we have $o(\psi(a)\psi(b)) = o(\psi(ab)) = o(ab)$. If $o(ab) = 1$, then $o(\psi(a)\psi(b)) = 1$ and hence $\psi(a)$ and $\psi(b)$ are adjacent in $\mathscr{P}\mathscr{D}(G_2)$. On the other hand, if $o(ab) = p$, then $o(\psi(a)\psi(b)) = p$ and hence $\psi(a)$ is adjacent to $\psi(b)$ in $\mathscr{P}\mathscr{D}(G_2)$. Conversely, if $\psi(a)$ and $\psi(b)$ are adjacent in $\mathscr{P}\mathscr{D}(G_2)$, one can easily check that a and b are adjacent in $\mathscr{P}\mathscr{D}(G_1)$. Hence $\mathscr{P}\mathscr{D}(G_1) \cong \mathscr{P}\mathscr{D}(G_2)$. ■

Remark 2.14. The converse of Theorem 2.13 is not true in general. For example we consider two non-isomorphic groups $\mathbb{Z}_{27} \oplus \mathbb{Z}_3$ and $\mathbb{Z}_9 \oplus \mathbb{Z}_9$. Then both the graphs $\mathscr{P}\mathscr{D}(\mathbb{Z}_{27} \oplus \mathbb{Z}_3)$ and $\mathscr{P}\mathscr{D}(\mathbb{Z}_9 \oplus \mathbb{Z}_9)$ have five components. Out of these five components one component is a 8-regular graph with 9 vertices and each of remaining four components is a 9-regular graph with 18 vertices. Therefore, $\mathscr{P}\mathscr{D}(\mathbb{Z}_{27} \oplus \mathbb{Z}_3) \cong \mathscr{P}\mathscr{D}(\mathbb{Z}_9 \oplus \mathbb{Z}_9)$ though $\mathbb{Z}_{27} \oplus \mathbb{Z}_3 \ncong \mathbb{Z}_9 \oplus \mathbb{Z}_9$.

Corollary 2.15. *For a finite group G, $Aut(G) \subseteq Aut(\mathscr{P}\mathscr{D}(G))$.*

Proof. Follows from Theorem 2.13. ■

Theorem 2.16. *Let G be a finite commutative group. Then $\mathscr{P}\mathscr{D}(G)$ has at least two pendant vertices if and only if $G \cong \mathbb{Z}_{2^r}$, for some positive integer r.*

Proof. Let $G \cong \mathbb{Z}_{2^r}$, for some positive integer r. Since $\overline{2^{r-1}} \in \mathbb{Z}_{2^r}$ is the unique element of order 2, we have $\deg(\overline{0}) = 1 = \deg(\overline{2^{r-1}})$ in $\mathscr{P}\mathscr{D}(\mathbb{Z}_{2^r})$. Hence $\mathscr{P}\mathscr{D}(\mathbb{Z}_{2^r})$ has at least two pendant vertices.

Conversely, let G be a finite commutative group of order n such that $\mathscr{P}\mathscr{D}(G)$ contains at least two pendant vertices. First we prove that n has no odd prime divisor. For this, let p be any odd prime divisor of n. Then G contains at least two elements x and y of order p. Now let $a \in G$ be any element. Then $a \leftrightarrow a^{-1}x$ and $a \leftrightarrow a^{-1}y$ in $\mathscr{P}\mathscr{D}(G)$ and thus $\deg(a) \geq 2$ in $\mathscr{P}\mathscr{D}(G)$. Hence $\mathscr{P}\mathscr{D}(G)$ contains no pendant vertices, a contradiction. Therefore, 2 is the only one divisor of n and hence $G \cong \mathbb{Z}_{2^{n_1}} \oplus \mathbb{Z}_{2^{n_2}} \oplus \cdots \oplus \mathbb{Z}_{2^{n_k}}$ for some positive integers $n_1, n_2, \ldots, n_k$. We now show that $k = 1$. On the contrary, let $k > 1$. Then G contains at least three elements of order 2 and hence degree of every vertex in $\mathscr{P}\mathscr{D}(G)$ is at least 3. Thus $\mathscr{P}\mathscr{D}(G)$ contains no pendant vertices, which is again a contradiction. Therefore, $k = 1$ and consequently $G \cong \mathbb{Z}_{2^r}$, for some positive integer r. ■

Corollary 2.17. *For a finite commutative group G, $\mathscr{P}\mathscr{D}(G)$ is a forest if and only if either $G \cong \mathbb{Z}_2$ or $G \cong \mathbb{Z}_4$.*

Proof. First we assume that G is isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_4$. Then clearly $\mathscr{PD}(G)$ is a forest.

Conversely, let G be a finite commutative group such that $\mathscr{PD}(G)$ is a forest. We know that every tree with at least two vertices has at least two pendant vertices and every component of a forest is a tree. Hence by Theorem 2.16, G is of the form $\mathbb{Z}_{2^r}$, for some positive integer r. We now show that r must be equal to 1 or 2. On the contrary, let $r \geq 3$. Then $\overline{1} \leftrightarrow \overline{2^{r-1}-1} \leftrightarrow \overline{2^{r-1}+1} \leftrightarrow \overline{2^r-1} \leftrightarrow \overline{1}$ forms a cycle in $\mathscr{PD}(\mathbb{Z}_{2^r})$. Hence $\mathscr{PD}(\mathbb{Z}_{2^r})$ is not a forest, a contradiction. This contradiction ensures that $r \leq 2$. Consequently, G is isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_4$. ■

We need a result that follows from [5, Chapter 9, Section 9.5, Corollary 20].

Theorem 2.18. *Let $n \geq 2$ be an integer with factorization $n = {p_1}^{k_1}{p_2}^{k_2}\cdots{p_r}^{k_r}$ in $\mathbb{Z}$, where $p_1, p_2, \ldots, p_r$ are distinct primes. Then*

(i) $U(n) \cong U\left(p_1^{k_1}\right) \times U\left(p_2^{k_2}\right) \times \cdots \times U\left(p_r^{k_r}\right)$,

(ii) $U(2^\alpha) \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^{\alpha-2}}$, *for all* $\alpha \geq 2$,

(iii) $U(p^\alpha) \cong \mathbb{Z}_{p^{\alpha-1}(p-1)}$ *for any odd prime* p.

Theorem 2.19. *For any integer $n \geq 3$, the order prime divisor graph $\mathscr{PD}(U(2^n))$ has no pendant vertices.*

Proof. For any $n \geq 3$, $U(2^n) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{2^{n-2}}$ and hence by Theorem 2.16, it follows that $\mathscr{PD}(U(2^n))$ has no pendant vertices. ■

Corollary 2.20. *For any odd prime p and for any positive integer n, the order prime divisor graph $\mathscr{PD}(U(p^n))$ has no pendant vertices.*

Proof. For any odd prime p, $U(p^n) \cong \mathbb{Z}_{p^{n-1}(p-1)}$. Therefore, by Theorem 2.16, it follows that for any odd prime p and for positive integer n, the order prime divisor graph $\mathscr{PD}(U(p^n))$ has no pendant vertices. ■

3. ORDER PRIME DIVISOR GRAPH OF THE GROUP $\mathbb{Z}_n$

A graph G is said to be a k-regular graph (k is a non negative integer) if the degree of each vertex of G is k. In this section we study k-regular order prime divisor graph of the group $(\mathbb{Z}_n, +)$. From Corollary 2.2, it follows that order prime divisor graph of any group of prime order p is $(p-1)$-regular. In this section we study order prime divisor graph of any cyclic group of order $2p$, where p is prime.

Theorem 3.1. *For any cyclic group G of order $2p$, where p is an odd prime, $\mathscr{PD}(G)$ is a p-regular graph.*

Proof. Let G be a cyclic group of order $2p$. Then G contains exactly one element of order 1, $(p-1)$ elements of order p, one element of order 2 and $(p-1)$ elements of order $2p$. Hence G has exactly $(p-1)+1=p$ elements of prime order. If x is any element of G, then $o(x)=1$ or 2 or p or $2p$. We consider the following cases.

Case 1. Suppose $o(x)=1$, then $x=e$. Now $\deg(e)$ in $\mathscr{PD}(G)$ is exactly equal to the number of prime order elements in G. Therefore, $\deg(e)=p$ in $\mathscr{PD}(G)$.

Case 2. Assume that $o(x)=2$. Since G is cyclic so G has only one element of order 2. Clearly x is adjacent to e. Also for any other element $y\in G$ with $o(y)=p$, we see that x is adjacent to $x^{-1}y$. Thus the number of such y is $(p-1)$. Therefore the total number of adjacent vertices of x in $\mathscr{PD}(G)$ is $1+(p-1)=p$.

Case 3. Suppose $o(x)=p$. There are $(p-1)$ elements of order p. In this case e and x^{-1} are both adjacent to x. Moreover, for any other prime order element $z(\neq x,x^2)\in G$, we see that x is adjacent to $x^{-1}z$. The number of such z is $(p-2)$ [$(p-3)$ elements of order p and one element of order 2]. Therefore the total number of adjacent vertices of x in $\mathscr{PD}(G)$ is $1+1+(p-2)=p$.

Case 4. Suppose that $o(x)=2p$. Then obviously $o(x^2)=p$. Clearly, x^{-1} is adjacent to x. Moreover, for any other prime order element $u(\neq x^2)\in G$, we see that x is adjacent to $x^{-1}u$. The number of such u is $(p-1)$ [$(p-2)$ elements of order p and one element of order 2]. Hence the total number of adjacent vertices of x in $\mathscr{PD}(G)$ is $1+(p-1)=p$.

Therefore, considering all the cases we have $\deg(x)=p$ in $\mathscr{PD}(G)$. Since $x\in G$ is arbitrary, we must have $\mathscr{PD}(G)$ is a p-regular graph. ■

Corollary 3.2. *For any odd prime p, $\mathscr{PD}\big(\mathbb{Z}_{2p}\big)$ is a p-regular graph.*

Remark 3.3. Figure 1 shows that $\mathscr{PD}(\mathbb{Z}_4)$ is 1-regular and thus Theorem 3.1 is not true when p is even prime.

Lemma 3.4. *$\mathscr{PD}(\mathbb{Z}_{2^n})$ is disconnected for $n\geq 2$ and non regular for $n\geq 3$.*

Proof. For $n\geq 2$, the set $\{\overline{0},\overline{2^{n-1}}\}$ forms a connected component of $\mathscr{PD}(\mathbb{Z}_{2^n})$ and thus $\mathscr{PD}(\mathbb{Z}_{2^n})$ is a disconnected graph.

Now, we consider the graph $\mathscr{PD}(\mathbb{Z}_{2^n})$ for $n\geq 3$. In this graph $\deg(\overline{1})=2$ because $\overline{1}$ is adjacent to $\overline{2^n-1}$ and $\overline{2^{n-1}-1}$, whereas $\deg(\overline{2^{n-1}})=1$, since $o(\overline{2^{n-1}})=2$ in the group $\mathbb{Z}_{2^n}$. Hence $\mathscr{PD}(\mathbb{Z}_{2^n})$ is non regular. ■

Lemma 3.5. *For any odd prime p and any integer $n\geq 2$, $\mathscr{PD}(\mathbb{Z}_{p^n})$ is a non regular graph.*

Proof. Since $\mathbb{Z}_{p^n}$ is a cyclic group, the number of elements of order p is $\varphi(p)=p-1$. This implies $\deg(\overline{0})=p-1$ in $\mathscr{PD}(\mathbb{Z}_{p^n})$. Again, $\overline{1}$ is adjacent to

$\overline{p^{n-1}-1}, \overline{2p^{n-1}-1}, \overline{3p^{n-1}-1}, \ldots, \overline{(p-1)p^{n-1}-1}$ and $\overline{p^n-1}$ in $\mathscr{PD}(\mathbb{Z}_{p^n})$. Therefore, $\deg(\overline{1}) = p$ in $\mathscr{PD}(\mathbb{Z}_{p^n})$. Hence $\mathscr{PD}(\mathbb{Z}_{p^n})$ is not a regular graph. ■

Lemma 3.6. *If $n \neq p, 2p$ for some prime p and if $n = p_1^{r_1} p_2^{r_2} \cdots p_m^{r_m}$, where $p_1, p_2, \ldots, p_m$ are distinct primes and $r_1, r_2, \ldots, r_m$ are positive integers, then $\mathscr{PD}(\mathbb{Z}_n)$ is not a regular graph.*

Proof. Similar to the proof of Lemma 3.5, we can prove that $\deg(\overline{0}) = \varphi(p_1) + \varphi(p_2) + \cdots + \varphi(p_m)$ and $\deg(\overline{1}) = \varphi(p_1) + \varphi(p_2) + \cdots + \varphi(p_m) + 1$ in $\mathscr{PD}(\mathbb{Z}_n)$. Hence $\mathscr{PD}(\mathbb{Z}_n)$ is not a regular graph. ■

We are now in a position to characterize all finite cyclic groups $\mathbb{Z}_n$ for which $\mathscr{PD}(\mathbb{Z}_n)$ is regular.

Theorem 3.7. *$\mathscr{PD}(\mathbb{Z}_n)$ is a regular graph if and only if $n = p$ or $2p$ for some prime p.*

Proof. First we assume that $n = p$ for some prime p. Then by Theorem 2.1, it follows that $\mathscr{PD}(\mathbb{Z}_p)$ is a complete graph with p vertices and hence $\mathscr{PD}(\mathbb{Z}_p)$ is a $(p-1)$-regular graph. On the other hand if $n = 2p$ for some odd prime p, then by Corollary 3.2, we have $\mathscr{PD}(\mathbb{Z}_{2p})$ is a p-regular graph. Moreover, from Figure 1, we see that $\mathscr{PD}(\mathbb{Z}_4)$ is 1-regular.

Converse part follows from Lemma 3.6. ■

Corollary 3.8. *For any positive integer $n (\neq 4)$, if $\mathscr{PD}(\mathbb{Z}_n)$ is regular then it is connected.*

Proof. From Theorem 3.7, we have $\mathscr{PD}(\mathbb{Z}_n)$ is regular if and only if $n = p$ or $2p$ for some prime p. Now for any prime p, $\mathscr{PD}(\mathbb{Z}_p)$ is complete and hence it is connected. If p is an odd prime, then by Theorem 3.7, we have $\mathscr{PD}(\mathbb{Z}_{2p})$ is a p-regular graph with $2p$ vertices and hence it is connected. ■

Remark 3.9. The converse of Corollary 3.8 is not true in general. The following graph shows that $\mathscr{PD}(\mathbb{Z}_{15})$ is connected but not regular.

Definition 3.10 [8]**.** The line graph of a graph G, written $L(G)$, is the graph whose vertices are the edges of G, with $ef \in E(L(G))$ when $e = uv$ and $f = vw$ in G.

Theorem 3.11. *For any odd prime p,*

(i) *the line graph $L(\mathscr{PD}(\mathbb{Z}_p))$ of $\mathscr{PD}(\mathbb{Z}_p)$ is $(2p-4)$-regular,*

(ii) *the line graph $L(\mathscr{PD}(\mathbb{Z}_{2p}))$ of $\mathscr{PD}(\mathbb{Z}_{2p})$ is $(2p-2)$-regular.*

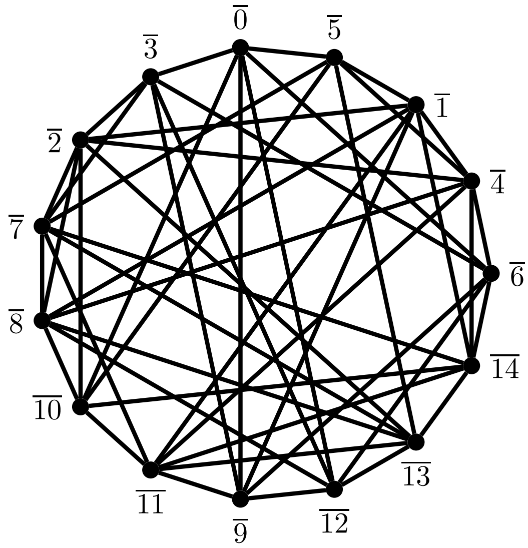

Figure 5. $\mathscr{PD}(\mathbb{Z}_{15})$

Proof. (i) Let p be an odd prime. Then $\mathscr{PD}(\mathbb{Z}_p)$ is $(p-1)$-regular. Let e be any edge of $\mathscr{PD}(\mathbb{Z}_p)$. Then e represents a vertex e_v in $L(\mathscr{PD}(\mathbb{Z}_p))$. Let a_e and b_e be the end vertices of the edge e in $\mathscr{PD}(\mathbb{Z}_p)$. Again $\deg(a_e) = deg(b_e) = (p-1)$ in $\mathscr{PD}(\mathbb{Z}_p)$. Since $\mathscr{PD}(\mathbb{Z}_p)$ is simple, the edge e has the vertex a_e in common with $(p-2)$ edges and the vertex b_e in common with $(p-2)$ edges in $\mathscr{PD}(\mathbb{Z}_p)$. Hence $\deg(e_v) = (p-2) + (p-2) = (2p-4)$ in $L(\mathscr{PD}(\mathbb{Z}_p))$. Therefore $L(\mathscr{PD}(\mathbb{Z}_p))$ is $(2p-4)$-regular graph.

(ii) Since $\mathscr{PD}(\mathbb{Z}_{2p})$, where p is any odd prime, is p-regular, so by the similar argument as in (i) of this theorem, we have the line graph $L(\mathscr{PD}(\mathbb{Z}_{2p}))$ of $\mathscr{PD}(\mathbb{Z}_{2p})$ is $(2p-2)$-regular. ■

Theorem 3.12 (Dirac [8])**.** *Let G be a simple graph with $n(> 2)$ vertices. If $\deg(v) \geq \frac{n}{2}$ for every vertex v of G, then G is Hamiltonian.*

Theorem 3.13. *If $n = p$ or $2p$, where p an odd prime, then $\mathscr{PD}(\mathbb{Z}_n)$ is a Hamiltonian graph.*

Proof. Let p be an odd prime. From Theorem 2.1, it follows that $\mathscr{PD}(\mathbb{Z}_p)$ is complete and hence it is Hamiltonian. On the other hand from Theorem 3.7, we have $\mathscr{PD}(\mathbb{Z}_{2p})$ is a p-regular graph and by Theorem 3.12, it follows that $\mathscr{PD}(\mathbb{Z}_{2p})$ is a Hamiltonian graph. ■

Theorem 3.14. *For a finite commutative group G, $\mathscr{PD}(G)$ is a bipartite graph if and only if $G \cong \mathbb{Z}_{2^r}$, for some positive integer r.*

Proof. Let $G \cong \mathbb{Z}_{2^r}$, for some positive integer r. If G is isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_{2^2}$ then clearly $\mathscr{PD}(G)$ is a bipartite graph. We now consider $r \geq 3$. Then in the order prime divisor graph $\mathscr{PD}(\mathbb{Z}_{2^r})$, $\deg(\overline{0}) = \deg(\overline{2^{r-1}}) = \deg(\overline{2^{r-2}}) = \deg(\overline{3 \cdot 2^{r-2}}) = 1$ and also $\{\overline{0}, \overline{2^{r-1}}\}$, $\{\overline{2^{r-2}}, \overline{3 \cdot 2^{r-2}}\}$ form two components. Let $\overline{x} \in \mathbb{Z}_{2^r}$ such that $\overline{x} \notin \{\overline{0}, \overline{2^{r-1}}, \overline{2^{r-2}}, \overline{3 \cdot 2^{r-2}}\}$. Then $\{\overline{x}, \overline{2^{r-1} - x}, \overline{2^{r-1} + x}, \overline{2^r - x}\}$ forms a component which is isomorphic to C_4. Hence every component of $\mathscr{PD}(\mathbb{Z}_{2^r})$ is isomorphic to either K_2 or C_4. Hence $\mathscr{PD}(\mathbb{Z}_{2^r})$ has no odd cycle and consequently, $\mathscr{PD}(G)$ is a bipartite graph.

Conversely, let G be a finite commutative group of order n such that $\mathscr{PD}(G)$ is a bipartite graph. We now show that n is not divisible by any odd prime p. If any odd prime p divides n, then G contains an element x of order p. Since p is an odd prime, we have $x \neq x^{-1}$ and $o(x^{-1}) = p$. This implies $x \leftrightarrow e \leftrightarrow x^{-1} \leftrightarrow x$ forms a cycle in $\mathscr{PD}(G)$ of length 3. Therefore $\mathscr{PD}(G)$ contains an odd cycle, which contradicts that $\mathscr{PD}(G)$ is a bipartite graph. Hence n is of the form 2^r, for some positive integer r. Therefore $G \cong \mathbb{Z}_{2^{n_1}} \oplus \mathbb{Z}_{2^{n_2}} \oplus \cdots \oplus \mathbb{Z}_{2^{n_k}}$ for some positive integers $n_1, n_2, \ldots, n_k$. We now show that $k = 1$. On the contrary, let $k > 1$. Then $\{(\overline{2^{n_1-1}}, \overline{0}, \ldots, \overline{0}), (\overline{0}, \overline{2^{n_2-1}}, \ldots, \overline{0}), (\overline{2^{n_1-1}}, \overline{2^{n_2-1}}, \ldots, \overline{0})\}$ forms a triangle. Thus $\mathscr{PD}(G)$ is not bipartite graph, which is again a contradiction. Therefore, $k = 1$ and thus $G \cong \mathbb{Z}_{2^r}$, for some positive integer r. ■

Theorem 3.15. *Let p be an odd prime and $n \geq 2$ is an integer. Then $\mathscr{PD}(\mathbb{Z}_{p^n})$ have one $(p-1)$-regular component and $\frac{p^{n-1}-1}{2}$ components each of them is p-regular.*

Proof. In $\mathscr{PD}(\mathbb{Z}_{p^n})$, $\{\overline{0}, \overline{p^{n-1}}, \overline{2p^{n-1}}, \ldots, \overline{(p-1)p^{n-1}}\}$ forms a component with p vertices and this component is of $(p-1)$-regular.

Let $A_i = \{\overline{i}, \overline{p^{n-1} - i}, \overline{2p^{n-1} - i}, \ldots, \overline{p^n - i}, \overline{p^{n-1} + i}, \overline{2p^{n-1} + i}, \ldots, \overline{(p-1)p^{n-1} + i}\} = U_i \cup V_i$, where $U_i = \{\overline{p^{n-1} - i}, \overline{2p^{n-1} - i}, \ldots, \overline{p^n - i}\}$ and $V_i = \{\overline{i}, \overline{p^{n-1} + i}, \overline{2p^{n-1} + i}, \ldots, \overline{(p-1)p^{n-1} + i}\}$, for all $i = 1, 2, \ldots, \frac{p^{n-1}-1}{2}$.

We now show that every A_i, for all $i = 1, 2, \ldots, \frac{p^{n-1}-1}{2}$, contains $2p$ elements. Let $\overline{u_1}, \overline{u_2} \in U_i$ be any two elements of U_i. Let $\overline{u_1} = \overline{rp^{n-1} - i}, \overline{u_2} = \overline{sp^{n-1} - i}$ for some $r, s \in \{1, 2, \ldots, p\}$. If possible let $\overline{u_1} = \overline{u_2}$, i.e., $rp^{n-1} - i + t_1 p^n = sp^{n-1} - i + t_2 p^n$, i.e., $(r - s) = (t_2 - t_1)p$ which is possible only when $r = s$ and hence U_i contains p elements. Similarly, we can show that V_i contains p elements. Now we need to show that $U_i \cap V_i = \emptyset$. If possible let $\overline{u} \in U_i \cap V_i$. Then $u = cp^{n-1} - i + t_3 p^n = dp^{n-1} + i + t_4 p^n$, where $c \in \{1, 2, \ldots, p\}$ and $d \in \{0, 1, \ldots, (p-1)\}$, i.e., $(c - d)p^{n-1} + (t_3 - t_4)p^n = 2i$, i.e., $2i$ is divisible by p^{n-1}, a contradiction since $2i \leq p^{n-1} - 1 < p^{n-1}$. Therefore, $U_i \cap V_i = \emptyset$ and hence A_i contains $2p$ elements.

Let $\overline{rp^{n-1} - i} \in U_i$ for some $r \in \{1, 2, \ldots, p\}$, and $\overline{sp^{n-1} + i} \in V_i$ for some $s \in \{0, 1, \ldots, (p-1)\}$, be any two elements. In $\mathbb{Z}_{p^n}$, $o(\overline{rp^{n-1} - i + sp^{n-1} + i}) = p$

and thus every element of U_i is adjacent to each element of V_i. But for any two elements $\overline{rp^{n-1}-i}, \overline{tp^{n-1}-i} \in U_i$ $(r, t \in \{1, 2, \ldots, p\})$, we have $o(\overline{r_1 p^{n-1}-i} + \overline{r_2 p^{n-1}-i}) \neq p$ and thus no two elements of U_i are adjacent. Similarly we can show that no two elements in V_i are adjacent. Hence degree of every vertex in A_i, for all $i = 1, 2, \ldots, \frac{p^{n-1}-1}{2}$, is p in $\mathscr{PD}(\mathbb{Z}_{p^n})$.

We now show that every A_i, for all $i = 1, 2, \ldots, \frac{p^{n-1}-1}{2}$, forms a component of $\mathscr{PD}(\mathbb{Z}_{p^n})$. Let $x \in A_i$. Then x is adjacent to either $\overline{i}$ or $\overline{p^{n-1}-i}$ in $\mathscr{PD}(\mathbb{Z}_{p^n})$. Therefore every element of A_i is connected to each other by a path of length at most 2. It is easy to verify that there is no edge between a vertex in A_i and a vertex in A_j for $i \neq j$ and $i, j \in \left\{1, 2, \ldots, \frac{p^{n-1}-1}{2}\right\}$. Therefore, every A_i, for all $i = 1, 2, \ldots, \frac{p^{n-1}-1}{2}$, forms a p-regular component with $2p$ vertices. Hence the theorem. ■

4. Order prime divisor graph of the group D_n

For each positive integer $n \geq 3$, the dihedral group of degree n, denoted by D_n, is a non-commutative group containing $2n$ elements and is defined by $D_n = \{\langle a, b\rangle : o(a) = n, o(b) = 2, ba = a^{-1}b\}$. The study of dihedral group D_n helps us to characterize non commutative groups. In this section, we establish some graph-theoretic properties of $\mathscr{PD}(D_n)$.

Before going to our results, we first consider the order prime divisor graph $\mathscr{PD}(D_8)$.

Example 4.1.

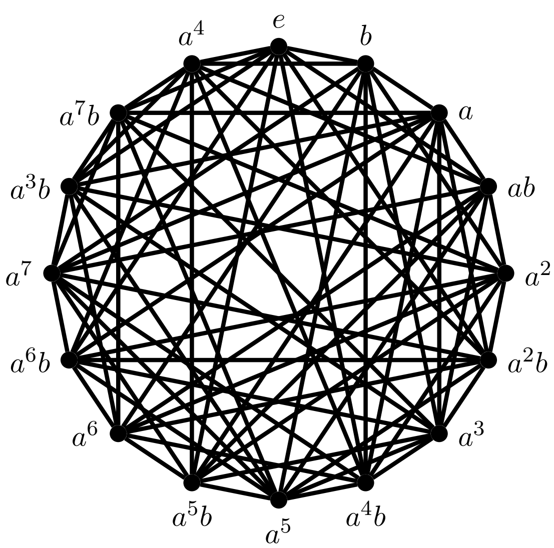

Figure 6. $\mathscr{PD}(D_8)$

From Figure 6, we have $\deg(a) = \deg(a^3) = \deg(a^5) = \deg(a^7) = 10$ and $\deg(e) = \deg(a^2) = \deg(a^4) = \deg(a^6) = \deg(b) = \deg(ab) = \deg(a^2b) = \deg(a^3b) = \deg(a^4b) = \deg(a^5b) = \deg(a^6b) = \deg(a^7b) = 9$ in order prime divisor graph $\mathscr{PD}(D_8)$. Here $8 = 2^3$ and note that degree of any vertex in $\mathscr{PD}(D_8)$ is either $\varphi(2) + 8$ or $\varphi(2) + 8 + 1$.

Theorem 4.2. *Let $n(\geq 3)$ be a number and $n = p_1^{r_1} p_2^{r_2} \cdots p_m^{r_m}$ be the factorization of n as product of distinct primes and their positive powers. Then the degree of a vertex of $\mathscr{PD}(D_n)$ is either $\varphi(p_1) + \varphi(p_2) + \cdots + \varphi(p_m) + n$ or $\varphi(p_1) + \varphi(p_2) + \cdots + \varphi(p_m) + n + 1$.*

Proof. Now $D_n = \{\langle a, b\rangle : o(a) = n, o(b) = 2, ba = a^{-1}b\}$. Let $H = \langle a\rangle$ and $K = Hb$. Then $D_n = H \cup K$ and all the n elements of K are of order 2. Also the subgroup H contains $\varphi(p_1)$ elements of order p_1, $\varphi(p_2)$ elements of order p_2, and so on, $\varphi(p_m)$ elements of order p_m. Now $\deg(e)$ in $\mathscr{PD}(D_n)$ is exactly equal to the number of prime order elements in D_n. Therefore, $\deg(e) = \varphi(p_1) + \varphi(p_2) + \cdots + \varphi(p_m) + n$ in $\mathscr{PD}(D_n)$. Let $x(\neq e) \in D_n$ be any element.

If $o(x) = 2$, then e is adjacent to x. Also for any other element $y(\neq x) \in G$ with prime order, we see that x is adjacent to $x^{-1}y$. Thus the total number of adjacent vertices of x in $\mathscr{PD}(D_n)$ is $\varphi(p_1) + \varphi(p_2) + \cdots + \varphi(p_m) + n$.

Suppose $o(x) = p$, where p is an odd prime. Then x^2 is again an element of order p. In this case e and x^{-1} are adjacent to x. Moreover, for any other element $z(\neq x, x^2) \in G$ with prime order, we see that x is adjacent to $x^{-1}z$. Thus the total number of adjacent vertices of x in $\mathscr{PD}(D_n)$ is $\varphi(p_1) + \varphi(p_2) + \cdots + \varphi(p_m) + n$.

Finally, suppose that $o(x)$ is composite. Then in this case x^{-1} is adjacent to x. If x^2 is an element of prime order, then for any other element $u(\neq x^2) \in G$ with prime order, we see that x is adjacent to $x^{-1}u$. Thus, the total number of adjacent vertices of x in $\mathscr{PD}(D_n)$ is $\varphi(p_1) + \varphi(p_2) + \cdots + \varphi(p_m) + n$. On the other hand, if $o(x^2)$ is composite, then for any other element $u \in G$ with prime order, we see that x is adjacent to $x^{-1}u$. Hence the total number of adjacent vertices of x in $\mathscr{PD}(D_n)$ is $\varphi(p_1) + \varphi(p_2) + \cdots + \varphi(p_m) + n + 1$. Thus the proof is completed. ■

Theorem 4.3. *$\mathscr{PD}(D_n)\,(n \geq 3)$ is a regular graph if and only if $n = p$ or $2p$ for some prime p.*

Proof. Let $p\,(\geq 3)$ be a prime. Now D_p is a non-commutative group of order $2p$ such that every non identity element is of prime order. Hence by Theorem 2.1, it follows that $\mathscr{PD}(D_p)$ is complete and thus it is $(2p - 1)$-regular.

For $p = 2$, we have from Figure 3 that $\mathscr{PD}(D_{2p}) = \mathscr{PD}(D_4)$ is 5-regular.

We now establish the regularity of the graph $\mathscr{PD}(D_{2p})$, where p is any odd prime. Note that $D_{2p} = \{e, a, a^2, \ldots, a^{2p-1}, b, ab, a^2b, \ldots, a^{2p-1}b\} = H \cup K$, where $H = \langle a\rangle$ is the cyclic subgroup of D_{2p} generated by a and $K = Hb$ is a right coset of H, different from H. Here every element of K is of order 2. Moreover, H contains exactly $(p-1)$ elements of order p and a unique element of order 2. Let $a^r \in H$ and $a^s b \in K$. Then $a^r a^s b = a^{r+s(mod\ 2p)}b$, for $r, s = 1, 2, \dots, 2p$. Thus $a^r a^s b \in K$ and hence $o(a^r a^s b) = 2$. Therefore, every element of H is adjacent to every element of K in $\mathscr{PD}(D_{2p})$. Let $a^r, a^s \in H$, then $a^r a^s \in H$, for $r, s = 1, 2, \dots, 2p$. Finally, for any two elements $a^r b, a^s b \in K$, we have $a^r b a^s b \in H$, for $r, s = 1, 2, \dots, 2p$. We now show that every vertex of $\mathscr{PD}(D_{2p})$ is of degree $3p$. For this let x be any vertex of $\mathscr{PD}(D_{2p})$.

Suppose $x \in H$. Then x is adjacent to every element of K. Moreover, similar to the proof of Theorem 3.1, we can conclude that x is adjacent to exactly p elements of H. Hence, if $x \in H$, then $\deg(x) = 3p$. On the other hand, if $x \in K$, then all the $2p$ elements of H are adjacent to x. Since $x \in K$, we must have $x^2 = e$ and $K = Hx$. Also, $Kx = (Hx)x = Hx^2 = H$. Now H contains total p elements of prime order (exactly $p-1$ elements of order p and 1 element of order 2). Therefore, x is adjacent to exactly p elements of K. Hence total number of adjacent vertices is $3p$ and thus $\deg(x) = 3p$. Therefore, in either cases $\deg(x) = 3p$ in $\mathscr{PD}(D_{2p})$. Consequently, $\mathscr{PD}(D_{2p})$ is a $3p$-regular graph.

Conversely, we assume that $\mathscr{PD}(D_n)$ is regular. We show that $n = p$ or $2p$ for some prime p. On the contrary we let $n \neq p, 2p$ for any prime p. Let $n = p_1^{r_1} p_2^{r_2} \cdots p_m^{r_m}$, where $p_1, p_2, \dots, p_m$ are distinct primes and $r_1, r_2, \dots, r_m$ are positive integers. Then by Theorem 4.2, it follows that $\deg(e) = \varphi(p_1) + \varphi(p_2) + \cdots + \varphi(p_m) + n$ whereas $\deg(a) = \varphi(p_1) + \varphi(p_2) + \cdots + \varphi(p_m) + n + 1$ in $\mathscr{PD}(D_n)$. This leads to $\mathscr{PD}(D_n)$ is not regular, which is a contradiction. Consequently, $n = p$ or $2p$ for some prime p. ∎

From Corollary 3.2, Theorem 4.3, Figure 1 and Figure 2, we have the following result.

Theorem 4.4. *If G is a group of order p or $2p$, where p is prime, then $\mathscr{PD}(G)$ is regular.*

Theorem 4.5. *Let p be an odd prime, then*

(i) *the line graph $L(\mathscr{PD}(D_p))$ of $\mathscr{PD}(D_p)$ is $(4p-4)$-regular.*

(ii) *the line graph $L(\mathscr{PD}(D_{2p}))$ of $\mathscr{PD}(D_{2p})$ is $(6p-2)$-regular.*

Proof. (i) For any prime $p \geq 3$, $\mathscr{PD}(D_p)$ is $(2p-1)$-regular graph. Thus by the similar argument of the proof of Theorem 3.11(i), it follows that the line graph $L(\mathscr{PD}(D_p))$ of $\mathscr{PD}(D_p)$ is $(4p-4)$-regular.

(ii) For any odd prime p, the graph $\mathscr{PD}(D_{2p})$ is $3p$-regular. So by the similar argument of the proof of Theorem 3.11(i), we have the line graph $L(\mathscr{PD}(D_{2p}))$ of $\mathscr{PD}(D_{2p})$ is $(6p-2)$-regular. ∎

Corollary 4.6. *The line graph $L(\mathscr{PD}(D_4))$ of $\mathscr{PD}(D_4)$ is 8-regular graph.*

Proof. Since $\mathscr{PD}(D_4)$ is 5-regular graph, so by the similar argument of the proof of Theorem 3.11(i), it follows that the line graph $L(\mathscr{PD}(D_4))$ of $\mathscr{PD}(D_4)$ is 8-regular. ∎

Theorem 4.7. *For any integer $n(\geq 3)$, $\mathscr{PD}(D_n)$ is connected as well as Hamiltonian.*

Proof. Let $n(\geq 3)$ and $n = p_1^{r_1} p_2^{r_2} \cdots p_m^{r_m}$ be the factorization of n as product of distinct primes and their positive powers. Then $\mathscr{PD}(D_n)$ is a graph with $2n$ vertices and by Theorem 4.2, we have degree of each vertex is either $\varphi(p_1) + \varphi(p_2) + \cdots + \varphi(p_m) + n$ or $\varphi(p_1) + \varphi(p_2) + \cdots + \varphi(p_m) + n + 1$. Thus $\deg(v) \geq n$ for every vertex in $\mathscr{PD}(D_n)$. Therefore, by Theorem 3.12, it follows that $\mathscr{PD}(D_n)$ is Hamiltonian and hence connected. ∎

Theorem 4.8. *The graph $\mathscr{PD}(D_n)$ is not Eulerian for any positive integer $n(\geq 3)$.*

Proof. Let $n(\geq 3)$ and $n = p_1^{r_1} p_2^{r_2} \cdots p_m^{r_m}$ be the factorization of n as product of distinct primes and their positive powers. Then $\mathscr{PD}(D_n)$ is a graph with $2n$ vertices and by Theorem 4.2, we have degree of each vertex is either $\varphi(p_1)+\varphi(p_2)+ \cdots + \varphi(p_m) + n$ or $\varphi(p_1) + \varphi(p_2) + \cdots + \varphi(p_m) + n + 1$. Now $\varphi(p_1) + \varphi(p_2) + \cdots + \varphi(p_m)+n$ or $\varphi(p_1)+\varphi(p_2)+\cdots+\varphi(p_m)+n+1$ are two consecutive positive integers. So one of them must be even and another must be odd. Therefore, $\mathscr{PD}(D_n)$ contains odd degree vertices. Consequently, $\mathscr{PD}(D_n)$ is not Eulerian. ∎

Remark 4.9. For any odd prime p, the graph $\mathscr{PD}(D_p)$ is complete and hence $diam(\mathscr{PD}(D_p)) = 1$.

Theorem 4.10. *For any composite number $n\,(\geq 3)$, $diam(\mathscr{PD}(D_n)) = 2$.*

Proof. Now $D_n = \{\langle a, b\rangle : o(a) = n, o(b) = 2, ba = a^{-1}b\}$. Let $H = \langle a\rangle$, the cyclic subgroup generated by a and $K = Hb$ be the right coset of H different from H. Then $D_n = H \cup K$. Here, every element of K is of order 2. Let $a^r \in H$ and $a^s b \in K$. Then $a^r a^s b = a^{r+s(\text{mod } n)} b$, for $r, s = 1, 2, \ldots, n$. Hence $a^r a^s b \in K$ and thus $o(a^r a^s b) = 2$. Therefore, every element of H is adjacent to every element of K in $\mathscr{PD}(D_n)$. Let x, y be any two vertices of $\mathscr{PD}(D_n)$.

Case 1. If H contains exactly one of x or y and K contains the other, then $x \leftrightarrow y$ is a path in $\mathscr{PD}(D_n)$ of length 1.

Case 2. We now consider the other possibility. Without loss of generality, we assume that $x, y \in H$. Then two sub-cases arise.

Subcase (a). Suppose $xy = e$ or $o(xy) = p$ for some prime p. Then $x \leftrightarrow y$ is a path in $\mathscr{PD}(D_n)$ of length 1.

Subcase (b). If x and y are not adjacent in $\mathscr{PD}(D_n)$. Then for any $z \in K$, we get a path $x \leftrightarrow z \leftrightarrow y$ in $\mathscr{PD}(D_n)$ of length 2.

Therefore, there is a path between any two vertices of $\mathscr{PD}(D_n)$. Hence $\mathscr{PD}(D_n)$ is connected and $diam(\mathscr{PD}(D_n)) = 2$. ■

Theorem 4.11. *For any $n(\geq 3)$, the graph $\mathscr{PD}(D_n)$ is neither bipartite nor a tree. Moreover, $girth(\mathscr{PD}(D_n)) = 3$.*

Proof. For any $n \geq 3$, $D_n = \{\langle a, b\rangle : o(a) = n, o(b) = 2, ba = a^{-1}b\}$. Let p be a prime factor of n. Then $n = pq$ for some positive integer q. Now $a^q \in D_n$ such that $o(a^q) = p$. Then $a^q \leftrightarrow b \leftrightarrow e \leftrightarrow a^q$ forms a cycle of length 3. Since $\mathscr{PD}(D_n)$ is a simple graph, it follows that $girth(\mathscr{PD}(D_n)) = 3$. Since $\mathscr{PD}(D_n)$ contains an odd cycle, it follows that $\mathscr{PD}(D_n)$ is neither bipartite nor a tree. ■

Theorem 4.12. *For any integer $(n \geq 3)$, $\mathscr{PD}(D_n)$ is non planar.*

Proof. We prove that $\mathscr{PD}(D_n)$ is non planar for any integer $n(\geq 3)$. On the contrary, suppose $\mathscr{PD}(D_k)$ is planar for some integer $k(\geq 3)$. First we claim that 2 is only one prime factor of k. If not, let p be an odd prime factor of k and thus $k = p^m q$ for some positive integers m, q with $\gcd(p, q) = 1$. Now $D_k = \{\langle a, b\rangle : o(a) = k, o(b) = 2, ba = a^{-1}b\}$. Let $x \in D_k$ such that $o(x) = p$. Then the induced subgraph of $\mathscr{PD}(D_k)$ induced by the set of vertices $\{e, x, x^{-1}, b, xb\}$ forms the complete subgraph K_5. Hence by Kuratowski's Theorem, we conclude that $\mathscr{PD}(D_k)$ is non planar, which contradicts our assumption that $\mathscr{PD}(D_k)$ is planar. This contradiction ensures that 2 is the only one prime factor of k and hence $k = 2^r$ for some positive integer $r \geq 2$.

Now we show that $\mathscr{PD}(D_{2^r})$ is non planar. First we see from Fig. 3 that $\mathscr{PD}(D_4)$ contains complete bipartite graph $K_{3,3}$ with bipartition $\{e, a, a^3\}$ and $\{b, ab, a^3b\}$ as a subgraph. Hence by Kuratowski's Theorem, we have $\mathscr{PD}(D_4)$ is non planar. Moreover, for $r \geq 3$, we have D_{2^r} has a subgroup isomorphic to D_4. Thus $\mathscr{PD}(D_{2^r})$ has a subgraph isomorphic to $\mathscr{PD}(D_4)$. Since $\mathscr{PD}(D_4)$ is non planar, it follows that $\mathscr{PD}(D_{2^r}) = \mathscr{PD}(D_k)$ is also non planar. Hence the theorem. ■

Theorem 4.13 (Brook's Theorem [8]). *If G is a connected graph other than a complete graph or an odd cycle, then $\chi(G) \leq \Delta(G)$, where $\chi(G)$ and $\Delta(G)$ are the chromatic number and the maximum vertex degree of G respectively.*

Remark 4.14. If n is an odd prime, then $\mathscr{PD}(D_n)$ is a complete graph with $2n$ vertices and hence $\chi(\mathscr{PD}(D_n)) = 2n$.

Theorem 4.15. *Let $n(\geq 3)$ be a composite number and $n = p_1^{r_1} p_2^{r_2} \cdots p_m^{r_m}$ be the factorization of n, where $p_1, p_2, \ldots, p_m$ are distinct primes and $r_1, r_2, \ldots, r_m$ are positive integers. Then $\chi(\mathscr{PD}(D_n)) \leq \varphi(p_1) + \varphi(p_2) + \cdots + \varphi(p_m) + n + 1$.*

Proof. If $n(\geq 3)$ and $n = p_1^{r_1} p_2^{r_2} \cdots p_m^{r_m}$ be the factorization of n, where $p_1, p_2, \ldots, p_m$ are distinct primes and $r_1, r_2, \ldots, r_m$ are positive integers. Then by Theorem 4.2, it follows that degree of every vertex in $\mathscr{PD}(D_n)$ is either $\varphi(p_1) + \varphi(p_2) + \cdots + \varphi(p_m) + n$ or $\varphi(p_1) + \varphi(p_2) + \cdots + \varphi(p_m) + n + 1$. Therefore, $\Delta(\mathscr{PD}(D_n)) = \varphi(p_1) + \varphi(p_2) + \cdots + \varphi(p_m) + n + 1$. Moreover, $\mathscr{PD}(D_n)$ is connected graph which is neither complete nor an odd cycle. Hence by Theorem 4.13, it follows that $\chi(\mathscr{PD}(D_n)) \leq \varphi(p_1) + \varphi(p_2) + \cdots + \varphi(p_m) + n + 1$. ■

5. Order prime divisor graphs of small finite groups

Here we discuss all possible order prime divisor graphs $\mathscr{PD}(G)$, where G is a group of order at most 15. For this purpose, we first exhibit the order prime divisor graph $\mathscr{PD}(\mathbb{Z}_{12})$.

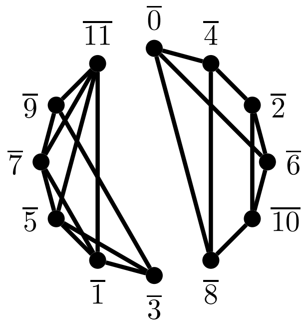

Figure 7. $\mathscr{PD}(\mathbb{Z}_{12})$

Order of Group G	Group G	Order Prime Divisor Graph $\mathscr{PD}(G)$
2	$\mathbb{Z}_2$	K_2
3	$\mathbb{Z}_3$	K_3
4	$\mathbb{Z}_4$ $\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$K_2 \cup K_2$ (Figure 1) K_4 (Figure 2)
5	$\mathbb{Z}_5$	K_5
6	$\mathbb{Z}_6$ S_3	3-regular connected graph K_6
7	$\mathbb{Z}_7$	K_7
8	$\mathbb{Z}_8$ $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ D_4 Q_8	$K_2 \cup K_2 \cup C_4$ $K_4 \cup K_4$ K_8 Figure 3 $K_2 \cup K_2 \cup K_2 \cup K_2$

9	$\mathbb{Z}_9$	Figure 4
	$\mathbb{Z}_3 \oplus \mathbb{Z}_3$	K_9
10	$\mathbb{Z}_{10}$	5-regular connected graph
	D_5	K_{10}
11	$\mathbb{Z}_{11}$	K_{11}
12	$\mathbb{Z}_{12}$	Figure 7
	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$	5-regular connected graph
	A_4	K_{12}
	D_6	9-regular connected graph
	T	Union of two 3-regular component
13	$\mathbb{Z}_{13}$	K_{13}
14	$\mathbb{Z}_{14}$	7-regular connected graph
	D_7	K_{14}
15	$\mathbb{Z}_{15}$	Figure 5

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