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$\mathcal{N}$-PRIME SPECTRUM OF STONE ALMOST DISTRIBUTIVE LATTICES

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Abstract

Introduced the notions of annulets and $\mathcal{N}$-filters in stone Almost Distributive Lattices and investigated their properties. Utilized annulets to characterize the $\mathcal{N}$-filters. Derived that every proper $\mathcal{N}$-filter is the intersection of all $\mathcal{N}$-prime filters containing it and also proved that the set $\mathcal{F}_{\mathcal{N}}(L)$ of all $\mathcal{N}$-filters is isomorphic to the class $Con_E(L)$ of all G-extentions of L. Given some topological properties of the space of all $\mathcal{N}$-prime filters. Derived a necessary and sufficient condition for the space of all $\mathcal{N}$-prime filters to be a Hausdorff space.

Keywords: Almost Distributive Lattice (ADL), stone ADL, ideal, filter, annulet, $\mathcal{N}$-filters, isomorphism, compact set, Hausdorff space.

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1. Introduction

After Boole's axiomatization of two valued propositional calculus as a Boolean algebra, a number of generalizations both ring theoretically and lattice theoretically have come into being. The concept of an Almost Distributive Lattice (ADL)

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was introduced by Swamy and Rao [6] as a common abstraction of many existing ring theoretic generalizations of a Boolean algebra on one hand and the class of distributive lattices on the other. In that paper, the concept of an ideal in an ADL was introduced analogous to that in a distributive lattice and it was observed that the set $PI(L)$ of all principal ideals of L forms a distributive lattice. This provided a path to extend many existing concepts of lattice theory to the class of ADLs. With this motivation, Swamy, Rao and Nanaji [7] introduced the concept of pseudo-complementation on an ADL. They observed that unlike in a distributive lattice, an ADL L can have more than one pseudo-complementation. If $*, \perp$ are two pseudo-complementations on L, it was observed that $x^* \vee x^{**}$ is maximal, for all $x \in L$ if and only if $x^{\perp} \vee x^{\perp\perp}$ is maximal, for all $x \in L$. With this motivation, in refswamy stone, the concept of a Stone ADL was introduced as an ADL with a pseudo-complementation $*$ satisfying the condition $x^* \vee x^{**}$ is maximal, for all $x \in L$. They studied the properties of pseudo-complemented ADLs and characterized Stone ADLs algebraically, topologically and by means of prime ideals. In [5], Rao and Ravi Kumar proved that some important results on minimal prime ideal of an ADL. In [1], Abd El-Mohsen Badawy introduced normal filters in a stone lattice and proved their properties. Also, the normal filters of a stone lattice are characterized in terms of annulets. In this paper, we extend the concept of $\mathcal{N}$-filters (normal filter) to a stone ADL, analogously and studied their properties. We characterize the $\mathcal{N}$-filters in terms of annulets. In addition to this, it was observed that a mapping Θ is an isomorphism of the set of all $\mathcal{N}$-filters of a stone ADL onto the set of all ideals of stone ADL. Some topological properties of the space $Spec_{\mathcal{N}_F}(L)$ of all $\mathcal{N}$-prime filters of an ADL L are observed. A set of equivalent conditions are derived for the space $Spec_{\mathcal{N}_F}(L)$ to become a T_1-space. A necessary and sufficient condition is obtained for $Spec_{\mathcal{N}_F}(L)$ to become a Hausdorff space.

2. Preliminaries

In this section, we recall certain definitions and important results, those will be required in the text of the paper.

Definition 2.1 [6]**.** An Almost Distributive Lattice with zero or simply ADL is an algebra $(L, \vee, \wedge, 0)$ of type $(2, 2, 0)$ satisfying:

(1) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$

(2) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

(3) $(x \vee y) \wedge y = y$

(4) $(x \vee y) \wedge x = x$

(5) $x \vee (x \wedge y) = x$

(6) $0 \wedge x = 0$

(7) $x \vee 0 = x$, for all $x, y, z \in L$.

Example 2.2. Every non-empty set X can be regarded as an ADL as follows. Let $x_0 \in X$. Define the binary operations $\vee, \wedge$ on X by

$$x \vee y = \begin{cases} x & \text{if } x \neq x_0 \\ y & \text{if } x = x_0 \end{cases} \qquad x \wedge y = \begin{cases} y & \text{if } x \neq x_0 \\ x_0 & \text{if } x = x_0. \end{cases}$$

Then $(X, \vee, \wedge, x_0)$ is an ADL (where x_0 is the zero) and is called a discrete ADL.

If $(L, \vee, \wedge, 0)$ is an ADL, for any $a, b \in L$, define $a \leq b$ if and only if $a = a \wedge b$ (or equivalently, $a \vee b = b$), then $\leq$ is a partial ordering on L.

Theorem 2.3 [6]**.** *If $(L, \vee, \wedge, 0)$ is an ADL, for any $a, b, c \in L$, we have the following:*

(1) $a \vee b = a \Leftrightarrow a \wedge b = b$

(2) $a \vee b = b \Leftrightarrow a \wedge b = a$

(3) $\wedge$ *is associative in* L

(4) $a \wedge b \wedge c = b \wedge a \wedge c$

(5) $(a \vee b) \wedge c = (b \vee a) \wedge c$

(6) $a \wedge b = 0 \Leftrightarrow b \wedge a = 0$

(7) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

(8) $a \wedge (a \vee b) = a$, $(a \wedge b) \vee b = b$ *and* $a \vee (b \wedge a) = a$

(9) $a \leq a \vee b$ *and* $a \wedge b \leq b$

(10) $a \wedge a = a$ *and* $a \vee a = a$

(11) $0 \vee a = a$ *and* $a \wedge 0 = 0$.

(12) *If* $a \leq c$, $b \leq c$, *then* $a \wedge b = b \wedge a$ *and* $a \vee b = b \vee a$

(13) $a \vee b = (a \vee b) \vee a$.

It can be observed that an ADL L satisfies almost all the properties of a distributive lattice except the right distributivity of $\vee$ over $\wedge$, commutativity of $\vee$, commutativity of $\wedge$. Any one of these properties make an ADL L a distributive lattice. That is

Theorem 2.4 [6]**.** *Let $(L, \vee, \wedge, 0)$ be an ADL with 0. Then the following are equivalent:*

(1) $(L, \vee, \wedge, 0)$ *is a distributive lattice*

(2) $a \vee b = b \vee a$, *for all* $a, b \in L$

(3) $a \wedge b = b \wedge a$, *for all* $a, b \in L$

(4) $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$, *for all* $a, b, c \in L$.

As usual, an element $m \in L$ is called maximal if it is a maximal element in the partially ordered set $(L, \leq)$. That is, for any $a \in L$, $m \leq a \Rightarrow m = a$.

Theorem 2.5 [6]. *Let L be an ADL and $m \in L$. Then the following are equivalent:*

(1) *m is maximal with respect to $\leq$*

(2) *$m \vee a = m$, for all $a \in L$*

(3) *$m \wedge a = a$, for all $a \in L$*

(4) *$a \vee m$ is maximal, for all $a \in L$.*

As in distributive lattices [2, 3], a non-empty sub set I of an ADL L is called an ideal of L if $a \vee b \in I$ and $a \wedge x \in I$ for any $a, b \in I$ and $x \in L$. Also, a non-empty subset F of L is said to be a filter of L if $a \wedge b \in F$ and $x \vee a \in F$ for $a, b \in F$ and $x \in L$.

The set $I(L)$ of all ideals of L is a bounded distributive lattice with least element $\{0\}$ and greatest element L under set inclusion in which, for any $I, J \in I(L)$, $I \cap J$ is the infimum of I and J while the supremum is given by $I \vee J := \{a \vee b \mid a \in I, b \in J\}$. A proper ideal P of L is called a prime ideal if, for any $x, y \in L$, $x \wedge y \in P \Rightarrow x \in P$ or $y \in P$. A proper ideal M of L is said to be maximal if it is not properly contained in any proper ideal of L. It can be observed that every maximal ideal of L is a prime ideal. Every proper ideal of L is contained in a maximal ideal. For any subset S of L the smallest ideal containing S is given by $(S] := \{(\bigvee_{i=1}^{n} s_i) \wedge x \mid s_i \in S, x \in L \text{ and } n \in N\}$. If $S = \{s\}$, we write $(s]$ instead of $(S]$. Similarly, for any $S \subseteq L$, $[S) := \{x \vee (\bigwedge_{i=1}^{n} s_i) \mid s_i \in S, x \in L \text{ and } n \in N\}$. If $S = \{s\}$, we write $[s)$ instead of $[S)$.

Theorem 2.6 [6]. *For any x, y in L the following are equivalent:*

(1) $(x] \subseteq (y]$

(2) $y \wedge x = x$

(3) $y \vee x = y$

(4) $[y) \subseteq [x)$.

For any $x, y \in L$, it can be verified that $(x] \vee (y] = (x \vee y]$ and $(x] \wedge (y] = (x \wedge y]$. Hence the set $PI(L)$ of all principal ideals of L is a sublattice of the distributive lattice $I(L)$ of ideals of L.

Theorem 2.7 [4]. *Let I be an ideal and F a filter of L such that $I \cap F = \emptyset$. Then there exists a prime ideal P such that $I \subseteq P$ and $P \cap F = \emptyset$.*

Definition 2.8. Let $(L, \vee, \wedge, 0)$ be an ADL. Then a unary operation $a \longrightarrow a^*$ on L is called a pseudo-complementation on L if, for any $a, b \in L$, it satisfies the following conditions:

(1) $a \wedge b = 0 \Rightarrow a^* \wedge b = b$

(2) $a \wedge a^* = 0$

(3) $(a \vee b)^* = a^* \wedge b^*$.

Then $(L, \vee, \wedge, ^*, 0)$ is called a pseudo-complemented ADL.

Theorem 2.9. *Let L be an ADL and $*$ a pseudo-complementation on L. Then, for any $a, b \in L$, we have the following:*

(1) $0^{**} = 0$

(2) $0^* \wedge a = a$

(3) $a^{**} \wedge a = a$

(4) $a^{***} = a^*$

(5) $a \leq b \Rightarrow b^* \leq a^*$

(6) $a^* \wedge b^* = b^* \wedge a^*$

(7) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$

(8) $a^* \wedge b = (a \wedge b)^* \wedge b^*$.

For any pseudo-complemented ADL L, let us denote the set of all elements of the form $x^* = 0$ by $D(L)$. Then the following lemma is a direct consequence.

Definition 2.10 [8]**.** Let L be an ADL and $*$ a pseudo-complementation on L.Then L is called Stone ADL if, for any $x \in L,\ x^* \vee x^{**} = 0^*$.

Lemma 2.11 [8]**.** *Let L be a Stone ADL and $a, b \in L$. Then the following conditions hold:*

(1) $0^* \wedge a = a$ *and* $0^* \vee a = 0^*$

(2) $(a \wedge b)^* = a^* \vee b^*$.

3. Annulets of Stone ADLs

Definition 3.1. Let S be a non-empty subset of a stone ADL L, which is closed under $\wedge$. Define $S^\perp$ as $S^\perp = \{a \in L \mid a^* \wedge s = 0,$ for some $s \in S\}$.

Lemma 3.2. *Let S, T be any two non-empty subsets of a stone ADL L with maximal elements, which are closed under $\wedge$. Then we have the following:*

(1) *$S^\perp$ is a filter of L containing S*

(2) $[S) \subseteq S^{\perp}$ *and* $D(L) \subseteq S^{\perp}$

(3) $S^{\perp} = [S) \vee D(L)$

(4) $S \subseteq T \Rightarrow S^{\perp} \subseteq T^{\perp}$

(5) $S^{\perp\perp} = S^{\perp}$

(6) $[S)^{\perp} = S^{\perp}$

(7) $L^{\perp} = L$.

Proof. (1) Let m be any maximal element of L. Clearly, $m \in S^{\perp}$ and hence $S^{\perp} \neq \emptyset$. We have that $s^* \wedge s = 0$, for all $s \in S$. That implies $s \in S^{\perp}$, for all $s \in S$. Therefore $S \subseteq S^{\perp}$. Let $a, b \in S^{\perp}$. Then there exist elements $s_1, s_2 \in S$ such that $a^* \wedge s_1 = 0$ and $b^* \wedge s_2 = 0$. Since $s_1, s_2 \in S$, we get that $s_1 \wedge s_2 \in S$. Now, $(a \wedge b)^* \wedge s_1 \wedge s_2 = (a^* \vee b^*) \wedge s_1 \wedge s_2 = (a^* \wedge s_1 \wedge s_2) \vee (b^* \wedge s_1 \wedge s_2) = 0$. That implies $a \wedge b \in S^{\perp}$. Let $a \in S^{\perp}$. Then there exists an element $s \in S$ such that $a^* \wedge s = 0$. Let r be any element of L. Since $(r \vee a)^* \leq a^*$, we get that $(r \vee a)^* \wedge s = 0$. That implies $r \vee a \in S^{\perp}$. Therefore $S^{\perp}$ is a filter of L containing S.

(2) Clearly, we have that $S \subseteq [S)$. By (1), we have that $S \subseteq S^{\perp}$. Since $[S)$ is the smallest filter containing S and $S^{\perp}$ is a filter of L containing S, we get that $[S) \subseteq S^{\perp}$. Let $a \in D(L)$. Then $a^* = 0$ and hence $a^* \wedge s = 0$, for all $s \in S$. Therefore $a \in S^{\perp}$. Thus $D(L) \subseteq S^{\perp}$.

(3) By (2), we have that $[S) \vee D(L) \subseteq S^{\perp}$. Let $a \in S^{\perp}$. Then there exists an element $s \in S$ such that $a^* \wedge s = 0$. Now, $a = a \vee 0 = a \vee (s \wedge a^*) = (a \vee s) \wedge (a \vee a^*)$. Since $a \vee a^* \in D(L)$ and $a \vee s \in [s) \subseteq [S)$, we get that $(a \vee s) \wedge (a \vee a^*) \in [S) \vee D(L)$ and hence $a \in [S) \vee D(L)$. Therefore $S^{\perp} \subseteq [S) \vee D(L)$. Thus $S^{\perp} = [S) \vee D(L)$.

(4) Assume that $S \subseteq T$. Let $a \in S^{\perp}$. Then there exists an element $s \in S$ such that $a^* \wedge s = 0$. Since $S \subseteq T$, we get that $a \in T^{\perp}$. Therefore $S^{\perp} \subseteq T^{\perp}$.

(5) By (1) and (4), we get that $S^{\perp} \subseteq S^{\perp\perp}$. Let $a \in S^{\perp\perp}$. Then there exists an element $s \in S^{\perp}$ such that $a^* \wedge s = 0$. That implies $s^* \wedge a^* = a^*$. Since $s \in S^{\perp}$, there exists an element $t \in S$ such that $s^* \wedge t = 0$. Since $s, t \in S$, we get that $s \wedge t \in S$. Now $a^* \wedge t = s^* \wedge a^* \wedge t = a^* \wedge s^* \wedge t = 0$. That implies $a \in S^{\perp}$. Therefore $S^{\perp\perp} \subseteq S^{\perp}$ and hence $S^{\perp\perp} = S^{\perp}$.

(6) By (2), we have that $[S) \subseteq S^{\perp}$. So that $[S)^{\perp} \subseteq S^{\perp\perp} = S^{\perp}$ and hence $[S)^{\perp} \subseteq S^{\perp}$. Let $a \in S^{\perp}$. Then there exists an element $s \in S$ such that $a^* \wedge s = 0$. Since $s \in S \subseteq [S)$, we get that $a \in [S)^{\perp}$. Therefore $S^{\perp} \subseteq [S)^{\perp}$ and hence $S^{\perp} = [S)^{\perp}$.

(7) It is clear. ∎

Theorem 3.3. *Let L be a stone ADL. Then $\mathfrak{F}^{\perp}(L) = \{F^{\perp} \mid F$ is a filter of $L\}$ is a bounded distributive lattice.*

Proof. Let F_1, F_2 be any two filters of L. By the above result, we have that $F_1^{\perp} \subseteq (F_1 \vee F_2)^{\perp}$ and $F_2^{\perp} \subseteq (F_1 \vee F_2)^{\perp}$. That implies $(F_1 \vee F_2)^{\perp}$ is an upper bound of $F_1^\perp$ and $F_2^\perp$. Let $F_3^\perp$ be any upper bound of $F_1^\perp$ and $F_2^\perp$. Then $F_1^\perp \subseteq F_3^\perp$ and $F_2^\perp \subseteq F_3^\perp$. That implies $F_1 \subseteq F_3^\perp$ and $F_2 \subseteq F_3^\perp$. That implies $(F_1 \vee F_2)^\perp \subseteq F_3^{\perp\perp} = F_3^\perp$. Therefore $(F_1 \vee F_2)^\perp$ is a least upper bound of $F_1^\perp$ and $F_2^\perp$. Hence $(F_1 \vee F_2)^\perp = F_1^\perp \vee F_2^\perp$. Clearly, we have that $F_1 \cap F_2 \subseteq F_1$ and $F_1 \cap F_2 \subseteq F_2$. By the above result, we have that $(F_1 \cap F_2)^\perp \subseteq F_1^\perp$ and $(F_1 \cap F_2)^\perp \subseteq F_2^\perp$. Therefore $(F_1 \cap F_2)^\perp \subseteq F_1^\perp \cap F_2^\perp$. Let $a \in F_1^\perp \cap F_2^\perp$. Then $a \in F_1^\perp$ and $a \in F_2^\perp$. Then there exist elements $f_1 \in F_1$ and $f_2 \in F_2$ such that $a^* \wedge f_1 = 0$ and $a^* \wedge f_2 = 0$. Since F_1, F_2 are filter of L and $f_1 \in F_1^\perp$, $f_2 \in F_2^\perp$, we get that $f_1 \vee f_2 \in F_1 \cap F_2$. Now, $a^* \wedge (f_1 \vee f_2) = (a^* \wedge f_1) \vee (a^* \wedge f_2) = 0$. That implies $a \in (F_1 \cap F_2)^\perp$. Therefore $F_1^\perp \cap F_2^\perp \subseteq (F_1 \cap F_2)^\perp$ and hence $(F_1 \cap F_2)^\perp = F_1^\perp \cap F_2^\perp$. Thus $\mathfrak{F}^\perp(L)$ is a sublattice of $\mathfrak{F}(L)$. Since $\mathfrak{F}(L)$ is a distributive lattice, we get that $\mathfrak{F}^\perp(L)$ is a distributive lattice with least element $D(L)$ and greatest element L. Therefore $(\mathfrak{F}^\perp(L), \vee, \cap, D(L), L)$ is a bounded distributive lattice. ■

For any subset $S = \{x\}$ of L, $\{x\}^\perp = \{a \in L \mid a^* \wedge x = 0\}$. We write $(x)^\perp$ instead of $\{x\}^\perp$. Clearly, $(x)^\perp$ is a filter of L. Then it called the annulet of x.

Lemma 3.4. *Let x, y be any two elements of a Stone ADL L with maximal elements. Then we have the following:*

(1) $x \in (x)^\perp$
(2) $[x) \subseteq (x)^\perp$
(3) $x \leq y \Rightarrow (y)^\perp \subseteq (x)^\perp$
(4) $x \in (y)^\perp \Rightarrow (x)^\perp \subseteq (y)^\perp$
(5) $(x)^{\perp\perp} = (x)^\perp$
(6) $[x)^\perp = (x)^\perp$
(7) $(x)^\perp = L \Leftrightarrow x = 0$
(8) $D(L) \subseteq (x)^\perp$, *for all* $x \in L$.
(9) *For any maximal element m of L,* $(m)^\perp = D(L)$
(10) $(x)^\perp = [x) \vee D(L) = [x^{**}) \vee D(L)$.
(11) *If $D(L)$ is a principal filter of L then $(x)^\perp$ is a principal filter of L, for all $x \in L$*
(12) $(x \vee y)^\perp = (y \vee x)^\perp$ *and* $(x \wedge y)^\perp = (y \wedge x)^\perp$
(13) $(x \vee y)^\perp = (x)^\perp \cap (y)^\perp$.
(14) *If $(x)^\perp = (y)^\perp$, then $(x \vee z)^\perp = (y \vee z)^\perp$ and $(x \wedge z)^\perp = (y \wedge z)^\perp$, for all $z \in L$.*
(15) *If $x \wedge y = 0$ then* $(x)^\perp \vee (y)^\perp = L$
(16) $(x)^\perp = (y)^\perp$ *if and only if* $x^* = y^*$
(17) $(x)^\perp = (x^{**})^\perp$

(18) $(x \vee (y \vee z))^{\perp} = ((x \vee y) \vee z)^{\perp}$.

Proof. (1) Since $x^* \wedge x = 0$, we get that $x \in (x)^{\perp}$.

(2) Since $(x)^{\perp}$ is a filter of L and $x \in (x)^{\perp}$, we get that $[x) \subseteq (x)^{\perp}$.

(3) Assume that $x \leq y$. Then $x \wedge y = x$. Let $a \in (y)^{\perp}$. Then $a^* \wedge y = 0$ and hence $a^* \wedge x = 0$. Therefore $a \in (x)^{\perp}$. Thus $(y)^{\perp} \subseteq (x)^{\perp}$.

(4) Assume that $x \in (y)^{\perp}$. Then $x^* \wedge y = 0$. Let $a \in (x)^{\perp}$. Then $a^* \wedge x = 0$ and hence $x^* \wedge a^* = a^*$. Now, $a^* \wedge y = x^* \wedge a^* \wedge y = 0$. Then $a \in (y)^{\perp}$. Therefore $(x)^{\perp} \subseteq (y)^{\perp}$.

(5) Clearly, we have $x \in (x)^{\perp}$. By (4), we have that $(x)^{\perp} \subseteq (x)^{\perp\perp}$. Let $a \in (x)^{\perp\perp}$. Then there exists an element $b \in (x)^{\perp}$ such that $a^* \wedge b = 0$. That implies $b^* \wedge a^* = a^*$. Since $b \in (x)^{\perp}$, we have $b^* \wedge x = 0$. Now, $a^* \wedge x = b^* \wedge a^* \wedge x = 0$. That implies $a \in (x)^{\perp}$ and hence $(x)^{\perp\perp} \subseteq (x)^{\perp}$. Therefore $(x)^{\perp\perp} = (x)^{\perp}$.

(6) By (2), we have that $[x) \subseteq (x)^{\perp}$. Then $(x)^{\perp\perp} \subseteq [x)^{\perp}$. By (5), we get that $(x)^{\perp} \subseteq [x)^{\perp}$. Let $a \in [x)^{\perp}$. Then there exists an element $b \in [x)$ such that $a^* \wedge b = 0$. Since $b \in [x)$, we have that $b \vee x = b$ and hence $b \wedge x = x$. Now, $a^* \wedge x = a^* \wedge b \wedge x = 0$. Therefore $a \in (x)^{\perp}$ and hence $[x)^{\perp} \subseteq (x)^{\perp}$. Thus $[x)^{\perp} = (x)^{\perp}$.

(7) Assume that $(x)^{\perp} = L$. Then $0 \in (x)^{\perp}$. That implies $0^* \wedge x = 0$ and hence $x = 0$. Assume that $x = 0$. Clearly, we have that $a^* \wedge x = 0$, for all $a \in L$. That implies $a \in (x)^{\perp}$, for all $a \in L$. Therefore $(x)^{\perp} = L$.

(8) Let $a \in D(L)$. Then $a^* = 0$. Clearly, we have that $a^* \wedge x = 0$, for all $x \in L$. That implies $a \in (x)^{\perp}$, for all $x \in L$. Therefore $D(L) \subseteq (x)^{\perp}$, for all $x \in L$.

(9) Let m be any maximal element of L. By (8), we have that $D(L) \subseteq (m)^{\perp}$. Let $a \in (m)^{\perp}$. Then $a^* \wedge m = 0$ and hence $m \wedge a^* = 0$. That implies $a^* = 0$. Therefore $a \in D(L)$ and hence $(m)^{\perp} \subseteq D(L)$. Thus $(m)^{\perp} = D(L)$.

(10) Clearly, we have that $[x) \vee D(L) \subseteq (x)^{\perp}$. Let $a \in (x)^{\perp}$. Then $a^* \wedge x = 0$. Now, $a = a \vee 0 = a \vee (x \wedge a^*) = (a \vee x) \wedge (a \vee a^*)$. Since $a \vee x \in [x)$ and $a \vee a^* \in D(L)$, we get that $(a \vee x) \wedge (a \vee a^*) \in [x) \vee D(L)$ and hence $a \in [x) \vee D(L)$. Therefore $(x)^{\perp} = [x) \vee D(L)$. Let $a \in [x^{**})$. Then $a \vee x^{**} = a$. That implies $(a \vee x^{**})^* = a^*$. That implies $a^* \wedge x^* = a^*$ and hence $a^* \wedge x = 0$. Therefore $a \in (x)^{\perp}$. Thus $[x^{**}) \subseteq (x)^{\perp}$. Clearly, we have that $[x^{**}) \vee D(L) \subseteq (x)^{\perp}$. Let $a \in (x)^{\perp}$. Then $a^* \wedge x = 0$. That implies $x^* \wedge a^* = a^*$ and hence $x^{**} \vee a^{**} = a^{**}$. Now, $a = a \vee 0 = a \vee (a^{**} \wedge a^*) = (a \vee a^{**}) \wedge (a \vee a^*) = a^{**} \wedge (a \vee a^*) = (x^{**} \vee a^{**}) \wedge (a \vee a^*) = (a^{**} \vee x^{**}) \wedge (a \vee a^*)$. Since $a^{**} \vee x^{**} \in [x^{**})$ and $a \vee a^* \in D(L)$, we get that $(a^{**} \vee x^{**}) \wedge (a \vee a^*) \in [x^{**}) \vee D(L)$. Therefore $(x)^{\perp} \subseteq [x^{**}) \vee D(L)$ and hence $(x)^{\perp} = [x^{**}) \vee D(L)$.

(11) Let $D(L)$ be a principal filter of L. Then there exists an element $a \in L$ such that $D(L) = [a)$. By (10), we have that $(x)^{\perp} = [x^{**}) \vee D(L)$, for all $x \in L$. That implies $(x)^{\perp} = [x^{**}) \vee [a) = [x^{**} \wedge a)$. Therefore $(x)^{\perp}$ is a principal filter of L, for all $x \in L$.

(12) Clear.

(13) Clearly, we have that $(x \vee y)^{\perp} \subseteq (x)^{\perp} \cap (y)^{\perp}$. Conversely, let $a \in (x)^{\perp} \cap (y)^{\perp}$. Then $a \in (x)^{\perp}$ and $a \in (y)^{\perp}$. That implies $a^* \wedge x = 0$ and $a^* \wedge y = 0$. That implies $a^* \wedge (x \vee y) = 0$ and hence $a \in (x \vee y)^{\perp}$. Therefore $(x)^{\perp} \cap (y)^{\perp} \subseteq (x \vee y)^{\perp}$. Thus $(x \vee y)^{\perp} = (x)^{\perp} \cap (y)^{\perp}$.

(14) Let z be any element of a stone ADL L. Assume that $(x)^{\perp} = (y)^{\perp}$.

Now, $a \in (x \vee z)^{\perp} \Leftrightarrow a^* \wedge (x \vee z) = 0 \Leftrightarrow (a^* \wedge x) \vee (a^* \wedge z) = 0 \Leftrightarrow a^* \wedge x = 0$ and $a^* \wedge z = 0 \Leftrightarrow a \in (x)^{\perp} = (y)^{\perp}$ and $a^* \wedge z = 0 \Leftrightarrow a^* \wedge y = 0$ and $a^* \wedge z = 0 \Leftrightarrow (a^* \wedge y) \vee (a^* \wedge z) = 0 \Leftrightarrow a^* \wedge (y \vee z) = 0 \Leftrightarrow a \in (y \vee z)^{\perp}$. Therefore $(x \vee z)^{\perp} = (y \vee z)^{\perp}$. Now, $a \notin (x \wedge z)^{\perp} \Leftrightarrow a^* \wedge x \wedge z \neq 0 \Leftrightarrow a^* \wedge x \wedge a^* \wedge z \neq 0 \Leftrightarrow a^* \wedge x \neq 0$ and $a^* \wedge z \neq 0 \Leftrightarrow a \notin (x)^{\perp} = (y)^{\perp}$ and $a^* \wedge z \neq 0 \Leftrightarrow a^* \wedge y \neq 0$ and $a^* \wedge z \neq 0 \Leftrightarrow a^* \wedge y \wedge z \neq 0 \Leftrightarrow a \notin (y \wedge z)^{\perp}$. Therefore $(x \wedge z)^{\perp} = (y \wedge z)^{\perp}$.

(15) Assume that $x \wedge y = 0$. We prove that $(x)^{\perp} \vee (y)^{\perp} = L$. Suppose $(x)^{\perp} \vee (y)^{\perp} \neq L$. Then there exists a maximal filter M of L such that $(x)^{\perp} \vee (y)^{\perp} \subseteq M$. That implies $(x)^{\perp} \subseteq M$ and $(y)^{\perp} \subseteq M$. That implies $x, y \in M$ and hence $x \wedge y \in M$. That implies $0 \in M$. That implies $M = L$, which is a contradiction to proper filter M of L. Therefore $(x)^{\perp} \vee (y)^{\perp} = L$.

(16) Assume that $(x)^{\perp} = (y)^{\perp}$. Then $x \in (y)^{\perp}$ and $y \in (x)^{\perp}$. That implies $x^* \wedge y = 0$ and $y^* \wedge x = 0$. That implies $y^* \wedge x^* = x^*$ and $x^* \wedge y^* = y^*$. That implies $(y \vee x)^* = x^*$ and $(x \vee y)^* = y^*$. Since $(x \vee y)^* = (y \vee x)^*$, we get that $x^* = y^*$. Conversely, assume that $x^* = y^*$. Now, $a \in (x)^{\perp} \Leftrightarrow a^* \wedge x = 0 \Leftrightarrow (a^* \wedge x)^* = 0^* \Leftrightarrow a^{**} \vee x^* = 0^* \Leftrightarrow a^{**} \vee y^* = 0^* \Leftrightarrow (a^* \wedge y)^* = 0^* \Leftrightarrow (a^* \wedge y)^{**} = 0^{**} = 0 \Leftrightarrow (a^* \wedge y)^{**} \wedge (a^* \wedge y) = 0 \Leftrightarrow a^* \wedge y = 0 \Leftrightarrow a \in (y)^{\perp}$. Therefore $(x)^{\perp} = (y)^{\perp}$.

(17) Now, $a \in (x)^{\perp} \Leftrightarrow a^* \wedge x = 0 \Leftrightarrow (a^* \wedge x)^{**} = 0 \Leftrightarrow a^* \wedge x^{**} = 0 \Leftrightarrow a \in (x^{**})^{\perp}$. Hence $(x)^{\perp} = (x^{**})^{\perp}$.

(18) Clear. ■

We denote the set of all annulets of a stone ADL L by $\mathfrak{A}^{\perp}(L)$, i.e., $\mathfrak{A}^{\perp}(L) = \{(x)^{\perp} \mid x \in L\}$.

Theorem 3.5. *Let L be a stone ADL with maximal element m. Then $(\mathfrak{A}^{\perp}(L), +, \bullet, (m)^{\perp}, (0)^{\perp})$ can be made into a Boolean ring, where $+$ is the additive operation and $\bullet$ is the multiplication operation are defined as $(x)^{\perp} + (y)^{\perp} = ((x \vee y^*) \wedge (y \vee x^*))^{\perp}$ and $(x)^{\perp} \bullet (y)^{\perp} = (x \vee y)^{\perp}$, for all $x, y \in L$.*

Proof. Let $(x)^{\perp}, (y)^{\perp}, (z)^{\perp} \in \mathfrak{A}^{\perp}(L)$. Now,

$$
\begin{aligned}
&(x)^{\perp} + ((y)^{\perp} + (z)^{\perp}) \\
&= (x)^{\perp} + ((y \vee z^*) \wedge (z \vee y^*))^{\perp} \\
&= ((x \vee ((y \vee z^*) \wedge (z \vee y^*))^*) \wedge (((y \vee z^*) \wedge (z \vee y^*)) \vee x^*))^{\perp} \\
&= ((x \vee (y^* \wedge z^{**}) \vee (z^* \wedge y^{**})) \wedge ((y \vee z^*) \wedge (z \vee y^*)) \vee x^*)^{\perp} \\
&= ((((x \vee y^*) \wedge (x \vee z^{**})) \vee (z^* \wedge y^{**})) \wedge (((y \vee z^*) \wedge (z \vee y^*)) \vee x^*))^{\perp}
\end{aligned}
$$

$$= ((x \vee y^* \vee (z^* \wedge y^{**})) \wedge (x \vee z^{**} \vee (z^* \wedge y^{**})) \wedge (((y \vee z^*) \wedge (z \vee y^*)) \vee x^*))^\perp$$
$$= (((x \vee y^* \vee z^*) \wedge (x \vee y^* \vee y^{**}) \wedge (x \vee z^{**} \vee z^*) \wedge (x \vee z^{**} \vee y^{**}))$$
$$\wedge\ (((y \vee z^*) \wedge (z \vee y^*)) \vee x^*))^\perp$$
$$= (((x \vee y^* \vee z^*) \wedge (x \vee z^{**} \vee y^{**})) \wedge (((y \vee z^*) \wedge (z \vee y^*)) \vee x^*))^\perp$$
$$= ((x \vee y^* \vee z^*) \wedge (x \vee z^{**} \vee y^{**}) \wedge (y \vee z^* \vee x^*) \wedge (z \vee y^* \vee x^*))^\perp \longrightarrow (1)$$

Now,

$$((x)^\perp + (y)^\perp) + (z)^\perp$$
$$= ((x \vee y^*) \wedge (y \vee x^*))^\perp + (z)^\perp$$
$$= ((x \vee y^*) \wedge (x^* \vee y))^\perp + (z)^\perp$$
$$= ((((x \vee y^*) \wedge (x^* \vee y)) \vee z^*) \wedge ((z \vee ((x \vee y^*) \wedge (x^* \vee y))^*)))^\perp$$
$$= ((((x \vee y^*) \wedge (x^* \vee y)) \vee z^*) \wedge ((((x \vee y^*) \wedge (x^* \vee y))^*) \vee z))^\perp$$
$$= ((x \vee y^* \vee z^*) \wedge (x^* \vee y \vee z^*) \wedge (((x^* \wedge y^{**}) \vee (x^{**} \wedge y^*)) \vee z))^\perp$$
$$= ((x \vee y^* \vee z^*) \wedge (x^* \vee y \vee z^*) \wedge (((x^* \wedge y^{**}) \vee x^{**}) \wedge ((x^* \wedge y^{**}) \vee y^*) \vee z))^\perp$$
$$= ((x \vee y^* \vee z^*) \wedge (x^* \vee y \vee z^*) \wedge (((x^* \vee x^{**}) \wedge (y^{**} \vee x^{**}) \wedge$$
$$(x^* \vee y^*) \wedge (y^{**} \vee y^*)) \vee z))^\perp$$
$$= ((x \vee y^* \vee z^*) \wedge (x^* \vee y \vee z^*) \wedge (y^{**} \vee x^{**} \vee z) \wedge (x^* \vee y^* \vee z))^\perp \longrightarrow (2)$$

Now,

$$((x \vee y^* \vee z^*) \wedge (x \vee z^{**} \vee y^{**}) \wedge (y \vee z^* \vee x^*) \wedge (z \vee y^* \vee x^*))^*$$
$$= (x^* \wedge y^{**} \wedge z^{**}) \vee (x^* \wedge z^* \wedge y^*) \vee (y^* \wedge z^{**} \wedge x^{**}) \vee (z^* \wedge y^{**} \wedge x^{**}) \longrightarrow (3).$$

Now,

$$((x \vee y^* \vee z^*) \wedge (x^* \vee y \vee z^*) \wedge (y^{**} \vee x^{**} \vee z) \wedge (x^* \vee y^* \vee z))^*$$
$$= ((x \vee y^* \vee z^*) \wedge (y^{**} \vee x^{**} \vee z) \wedge (x^* \vee y \vee z^*) \wedge (x^* \vee y^* \vee z))^*$$
$$= (x^* \wedge y^{**} \wedge z^{**}) \vee (x^* \wedge z^* \wedge y^*) \vee (y^* \wedge z^{**} \wedge x^{**}) \vee (z^* \wedge y^{**} \wedge x^{**}) \longrightarrow (4).$$

Therefore (3) and (4) are same. So that $((x \vee y^* \vee z^*) \wedge (x \vee z^{**} \vee y^{**}) \wedge (y \vee z^* \vee x^*) \wedge (z \vee y^* \vee x^*))^* = ((x \vee y^* \vee z^*) \wedge (x^* \vee y \vee z^*) \wedge (y^{**} \vee x^{**} \vee z) \wedge (x^* \vee y^* \vee z))^*$. By Lemma 3.4(16), we get that $((x \vee y^* \vee z^*) \wedge (x \vee z^{**} \vee y^{**}) \wedge (y \vee z^* \vee x^*) \wedge (z \vee y^* \vee x^*))^\perp = ((x \vee y^* \vee z^*) \wedge (x^* \vee y \vee z^*) \wedge (y^{**} \vee x^{**} \vee z) \wedge (x^* \vee y^* \vee z))^\perp$. Hence $(x)^\perp + ((y)^\perp + (z)^\perp) = ((x)^\perp + (y)^\perp) + (z)^\perp$. Thus $+$ is associative. Let $(x)^\perp$ be any element of $\mathfrak{A}^\perp(L)$. Now, $(x)^\perp + (0^*)^\perp = ((x \vee 0^{**}) \wedge (x^* \vee 0^*))^\perp = ((x \vee 0) \wedge (x \wedge 0)^*)^\perp = (x \wedge 0^*)^\perp = (x)^\perp$. Therefore $(0^*)^\perp$ is the additive identity. Let $(x)^\perp$ be any element of $\mathfrak{A}^\perp(L)$. Now $(x)^\perp + (x)^\perp = ((x \vee x^*) \wedge (x^* \vee x))^\perp = (x \vee x^*)^\perp$. Now $(x \vee x^*)^* = x^* \wedge x^{**} = 0 = (0^*)^*$. That implies $(x \vee x^*)^* = (0^*)^*$. By Lemma

3.4(16), we get that $(x\vee x^*)^{\perp} = (0^*)^{\perp}$. Since $(x)^{\perp}+(x)^{\perp} = (x\vee x^*)^{\perp} = (0^*)^{\perp}$, we have that $(x)^{\perp}$ is the additive inverse of $(x)^{\perp}$ in $\mathfrak{A}^{\perp}(L)$. Let $(x)^{\perp}, (y)^{\perp} \in \mathfrak{A}^{\perp}(L)$. Now, $(x)^{\perp} + (y)^{\perp} = ((x\vee y^*)\wedge(y\vee x^*))^{\perp} = ((y\vee x^*)\wedge(x\vee y^*))^{\perp} = (y)^{\perp}+(x)^{\perp}$. Therefore $+$ is commutative. Let $(x)^{\perp}$ be any element of $\mathfrak{A}^{\perp}(L)$. Now, $(x)^{\perp}\bullet(0)^{\perp} = (x\vee 0)^{\perp} = (x)^{\perp}$. Therefore $(0)^{\perp}$ is the multiplicative identity of $\mathfrak{A}^{\perp}(L)$. Let $(x)^{\perp}, (y)^{\perp}, (z)^{\perp} \in \mathfrak{A}^{\perp}(L)$. Now, $(x)^{\perp}\bullet((y)^{\perp}\bullet(z)^{\perp}) = (x)^{\perp}\bullet(y\vee z)^{\perp} = (x\vee(y\vee z))^{\perp} = ((x\vee y)\vee z)^{\perp} = (x\vee y)^{\perp}\bullet(z)^{\perp} = ((x)^{\perp}\bullet(y)^{\perp})\bullet(z)^{\perp}$. Therefore $\bullet$ is associative. Let $(x)^{\perp}, (y)^{\perp} \in \mathfrak{A}^{\perp}(L)$. Now, $(x)^{\perp}\bullet(y)^{\perp} = (x\vee y)^{\perp} = (y\vee x)^{\perp} = (y)^{\perp}\bullet(x)^{\perp}$. Therefore $\bullet$ is commutative. Let $(x)^{\perp}, (y)^{\perp}, (z)^{\perp} \in \mathfrak{A}^{\perp}(L)$. Now, $(x)^{\perp}\bullet((y)^{\perp}+(z)^{\perp}) = (x)^{\perp}\bullet((y\vee z^*)\wedge(z\vee y^*))^{\perp} = (x\vee((y\vee z^*)\wedge(y^*\vee z)))^{\perp} = ((x\vee y\vee z^*)\wedge(x\vee y^*\vee z))^{\perp} \longrightarrow (5)$. Now, $(x)^{\perp}\bullet(y)^{\perp}+(x)^{\perp}\bullet(z)^{\perp} = (x\vee y)^{\perp} + (x\vee z)^{\perp} = (((x\vee y)\vee(x\vee z)^*)\wedge((x\vee z)\vee(x\vee y)^*))^{\perp} = ((x\vee y\vee x^*)\wedge(x\vee y\vee z^*)\wedge(x^*\vee x\vee z)\wedge(y^*\vee x\vee z))^{\perp} \longrightarrow (6)$. Now, $((x\vee y\vee x^*)\wedge(x\vee y\vee z^*)\wedge(x^*\vee x\vee z)\wedge(y^*\vee x\vee z))^* = (x^*\wedge y^*\wedge x^{**})\vee(x^*\wedge y^*\wedge z^{**})\vee(x^{**}\wedge x^*\wedge z^*)\vee(y^{**}\wedge x^*\wedge z^*) = (x^*\wedge y^*\wedge z^{**})\vee(y^{**}\wedge x^*\wedge z^*) \longrightarrow (7)$. Now, $((x\vee y\vee z^*)\wedge(x\vee y^*\vee z))^* = (x^*\wedge y^*\wedge z^{**})\vee(y^{**}\wedge x^*\wedge z^*) \longrightarrow (8)$. Therefore (7) and (8) are equal. That is $((x\vee y\vee x^*)\wedge(x\vee y\vee z^*)\wedge(x^*\vee x\vee z)\wedge(y^*\vee x\vee z))^* = ((x\vee y\vee x^*)\wedge(x\vee y^*\vee z))^*$. By Lemma 3.4(16), we get that $((x\vee y\vee z^*)\wedge(x\vee y\vee z^*)\wedge(x^*\vee x\vee z)\wedge(y^*\vee x\vee z))^{\perp} = ((x\vee y\vee z^*)\wedge(x\vee y^*\vee z))^{\perp}$. Therefore (5) and (6) are equal. Hence $(x)^{\perp}\bullet((y)^{\perp}+(z)^{\perp}) = (x)^{\perp}\bullet(y)^{\perp}+(x)^{\perp}\bullet(z)^{\perp}$. Since $\bullet$ is commutative, we get that $((y)^{\perp}+(z)^{\perp})\bullet(x)^{\perp} = (y)^{\perp}\bullet(x)^{\perp}+(z)^{\perp}\bullet(x)^{\perp}$. Let $(x)^{\perp}$ be any element of $\mathfrak{A}^{\perp}(L)$. Now, $(x)^{\perp}\bullet(x)^{\perp} = (x\vee x)^{\perp} = (x)^{\perp}$. Therefore $(\mathfrak{A}^{\perp}(L), +, \bullet, (m)^{\perp}, (0)^{\perp})$ is a Boolean ring. ∎

Clearly, we have that there is a one-to-one correspondence between Boolean ring and Boolean lattice. So that we can convert the Boolean ring $(\mathfrak{A}^{\perp}(L), +, \bullet, (m)^{\perp}, (0)^{\perp})$ of a stone ADL L into a Boolean lattice in the following.

Corollary 3.6. *Let $(\mathfrak{A}^{\perp}(L), +, \bullet, (m)^{\perp}, (0)^{\perp})$ be a Boolean ring of all annulets of a stone ADL. Then $(\mathfrak{A}^{\perp}(L), \vee, \cap, ', (m)^{\perp}, (0)^{\perp})$ is a Boolean lattice, where* $(x)^{\perp}\vee(y)^{\perp} = (x\wedge y)^{\perp}$, $(x)^{\perp}\cap(y)^{\perp} = (x\vee y)^{\perp}$, $((x)^{\perp})' = (x^*)^{\perp}$.

4. $\mathcal{N}$-FILTERS OF STONE ADLS

Definition 4.1. A filter F of a stone ADL L is said to be an $\mathcal{N}$-filter if $F = F^{\perp}$.

Lemma 4.2. *Let L be a stone ADL. Then we have the following:*

(1) *For any $x \in L$, $(x)^{\perp}$ is an $\mathcal{N}$-filter of L.*

(2) *For any filter F, $F^{\perp}$ is an $\mathcal{N}$-filter of L.*

(3) *$D(L)$ is an $\mathcal{N}$-filter of L.*

Proof. (1) and (2) are clear.

(3) Clearly, we have that $D(L) \subseteq (D(L))^{\perp}$. Let $a \in (D(L))^{\perp}$. Then there exists an element $b \in D(L)$ such that $a^* \wedge b = 0$. That implies $b^* \wedge a^* = a^*$. Since $b \in D(L)$, we get that $b^* = 0$. Now $a^* = b^* \wedge a^* = 0 \wedge a^* = 0$. That implies $a^* = 0$ and hence $a \in D(L)$. That implies $(D(L))^{\perp} \subseteq D(L)$. Therefore $D(L) = (D(L))^{\perp}$. Thus $D(L)$ is an $\mathcal{N}$-filter of L. ∎

Theorem 4.3. *Let F be a filter of a stone ADL L. Then the following statements are equivalent:*

(1) *F is an $\mathcal{N}$-filter.*

(2) *For any $a \in L$, $a^{**} \in F \Rightarrow a \in F$.*

(3) *For any $a, b \in L$, $(a)^{\perp} = (b)^{\perp}$ and $a \in F \Rightarrow b \in F$.*

(4) *$F = \bigcup_{a \in F}(a)^{\perp}$.*

(5) *For any $a \in L$, $a \in F \Rightarrow (a)^{\perp} \subseteq F$.*

Proof. (1)⇒(2) Assume that F is an $\mathcal{N}$-filter of L. Then $F = F^{\perp}$. Let a be any element of L with $a^{**} \in F$. Then $a^{**} \in F^{\perp}$. Then there exists an element $b \in F$ such that $a^{***} \wedge b = 0$. That implies $a^* \wedge b = 0$ and hence $a \in F^{\perp}$. Therefore $a \in F$.

(2)⇒(3) Assume (2). Let $a, b \in L$ with $(a)^{\perp} = (b)^{\perp}$ and $a \in F$. Since $b^* \wedge b = 0$, we get that $b \in (b)^{\perp} = (a)^{\perp}$. That implies $b^* \wedge a = 0$ and hence $b^{**} \wedge a = a$. Since $a \in F$, we have that $b^{**} \wedge a \in F$. That implies $b^{**} \in F$. By our assumption we get that $b \in F$.

(3)⇒(4) Assume condition (3). Let $a \in F$. Clearly, we have that $[a) \subseteq (a)^{\perp}$. That implies $F = \bigcup_{a \in F}[a) \subseteq \bigcup_{a \in F}(a)^{\perp}$ and hence $F \subseteq \bigcup_{a \in F}(a)^{\perp}$. Let $b \in \bigcup_{a \in F}(a)^{\perp}$. Then there exists an element $c \in F$ such that $b \in (c)^{\perp}$. That implies $(b)^{\perp} \subseteq (c)^{\perp}$ and hence $(b)^{\perp} = (b)^{\perp} \cap (c)^{\perp} = (b \vee c)^{\perp}$. Since $c \in F$, we get that $b \vee c \in F$. By our assumption, we get that $b \in F$. That implies $\bigcup_{a \in F}(a)^{\perp} \subseteq F$. Therefore $F = \bigcup_{x \in F}(x)^{\perp}$.

(4)⇒(5) Clear.

(5)⇒(1) Assume (5). Clearly, we have $F \subseteq F^{\perp}$. Let $a \in F^{\perp}$. Then there exists an element $x \in F$ such that $a^* \wedge x = 0$. That implies $a \in (x)^{\perp}$. Since $x \in F$, we get that $(x)^{\perp} \subseteq F$. Since $a \in (x)^{\perp}$, we get that $a \in F$. Therefore $F^{\perp} \subseteq F$ and hence $F = F^{\perp}$. Thus F is an $\mathcal{N}$-filter of L. ∎

Theorem 4.4. *Let L be a stone ADL with maximal element m. Then we have:*

(1) *The set $\mathfrak{F}_{\mathcal{N}}(L)$ of all $\mathcal{N}$-filters of L is a bounded distributive lattice.*

(2) *$\mathfrak{A}^{\perp}(L)$ is a bounded sublattice of $\mathfrak{F}_{N}(L)$.*

Proof. (1) Let $F_1, F_2 \in \mathfrak{F}_{\mathcal{N}}(L)$. Then $F_1 = {F_1}^{\perp}$ and $F_2 = {F_2}^{\perp}$. Clearly, we have that $(F_1 \vee F_2)^{\perp} = {F_1}^{\perp} \vee {F_2}^{\perp} = F_1 \vee F_2$ and $(F_1 \cap F_2)^{\perp} = {F_1}^{\perp} \cap {F_2}^{\perp} = F_1 \cap F_2$.

Therefore $F_1 \vee F_2,\ F_1 \cap F_2 \in \mathfrak{F}_{\mathcal{N}}(L)$ and hence $\mathfrak{F}_{\mathcal{N}}(L)$ is a sublattice of $\mathfrak{F}(L)$. Clearly, we have that $D(L)$ is the smallest $\mathcal{N}$-filter of L and L is the greatest $\mathcal{N}$-filter of L. Since $\mathfrak{F}(L)$ is a distributive lattice, we get that $(\mathfrak{F}_{\mathcal{N}}(L), \vee, \cap, D(L), L)$ is a bounded distributive lattice.

(2) We have that every annulet is an $\mathcal{N}$-filter of L. So that $\mathfrak{A}^{\perp}(L) \subseteq \mathfrak{F}_{\mathcal{N}}(L)$. Now, we prove that $\mathfrak{A}^{\perp}(L)$ is a sublattice of $\mathfrak{F}_{\mathcal{N}}(L)$. We define $\vee$ and $\cap$ on $\mathfrak{A}^{\perp}(L)$ as $(x)^{\perp} \vee (y)^{\perp} = (x \wedge y)^{\perp}$ and $(x)^{\perp} \cap (y)^{\perp} = (x \vee y)^{\perp}$, for all $(x)^{\perp}, (y)^{\perp} \in \mathfrak{A}^{\perp}(L)$. Clearly, we get that $(\mathfrak{A}^{\perp}(L), \vee, \cap)$ is a sublattice of $\mathfrak{F}_{\mathcal{N}}(L)$. Since $(0^*)^{\perp} = D(L)$ and $(0)^{\perp} = L$, we get that $(\mathfrak{A}^{\perp}(L), \vee, \cap, (0^*)^{\perp}, (0)^{\perp})$ is a bounded sublattice of $\mathfrak{F}_{\mathcal{N}}(L)$. ∎

Definition 4.5. A congruence relation Θ on a Pseudo-complemented ADL L is called a Glivenko-type if $(a, b) \in \Theta$ implies $a^* = b^*$.

We denote the class of all Glivenko-type congruence on a pseudo-complemented ADL L by $Con_G(L)$.

Lemma 4.6. *Define a relation θ on a stone ADL L as follows $(a, b) \in \theta$ iff $(a)^{\perp} = (b)^{\perp}$. Then θ is a congruence relation on L.*

Definition 4.7. A congruence relation Θ on a stone ADL L is said to be a G-extention on L if $\alpha \subseteq \Theta$, for all $\alpha \in Con_G(L)$.

Theorem 4.8. *Let a be any element of a stone ADL L. Then a relation $\psi_{(a)^{\perp}} = \{(x,\ y) \in L \times L \mid x^* \wedge a = y^* \wedge a\}$ is a G-extension on L such that $[0^*]\psi_{(x)^{\perp}} = (x)^{\perp}$.*

Proof. Clearly, $\psi_{(a)^{\perp}}$ is an equivalence relation on L. Let $(x, y),\ (z, w) \in \psi_{(a)^{\perp}}$. Then $x^* \wedge a = y^* \wedge a$ and $z^* \wedge a = w^* \wedge a$. Now, $(x \wedge z)^* \wedge a = (x^* \vee z^*) \wedge a = (x^* \wedge a) \vee (z^* \wedge a) = (y^* \wedge a) \vee (w^* \wedge a) = (y^* \vee w^*) \wedge a = (y \wedge w)^* \wedge a$. Therefore $(x \wedge z,\ y \wedge w) \in \psi_{(a)^{\perp}}$. Similarly, we get $(z \wedge x,\ w \wedge y) \in \psi_{(a)^{\perp}}$. Now $(x \vee z)^* \wedge a = x^* \wedge z^* \wedge a = x^* \wedge a \wedge z^* \wedge a = y^* \wedge a \wedge w^* \wedge a = y^* \wedge w^* \wedge a = (y \vee w)^* \wedge a$. Therefore $(x \vee z,\ y \vee w) \in \psi_{(a)^{\perp}}$. Similarly, we get that $(z \vee x,\ w \vee y) \in \psi_{(a)^{\perp}}$. Let $(x, y) \in \psi_{(a)^{\perp}}$. Then $x^* \wedge a = y^* \wedge a$. That $(x^* \wedge a)^* = (y^* \wedge a)^*$. That implies $x^{**} \vee a^* = y^{**} \vee a^*$. That implies $(x^{**} \vee a^*) \wedge a = (y^{**} \vee a^*) \wedge a$. That implies $(x^{**} \wedge a) \vee (a^* \wedge a) = (y^{**} \wedge a) \vee (a^* \wedge a)$. That implies $x^{**} \wedge a = y^{**} \wedge a$. Therefore $(x^*, y^*) \in \psi_{(a)^{\perp}}$. Hence $\psi_{(a)^{\perp}}$ is a congruence relation on L. Let θ_1 be any relation on L define as $(x, y) \in \theta_1$ iff $x^* = y^*$. It is clear that θ_1 is a G-extension congruence on L. Let $(x, y) \in \theta_1$. Then $x^* = y^*$. That implies $x^* \wedge a = y^* \wedge a$, for all $a \in L$. That implies $(x, y) \in \psi_{(a)^{\perp}}$. Therefore $\theta_1 \subseteq \psi_{(a)^{\perp}}$ and hence $\psi_{(a)^{\perp}}$ is a G-extension on L. Now, $x \in [0^*]\psi_{(a)^{\perp}} \Leftrightarrow (x, 0^*) \in \psi_{(a)^{\perp}} \Leftrightarrow x^* \wedge a = (0^*)^* \wedge a \Leftrightarrow x^* \wedge a = 0 \Leftrightarrow x \in (a)^{\perp}$. Therefore $[0^*]\psi_{(a)^{\perp}} = (a)^{\perp}$. ∎

We denote the set $\{\psi_{(a)^{\perp}}\}_{a \in L}$ of all G-extensions on L by $Con^{\perp}_{E}(L)$.

Theorem 4.9. *Let L be a stone ADL. Then $(Con_E^{\perp}(L), \oplus, \odot, \psi_{(0^*)^{\perp}}, \psi_{(0)^{\perp}})$ is a Boolean ring, where $\psi_{(x)^{\perp}} \oplus \psi_{(y)^{\perp}} = \psi_{(x)^{\perp}+(y)^{\perp}}$ and $\psi_{(x)^{\perp}} \odot \psi_{(y)^{\perp}} = \psi_{(x)^{\perp}\bullet(y)^{\perp}}$. Moreover $Con_E^{\perp}(L)$ and $\mathfrak{A}^{\perp}(L)$ are isomorphic rings.*

Proof. It is easy to verify that $(Con_E^{\perp}(L), \oplus, \odot, \psi_{(0^*)^{\perp}}, \psi_{(0)^{\perp}})$ and $(\mathfrak{A}^{\perp}(L), +, \bullet, (0^*)^{\perp}, (0)^{\perp})$ are Boolean rings. Define $f : \mathfrak{A}^{\perp}(L) \longrightarrow Con_E^{\perp}(L)$ by $f((x)^{\perp}) = \psi_{(x)^{\perp}}$. Let $(x)^{\perp}, (y)^{\perp} \in \mathfrak{A}^{\perp}(L)$ with $(x)^{\perp} = (y)^{\perp}$. Then $x \in (y)^{\perp}$ and $y \in (x)^{\perp}$. That implies $x^* \wedge y = 0$ and $y^* \wedge x = 0$. Hence $x^{**} \wedge y = y$ and $y^{**} \wedge x = x$. Now, we prove that $\psi_{(x)^{\perp}} = \psi_{(y)^{\perp}}$. Let $(a, b) \in \psi_{(x)^{\perp}}$. Then $a^* \wedge x = b^* \wedge x$. Now $a^* \wedge y = a^* \wedge x^{**} \wedge y = a^{***} \wedge x^{**} \wedge y = (a^* \wedge x)^{**} \wedge y = (b^* \wedge x)^{**} \wedge y = b^{***} \wedge x^{**} \wedge y = b^* \wedge x^{**} \wedge y = b^* \wedge y$. That implies $(a, b) \in \psi_{(y)^{\perp}}$. Therefore $\psi_{(x)^{\perp}} \subseteq \psi_{(y)^{\perp}}$. Similarly, we get that $\psi_{(y)^{\perp}} \subseteq \psi_{(x)^{\perp}}$. Therefore $\psi_{(x)^{\perp}} = \psi_{(y)^{\perp}}$ and hence f is well defined. Let $x, y \in L$ with $\psi_{(x)^{\perp}} = \psi_{(y)^{\perp}}$. We prove that $(x)^{\perp} = (y)^{\perp}$. Now, $a \in (x)^{\perp} \Leftrightarrow a^* \wedge x = 0 = (0^{**}) \wedge x \Leftrightarrow (a, 0^*) \in \psi_{(x)^{\perp}} = \psi_{(y)^{\perp}} \Leftrightarrow a^* \wedge y = 0^{**} \wedge y = 0 \Leftrightarrow a \in (y)^{\perp}$. That implies $(x)^{\perp} = (y)^{\perp}$ and hence f is one-one. Clearly, f is onto. Let $(x)^{\perp}, (y)^{\perp} \in \mathfrak{A}^{\perp}(L)$. Now $f((x)^{\perp} + (y)^{\perp}) = \psi_{((x)^{\perp}+(y)^{\perp})} = \psi_{(x)^{\perp}} \oplus \psi_{(y)^{\perp}} = f((x)^{\perp}) + f((y)^{\perp})$. Now, $f((x)^{\perp} \bullet (y)^{\perp}) = \psi_{((x)^{\perp}\bullet(y)^{\perp})} = \psi_{(x)^{\perp}} \odot \psi_{(y)^{\perp}} = f((x)^{\perp}) \bullet f((y)^{\perp})$. We have that $f((0^*)^{\perp}) = \psi_{(0^*)^{\perp}}$ and $f((0)^{\perp}) = \psi_{(0)^{\perp}}$. Therefore f is homomorphism and hence f is isomorphism. ∎

Corollary 4.10. *Let $(Con_E^{\perp}(L), \oplus, \odot, \psi_{(0^*)^{\perp}}, \psi_{(0)^{\perp}})$ be a Boolean ring of all G-extensions on a stone ADL L. Then $(Con_E^{\perp}(L), \vee, \cap, ', \psi_{(0^*)^{\perp}}, \psi_{(0)^{\perp}})$ is a Boolean lattice, where $\psi_{(x)^{\perp}} \vee \psi_{(y)^{\perp}} = \psi_{(x\wedge y)^{\perp}}$, $\psi_{(x)^{\perp}} \cap \psi_{(y)^{\perp}} = \psi_{(x\vee y)^{\perp}}$ and $\psi'_{(x)^{\perp}} = \psi_{(x^*)^{\perp}}$. Further more, $\mathfrak{A}^{\perp}(L)$ and $Con_E^{\perp}(L)$ are isomorphic as a Boolean lattices.*

Proof. We prove that $(Con_E^{\perp}(L), \vee, \cap, ', \psi_{(0^*)^{\perp}}, \psi_{(0)^{\perp}})$ is a Boolean lattice. Define $\psi_{(x)^{\perp}} \vee \psi_{(y)^{\perp}} = \psi_{(x)^{\perp}} \oplus \psi_{(y)^{\perp}} \oplus (\psi_{(x)^{\perp}} \odot \psi_{(y)^{\perp}})$, $\psi_{(x)^{\perp}} \cap \psi_{(y)^{\perp}} = \psi_{(x)^{\perp}} \odot \psi_{(y)^{\perp}}$, $\psi'_{(x)^{\perp}} = \psi_{(0)^{\perp}} \oplus \psi_{(x)^{\perp}}$. By the known theorem, we have that $(x)^{\perp} + (y)^{\perp} = ((x \vee y)^* \wedge (y \vee x^*))^{\perp}$ and $(x)^{\perp} \bullet (y)^{\perp} = (x \vee y)^{\perp}$. Now. $\psi_{(x)^{\perp}} \vee \psi_{(y)^{\perp}} = \psi_{(x)^{\perp}} \oplus \psi_{(y)^{\perp}} \oplus (\psi_{(x)^{\perp}} \odot \psi_{(y)^{\perp}}) = \psi_{(x)^{\perp}+(y)^{\perp}} \oplus \psi_{(x)^{\perp}\bullet(y)^{\perp}} = \psi_{(x\wedge y)^{\perp}} \oplus \psi_{(x\vee y)^{\perp}} = \psi_{(x\wedge y)^{\perp}+(x\vee y)^{\perp}} = \psi_{((x\wedge y)\wedge(x\vee y))^{\perp}} = \psi_{(x\wedge y)^{\perp}}$. Now, $\psi_{(x)^{\perp}} \cap \psi_{(y)^{\perp}} = \psi_{(x)^{\perp}} \odot \psi_{(y)^{\perp}} = \psi_{(x)^{\perp}\bullet(y)^{\perp}} = \psi_{(x\vee y)^{\perp}}$. Now, $\psi'_{(x)^{\perp}} = \psi_{(0)^{\perp}} \oplus \psi_{(x)^{\perp}} = \psi_{(x)^{\perp}=(0)^{\perp}} = \psi_{(x^*)^{\perp}}$. Therefore $(Con_E^{\perp}(L), \vee, \cap, ', \psi_{(0^*)^{\perp}}, \psi_{(0)^{\perp}})$ is a Boolean lattice. Define $f : \mathfrak{A}^{\perp}(L) \longrightarrow Con_E^{\perp}(L)$ by $f((x)^{\perp}) = \psi_{(x)^{\perp}}$. Clearly, f is isomorphism. ∎

Definition 4.11. Let F be an $\mathcal{N}$-filter of a stone ADL L. Define a relation ψ_F on L as $\psi_F = \{(x, y) \in L \times L \mid x^* \wedge f = y^* \wedge f, \text{ for some } f \in F\}$.

Theorem 4.12. *Let F be an $\mathcal{N}$-filter of a stone ADL L. Then ψ_F is a G-extension on L such that $[0^*]\psi_F = F$.*

Proof. Clearly, we have that ψ_F is a congruence relation on L. Define a relation θ_1 on L as $(x, y) \in \theta_1$ iff $x^* = y^*$. Clearly, we get that $\theta_1 \in Con_G(L)$. Let $(x, y) \in \theta_1$. Then $x^* = y^*$. That implies $x^* \wedge 0^* = y^* \wedge 0^*$, where $0^* \in F$. Therefore $(x, y) \in \psi_F$ and hence $\theta_1 \subseteq \psi_F$. Now, $a \in [0^*]\psi_F \Leftrightarrow (a, 0^*) \in \psi_F \Leftrightarrow a^* \wedge f = (0^*)^* \wedge f$, for some $f \in F \Leftrightarrow a^* \wedge f = 0 \Leftrightarrow a \in (f)^\perp \Leftrightarrow a \in \bigcup_{f \in F}(f)^\perp = F \Leftrightarrow a \in F$. Therefore $[0^*]\psi_F = F$. ∎

Theorem 4.13. *Let F, G be any two $\mathcal{N}$-filters of a stone ADL L. Then we have the following:*

(1) *$F \subseteq G$ if and only if $\psi_F \subseteq \psi_G$.*

(2) *$\psi \subseteq \psi_F$, where ψ is the Glivenko congruence on L.*

(3) *$\psi_F = \psi$ if and only if $F = (0^*)^\perp$.*

(4) *$\psi_F = \nabla_L$ if and only if $F = (0)^\perp$.*

(5) *The quotient ADL L/ψ_F forms a Boolean Lattice.*

Proof. (1) Assume that $F \subseteq G$. Let $(a, b) \in \psi_F$. Then there exists an element f of F such that $a^* \wedge f = b^* \wedge f$. Since $F \subseteq G$, we get that $f \in G$. Since $a^* \wedge f = b^* \wedge f$, we get that $(a, b) \in \psi_G$. Therefore $\psi_F \subseteq \psi_G$. Conversely, assume that $\psi_F \subseteq \psi_G$. We prove that $F \subseteq G$. It enough to show that $[0^*]\psi_F \subseteq [0^*]\psi_G$. Let $a \in [0^*]\psi_F$. Then $(a, 0^*) \in \psi_F$. By our assumption, we get that $(a, 0^*) \in \psi_G$. That implies $a \in [0^*]\psi_G$ and hence $[0^*]\psi_F \subseteq [0^*]\psi_G$. Since ψ_F, ψ_G are G-extensions on$L$ and F, G are $\mathcal{N}$-filters of L, we get that $F = [0^*]\psi_F \subseteq [0^*]\psi_G = G$. Therefore $F \subseteq G$.

(2) Let ψ be the Glivenko congruence on a stone ADL L. Clearly, we have that ψ_F is a G-extension on L. Since F is an $\mathcal{N}$-filter of L, we get that $\psi \subseteq \psi_F$.

(3) Assume that $\psi_F = \psi$. We prove that $(0^*)^\perp = F$. Clearly, we have that $(0^*)^\perp \subseteq F$. Let $a \in F$. Since $a^* \wedge a = 0^{**} \wedge a$, we get that $(a, 0^*) \in \psi_F = \psi$. That implies $a^* = 0^{**} = 0$. That implies $a \in D(L) = (0^*)^\perp$. Therefore $F \subseteq (0^*)^\perp$ and hence $F = (0^*)^\perp$. Conversely, assume that $F = (0^*)^\perp$. By (2), we have that $\psi \subseteq \psi_F$. Let $(a, b) \in \psi_F$. Then there exists an element $f \in F$ such that $a^* \wedge f = b^* \wedge f$. Since $f \in F$, we get that $f \in (0^*)^\perp = D(L)$. That implies $f^* = 0$ and hence $f^{**} = 0^*$. Now $a^* = 0^* \wedge a^* = f^{**} \wedge a^{***} = (a^* \wedge f)^{**} = (b^* \wedge f)^{**} = f^{**} \wedge b^{***} = 0^* \wedge b^* = b^*$. That implies $a^* = b^*$ and hence $(a, b) \in \psi$. Therefore $\psi_F \subseteq \psi$. Thus $\psi_F = \psi$.

(4) Assume that $\psi_F = \nabla_L$. Then $F = [0^*]\psi_F = L$ and hence $F = L = (0)^\perp$. Conversely, assume that $F = (0)^\perp$. Then $F = [0^*]\psi_F = L$ and hence $\psi_F = \nabla_L$.

(5) Clearly, L/ψ_F is a Boolean lattice. ∎

Theorem 4.14. *Let Θ be any G-extesion on a stone ADL. Then we have the following:*

(1) *$[0^*]\Theta$ is an $\mathcal{N}$-filter of L.*

(2) *Θ can be expressed as ψ_F, for some $\mathcal{N}$-filter F of L.*

Proof. (1) Clearly, $0^* \in [0^*]\Theta$ and hence $[0^*]\Theta$ is a non-empty set. Let $a, b \in [0^*]\Theta$. Then $(a, 0^*), (b, 0^*) \in \Theta$. That implies $a^* = 0^{**}$ and $b^* = 0^{**}$. Now, $(a \wedge b)^* = a^* \vee b^* = 0^{**}$. That implies $(a \wedge b, 0^*) \in \Theta$ and hence $a \wedge b \in [0^*]\Theta$. Let $a \in [0^*]\Theta$. Then $(a, 0^*) \in \Theta$ and hence $a^* = 0^{**}$. Let r be and element of L. Now, $(r \vee a)^* = r^* \wedge a^* = 0^{**}$. That implies $(r \vee a, 0^*) \in \Theta$ and hence $r \vee a \in [0^*]\Theta$. Therefore $[0^*]\Theta$ is a filter of L. Clearly, we have that $[0^*]\Theta \subseteq ([0^*]\Theta)^{\perp}$. Let $a \in ([0^*]\Theta)^{\perp}$. Then there exists an element $x \in [0^*]\Theta$ such that $a^* \wedge x = 0$. That implies $x^* \wedge a^* = a^*$. Since $x \in [0^*]\Theta$, we get that $x^* = 0$. Since $x^* \wedge a^* = a^*$, we get that $a^* = 0 = 0^{**}$ and hence $a \in [0^*]\Theta$. Therefore $[0^*]\Theta = ([0^*]\Theta)^{\perp}$. Thus $[0^*]\Theta$ is an $\mathcal{N}$-filter of L.

(2) Let $(a, b) \in \Theta$. Then $a^* = b^*$. That implies $a^{**} = b^{**}$ and hence $(a^*, b^*) \in \Theta$. Since $(a, b) \in \Theta$, we get that $(a \vee a^*, b \vee b^*) \in \Theta$. That implies $(a^* \vee a^{**}, b^* \vee b^{**}) \in \Theta$ and $(a^* \vee b^{**}, b^* \vee b^{**}) \in \Theta$. That implies $(0^*, b^* \vee a^{**}) \in \Theta$ and $(a^* \vee b^{**}, 0^*) \in \Theta$. That implies $b^* \vee a^{**} \in [0^*]\Theta$ and $a^* \vee b^{**} \in [0^*]\Theta$. Since $[0^*]\Theta$ is an $\mathcal{N}$-filter of L, we have that $[0^*]\Theta = [0^*]\psi_{[0^*]\Theta}$ and hence $b^* \vee a^{**} \in [0^*]\psi_{[0^*]\Theta}$ and $a^* \vee b^{**} \in [0^*]\psi_{[0^*]\Theta}$. That implies $(b^* \vee a^{**}, 0^*) \in \psi_{[0^*]\Theta}$ and $(a^* \vee b^{**}, 0^*) \in \psi_{[0^*]\Theta}$. That implies $(a^* \vee b^{**}, b^* \vee a^{**}) \in \psi_{[0^*]\Theta}$. That implies $(a^* \wedge (a^* \vee b^{**}), a^* \wedge (b^* \vee a^{**})) \in \psi_{[0^*]\Theta}$ and $(b^* \wedge (a^* \vee b^{**}), b^* \wedge (b^* \vee a^{**})) \in \psi_{[0^*]\Theta}$. That implies $(a^*, a^* \wedge b^*) \in \psi_{[0^*]\Theta}$ and $(b^* \wedge a^*, b^*) \in \psi_{[0^*]\Theta}$. Therefore $(a^*, b^*) \in \psi_{[0^*]\Theta}$ and hence $(a^{**}, b^{**}) \in \psi_{[0^*]\Theta}$. Since $a \vee a^*,\ b \vee b^* \in D(L) \subseteq [0^*]\Theta = [0^*]\psi_{[0^*]\Theta}$, we get that $(a \vee a^*, b \vee b^*) \in \psi_{[0^*]\Theta}$. That implies $(a^{**} \wedge (a \vee a^*), b^{**} \wedge (b \vee b^*)) \in \psi_{[0^*]\Theta}$ and hence $(a, b) \in \psi_{[0^*]\Theta}$. Therefore $\Theta \subseteq \psi_{[0^*]\Theta}$. Let $(a, b) \in \psi_{[0^*]\Theta}$. Then $(a^*, b^*) \in \psi_{[0^*]\Theta}$. That implies $(a^* \vee a^{**}, b^* \vee a^{**}) \in \psi_{[0^*]\Theta}$ and $(a^* \vee b^{**}, b^* \vee b^{**}) \in \psi_{[0^*]\Theta}$. That implies $(0^*, b^* \vee a^{**}) \in \psi_{[0^*]\Theta}$ and $(a^* \vee b^{**}, 0^*) \in \psi_{[0^*]\Theta}$. That implies $b^* \vee a^{**} \in [0^*]\psi_{[0^*]\Theta} = [0^*]\Theta$ and $a^* \vee b^{**} \in [0^*]\psi_{[0^*]\Theta} = [0^*]\Theta$. That implies $(0^*, b^* \vee a^{**}) \in \Theta$ and $(a^* \vee b^{**}, 0^*) \in \Theta$. That implies $(b^* \vee a^{**}, a^* \vee b^{**}) \in \Theta$. That implies $(a^* \wedge (b^* \vee a^{**}), a^* \wedge (a^* \vee b^{**})) \in \Theta$ and $(b^* \wedge (b^* \vee a^{**}), b^* \wedge (a^* \vee b^{**})) \in \Theta$. That implies $(a^* \wedge b^*, a^*) \in \Theta$ and $(b^*, b^* \wedge a^*) \in \Theta$. That implies $(a^*, b^*) \in \Theta$ and hence $(a^{**}, b^{**}) \in \Theta$. Since $(a, b) \in \Theta$, we get that $(a \vee a^*, b \vee b^*) \in \Theta$. Since $(a^{**}, b^{**}) \in \Theta$, we get that $(a^{**} \wedge (a \vee a^*), b^{**} \wedge (b \vee b^*)) \in \Theta$. That implies $(a, b) \in \Theta$ and hence $\psi_{[0^*]\Theta} \subseteq \Theta$. Thus $\psi_{[0^*]\Theta} = \Theta$. ■

Theorem 4.15. *Let L be a stone ADL. Then there is one-to-one correspondence between the set $\mathfrak{F}_{\mathcal{N}}(L)$ of all $\mathcal{N}$-filters and the set $Con_E(L)$ of all G-extensions on L.*

Proof. Clearly, we have that $(\mathfrak{F}_{\mathcal{N}}(L), \subseteq)$ and $(Con_E(L), \subseteq)$ are posets. Now define $f : \mathfrak{F}_{\mathcal{N}}(L) \longrightarrow Con_E(L)$ by $f(G) = \psi_G$, for all $G \in \mathfrak{F}_{\mathcal{N}}(L)$. By Theorem 4.13(1), we have that f is one-one. Let $\Theta \in Con_E(L)$. Then by Theorem 4.14(1), $[0^*]\Theta$ is an $\mathcal{N}$-filter of L and hence $\Theta = \psi_{[0^*]\Theta} = f([0^*]\Theta)$. Therefore f is onto. By Theorem 4.13(1), we get that $G \subseteq H$ if and only if $f(G) \subseteq f(H)$. Therefore f is an order preserving mapping. Similarly, we get f^{-1} is also an order preserving mapping. Hence f is an order isomorphism. Thus $\mathfrak{F}_{\mathcal{N}}(L)$ and $Con_E(L)$ are isomorphic partially ordered sets. ■

Corollary 4.16. *Let L be a stone ADL. Then a poset $Con_E(L)$ forms a bounded distributive lattice.*

Now, we introduce the following two notations.

(1) For any filter F of L, define an operator α as $\alpha(F) = \{(a)^{\perp} \mid a \in F\}$.

(2) For any ideal I of $\mathfrak{A}^{\perp}(L)$, define an operator β as $\beta(I) = \{a \in L \mid (a)^{\perp} \in I\}$.

Lemma 4.17. *Let L be a stone ADL with maaimal elements. Then we have the following:*

(1) *For any filter F of $L, \alpha(F)$ is an ideal of $\mathfrak{A}^{\perp}(L)$.*

(2) *For any ideal I of $\mathfrak{A}^{\perp}(L)$, $\beta(I)$ is a filter of L.*

(3) *For any filters F, G of L, $F \subseteq G \Rightarrow \alpha(F) \subseteq \alpha(G)$.*

(4) *For any ideals I, J of $\mathfrak{A}^{\perp}(L)$, $I \subseteq J \Rightarrow \beta(I) \subseteq \beta(J)$.*

Proof. (1) Let F be a filter of L. Since $0^* \in F$, we get that $(0^*)^{\perp} \in \alpha(F)$ and hence $\alpha(F) \neq \emptyset$. Let $(a)^{\perp}, (b)^{\perp} \in \alpha(F)$. Now, $(a)^{\perp} \vee (b)^{\perp} = (a \wedge b)^{\perp} \in \alpha(F)$. Let $(a)^{\perp} \in \alpha(F)$ and $(r)^{\perp} \in \mathfrak{A}^{\perp}(L)$. Then $(a)^{\perp} \cap (r)^{\perp} = (a \vee r)^{\perp} \in \alpha(F)$. Therefore $\alpha(F)$ is an ideal in $\mathfrak{A}^{\perp}(L)$.

(2) Let I be an ideal of $\mathfrak{A}^{\perp}(L)$. Since $(0^*)^{\perp} \in I$, we get that $0^* \in \beta(I)$. Then $\beta(I) \neq \emptyset$. Let $a, b \in \beta(I)$. Then $(a)^{\perp}, (b)^{\perp} \in I$. Hence $(a \wedge b)^{\perp} = (a)^{\perp} \vee (b)^{\perp} \in I$. Thus $a \wedge b \in \beta(I)$. Let $a \in \beta(I)$ and $r \in L$. Then $(a)^{\perp} \in I$ and $(r)^{\perp} \in \mathfrak{A}^{\perp}(L)$. Since I is an ideal of $\mathfrak{A}^{\perp}(L)$, we get $(a \vee r)^{\perp} = (a)^{\perp} \cap (r)^{\perp} \in I$. Hence $a \vee r \in \beta(I)$. Therefore $\beta(I)$ is a filter of L.

(3) Let F, G be two filters of L. Suppose $F \subseteq G$. We prove that $\alpha(F) \subseteq \alpha(G)$. Let $(a)^{\perp} \in \alpha(F)$. Then $a \in F \subseteq G$. Hence $(a)^{\perp} \in \alpha(G)$. Thus $\alpha(F) \subseteq \alpha(G)$.

(4) Let I, J be two ideals of $\mathfrak{A}^{\perp}(L)$ such that $I \subseteq J$. We prove that $\beta(I) \subseteq \beta(J)$. Let $a \in \beta(I)$. Then $(a)^{\perp} \in I \subseteq J$ and hence $(a)^{\perp} \in J$. Therefore $a \in \beta(J)$. Thus $\beta(I) \subseteq \beta(J)$. ■

Proposition 4.1. *Let L be a stone ADL. Then the map $F \mapsto \beta \circ \alpha(F)$ is a closure operator on the filters of L, i.e.,*

(1) $F \subseteq \beta \circ \alpha(F)$

(2) *$F \subseteq G$ implies $\beta \circ \alpha(F) \subseteq \beta \circ \alpha(G)$*

(3) *$\beta \circ \alpha\{\beta \circ \alpha(F)\} = \beta \circ \alpha(F)$ for any filters F, G of L.*

Proof. (1) Let $a \in F$. Then we get $(a)^{\perp} \in \alpha(F)$. Hence $(a)^{\perp} = (b)^{\perp}$ for some $b \in F$. Since $\alpha(F)$ is an ideal of $\mathfrak{A}^{\perp}(L)$, we get that $a \in \beta \circ \alpha(F)$. Therefore $F \subseteq \beta \circ \alpha(F)$.

(2) Suppose $F \subseteq G$. Let $a \in \beta \circ \alpha(F)$. Then $(a)^{\perp} \in \alpha(F)$. Hence $(a)^{\perp} = (b)^{\perp}$ for some $b \in F \subseteq G$. Hence $(a)^{\perp} = (b)^{\perp} \in \alpha(G)$. Since $\alpha(G)$ is an ideal of $\mathfrak{A}^{\perp}(L)$, we get $a \in \beta \circ \alpha(G)$. Therefore $\beta \circ \alpha(F) \subseteq \beta \circ \alpha(G)$.

(3) Clearly, $\beta \circ \alpha(F) \subseteq \beta \circ \alpha\{\beta \circ \alpha(F)\}$. Let $a \in \beta \circ \alpha\{\beta \circ \alpha(F)\}$. Then $(a)^{\perp} \in \alpha\{\beta \circ \alpha(F)\}$. Hence $(a)^{\perp} = (b)^{\perp}$ for some $b \in \beta \circ \alpha(F)$. Now $b \in \beta \circ \alpha(F)$ implies that $(a)^{\perp} = (b)^{\perp} \in \alpha(F)$. Therefore $a \in \beta \circ \alpha(F)$. ∎

Theorem 4.18. *Let F be a filer of a stone ADL L. Then F is an $\mathcal{N}$-filter of L if and only if $\beta \circ \alpha(F) = F$.*

Proof. Assume that F is an $\mathcal{N}$-filter of L. Clearly, we have that $F \subseteq \beta \circ \alpha(F)$. Let $a \in \beta \circ \alpha(F)$. Then $(a)^{\perp} \in \alpha(F)$. Then there exists an element $b \in F$ such that $(a)^{\perp} = (b)^{\perp}$. Since F is an $\mathcal{N}$-filter, we get that $a \in F$ and hence $\beta \circ \alpha(F) \subseteq F$. Therefore $\beta \circ \alpha(F) = F$. Conversely, assume that $\beta \circ \alpha(F) = F$. We prove that F is an $\mathcal{N}$-filter of L. Let $a, b \in L$ with $(a)^{\perp} = (b)^{\perp}$ and $a \in F$. Since $a \in F$, we get that $a \in \beta \circ \alpha(F)$. Then $(a)^{\perp} \in \alpha(F)$. Then there exists an element $c \in F$ such that $(a)^{\perp} = (c)^{\perp}$. That implies $(b)^{\perp} = (c)^{\perp}$ and $c \in F$. That implies $(b)^{\perp} \in \alpha(F)$. That implies $b \in \beta \circ \alpha(F)$. That implies $b \in F$ and hence F is an $\mathcal{N}$-filter of L. ∎

Definition 4.19. For any prime filter P of a stone ADL L, define $\ell(P) = \{a \in L \mid a^* \notin P\}$.

Proposition 4.2. *Let P be a prime filter of a stone ADL L. Then $\ell(P)$ is a filter of L containing P.*

Proof. Clearly, $0^* \in \ell(P)$. Let $a, b \in \ell(P)$. Then $a^* \notin P$ and $b^* \notin P$. Since P is prime, we get that $a^* \vee b^* \notin P$ and hence $(a \wedge b)^* \notin P$. That implies $a \wedge b \in \ell(P)$. Let $a \in \ell(P)$. Then $a^* \notin P$. Let r be any element of L. We prove that $(r \vee a)^* \notin P$. Suppose $(r \vee a)^* \in P$. Then $r^* \wedge a^* \in P$ and hence $(r^* \wedge a^*) \vee a^* \in P$. That implies $a^* \in P$, which is a contradiction. Therefore $(r \vee a)^* \notin P$ and hence $r \vee a \in \ell(P)$. Thus $\ell(P)$ is a filter of L. Let $a \in P$. We prove that $a \in \ell(P)$. Suppose $a \notin \ell(P)$. Then $a^* \in P$ and hence $a^* \wedge a \in P$. That implies $0 \in P$. That implies $P = L$, which is a contradiction. Therefore $a \in \ell(P)$ and hence $P \subseteq \ell(P)$. ∎

Theorem 4.20. *Let P be a prime filter of a stone ADL L. For any $x \in L$, $(x)^{\perp} = \bigcap_{x \in P} \ell(P)$.*

Proof. Let x be any element of P with $a \in (x)^{\perp}$. Then $a^* \wedge x = 0$. We prove that $a \in \ell(P)$. Suppose $a \notin \ell(P)$. Then $a^* \in P$. Since $x \in P$, we get that $a^* \wedge x \in P$ and hence $0 \in P$. That implies $P = L$, which is a contradiction. Therefore $a \in \ell(P)$. Hence $(x)^{\perp} \subseteq \ell(P)$, for every $x \in P$. Thus $(x)^{\perp} \subseteq \bigcap_{x \in P} \ell(P)$. Let $a \in \bigcap_{x \in P} \ell(P)$. Then $a \in \ell(Q)$, for all prime filter Q of L with $x \in Q$. Then $a^* \notin Q$. We prove that $a^* \wedge x = 0$. Suppose $a^* \wedge x \neq 0$. Then there exists a maximal filter M of L such that $a^* \wedge x \in M$. That implies $a^*, x \in M$. Since $x \in M$, we get that $a \in \ell(M)$.

That implies $a^* \notin M$, which is a contradiction. Therefore $a^* \wedge x = 0$. That implies $a \in (x)^\perp$ and hence $\bigcap_{x\in P} \ell(P) \subseteq (x)^\perp$. Therefore $(x)^\perp = \bigcap_{x\in P} \ell(P)$. ■

Corollary 4.21. *Let P be a prime filter of a stone ADL. If $x \in P$ then $(x)^\perp \subseteq \ell(P)$.*

Definition 4.22. An $\mathcal{N}$-filter P of a stone ADL L is said to be an $\mathcal{N}$-prime filter if P is a prime filter of L.

Theorem 4.23. *Let F be an $\mathcal{N}$-filter and I, an ideal of L with $F \cap I = \emptyset$. There exists an $\mathcal{N}$-prime filter P of L such that $F \subseteq P$ and $P \cap I = \emptyset$.*

Proof. Consider $\mathfrak{F} = \{G \mid G$ is an $\mathcal{N}$-filter and $G \cap I = \emptyset\}$. Clearly $F \in \mathfrak{F}$ and $\mathfrak{F}$ satisfies the Zorn's lemma hypothesis. Then $\mathfrak{F}$ has a maximal element say N. Let $a, b \in L$ with $a \vee b \in N$. We prove that either $a \in N$ or $b \in N$. Suppose that $a \notin N$ and $b \notin N$. Then $N \subset N \vee [a) \subseteq \beta \circ \alpha(N \vee [a))$ and $N \subset N \vee [b) \subseteq \beta \circ \alpha(N \vee [b))$. That implies $N \subset \beta \circ \alpha(N \vee [a))$ and $N \subset \beta \circ \alpha(N \vee [b))$. Since $\beta \circ \alpha(M \vee [a))$ and $\beta \circ \alpha(N \vee [b))$ are $\mathcal{N}$-filters of L, we get that $\beta \circ \alpha(N \vee [a)) \cap I \neq \emptyset$ and $\beta \circ \alpha(N \vee [b)) \cap I \neq \emptyset$. Then choose $x \in \beta \circ \alpha(N \vee [a)) \cap I$ and $y \in \beta \circ \alpha(N \vee [b)) \cap I$. Therefore $x \vee y \in I$ and $x \vee y \in \beta \circ \alpha(N \vee [a)) \cap \beta \circ \alpha(N \vee [b)) = \beta \circ \alpha((N \vee [a)) \cap (N \vee [b))) = \beta \circ \alpha(N \vee [a \vee b)) = \beta \circ \alpha(N) = N$. Therefore $N \cap I \neq \emptyset$, which is a contradiction. Hence $a \in N$ or $b \in N$. Thus N is an $\mathcal{N}$-prime filter of L. ■

Theorem 4.24. *Let L be a stone ADL. Then every proper $\mathcal{N}$-filter of L is the intersection of all $\mathcal{N}$-prime filters containing it.*

Proof. Let F be a proper $\mathcal{N}$-filter of L. Consider the following set $F_0 = \bigcap\{P \mid P$ is an $\mathcal{N}$-prime filter and $F \subseteq P\}$. Clearly, $F \subseteq F_0$. Let $x \notin F$. Take $\mathfrak{F} = \{G \mid G$ is a $\mathcal{N}$-filter,$F \subseteq G, x \notin G\}$. Clearly, we have that $F \in \mathfrak{F}$ and $\mathfrak{F}$ satisfies the hypothesis of Zorn's lemma. That implies $\mathfrak{F}$ has a maximal element N, say. We prove that N is prime. Suppose that $a, b \in L$ with $a \notin N$ and $b \notin N$. Then $N \subset N \vee [a) \subseteq \beta \circ \alpha\{N \vee [a)\}$ and $N \subset N \vee [b) \subseteq \beta \circ \alpha\{N \vee [b)\}$. By maximality of N, we get $x \in \beta \circ \alpha\{N \vee [a)\}$ and $x \in \beta \circ \alpha\{N \vee [b)\}$. Hence we get that $x \in \beta \circ \alpha\{N \vee [a)\} \cap \beta \circ \alpha\{N \vee [b)\} = \beta \circ \alpha\{[N \vee [a)] \cap [N \vee [b)]\} = \beta \circ \alpha\{N \vee [a \vee b)\}$. If $a \vee b \in N$, then $x \in \beta \circ \alpha(N) = N$, which is a contradiction. Thus N is an $\mathcal{N}$-prime filter such that $x \notin N$. Therefore $x \notin F_0$ and hence $F = F_0$. Thus every proper $\mathcal{N}$-filter of L is the intersection of all $\mathcal{N}$-prime filters containing it. ■

5. THE SPACE OF $\mathcal{N}$-PRIME FILTERS

In this section, we discuss some topological concepts on the collection of $\mathcal{N}$-prime filters of a Stone ADL.

Let $Spec_{\mathcal{N}_F}(L)$ be the set of all $\mathcal{N}$-prime filters of a stone ADL L. For any $A \subseteq L$, define $h(A) = \{P \in Spec_{\mathcal{N}_F}(L) \mid A \nsubseteq P\}$ and for any $a \in L$; $h(a) = h(\{a\})$. For any two subsets A and B of L, it is obvious that $A \subseteq B$ implies $h(A) \subseteq h(B)$. The following observations can be verified directly.

Lemma 5.1. *For any $a, b \in L$, the following conditions holds.*

(1) $\bigcup_{a\in L} h(a) = Spec_{\mathcal{N}_F}(L)$

(2) $h(a) \cap h(b) = h(a \vee b)$

(3) $h(a) \cup h(b) = h(a \wedge b)$

(4) $h(a) = \emptyset \Leftrightarrow a$ *is maximal.*

From the above Lemma, it can be easily observed that the collection $\{h(a) \mid a \in L\}$ forms a base for a topology on $Spec_{\mathcal{N}_F}(L)$ which is called a hull-kernel topology.

Theorem 5.2. *For any filter F of L, $h(F) = h(\beta \circ \alpha(F))$.*

Proof. Clearly we get that $h(F) \subseteq h(\beta \circ \alpha(F))$. Let $P \in h(\beta \circ \alpha(F))$. Then $\beta \circ \alpha(F) \nsubseteq P$. Therefore we can choose an element $a \in \beta \circ \alpha(F)$ such that $a \notin P$. Since $a \in \beta \circ \alpha(F)$, we have $(a)^{\perp} \in \alpha(F)$ and hence there exists an element $b \in F$ such that $(a)^{\perp} = (b)^{\perp}$. Suppose $F \subseteq P$. Then $b \in P$. Since P is an $\mathcal{N}$-filter of L, we get that $a \in P$, which is a contradiction. Therefore $F \nsubseteq P$ and hence $P \in h(F)$. Thus $h(\beta \circ \alpha(F)) \subseteq h(F)$. ∎

In the following theorem, the compact open set of $Spec_{\mathcal{N}_F}(L)$ are characterized.

Theorem 5.3. *For any stone ADL, the set of all compact open sets of $Spec_{\mathcal{N}_F}(L)$ is the base $\{h(a) \mid a \in L\}$.*

Proof. Let $a \in L$ with $h(a) \subseteq \bigcup_{i\in\Delta} h(a_i)$. Let F be a filter generated by $\{a_i \mid i \in \Delta\}$. Suppose $a \notin \beta \circ \alpha(F)$. Since $\beta \circ \alpha(F)$ is an $\mathcal{N}$-filter of L, there exists an $\mathcal{N}$-prime filter P of L such that $a \notin P$ and $\beta \circ \alpha(F) \subseteq P$. Since $a \notin P$, we get that $P \in h(a) \subseteq \bigcup_{i\in\Delta} h(a_i)$. That implies $a_i \notin P$, for some $i \in \Delta$, which is a contradiction to that $F \subseteq \beta \circ \alpha(F) \subseteq P$. Therefore $a \in \beta \circ \alpha(F)$. That implies $(a)^{\perp} \in \alpha(F)$ and hence there exists an element $b \in F$ such that $(a)^{\perp} = (b)^{\perp}$. Since F is a filter generated by $\{a_i \mid i \in \Delta\}$, we get that $b = a_1 \wedge a_2 \wedge \cdots \wedge a_n$, for some $a_1, a_2, \ldots, a_n \in \{a_i \mid i \in \Delta\}$. That implies $(b)^{\perp} = (a_1 \wedge a_2 \wedge \cdots \wedge a_n)^{\perp}$. Let $P \in h(a)$. Then $a \notin P$. Suppose $P \notin \bigcup_{i\in\Delta} h(a_i)$. Then $a_i \in P$, for all $i = 1, 2, \ldots, n$ and hence $a_1 \wedge a_2 \wedge \cdots \wedge a_n \in P$. That implies $b \in P$, which is a contradiction. Therefore $P \in \bigcup_{i\in\Delta} h(a_i)$ and hence $h(a) \subseteq \bigcup_{i=1}^{n} h(a_i)$. Thus $h(a)$ is a compact space. It is enough to show that every compact open subset of $Spec_{\mathcal{N}_F}(L)$ is of the form $h(a)$, for some $a \in L$. Let C be a compact open subset of

$Spec_{\mathcal{N}_F}(L)$. Since C is open, we get that $C = \bigcup_{x \in A} h(x)$, for some $A \subseteq L$. Since C is compact, there exist $x_1, x_2, \ldots, x_n \in A$ such that $C = \bigcup_{i=1}^{n} h(x_i) = h(\bigwedge_{i=1}^{n} x_i)$. Therefore $C = h(a)$, for some $a \in L$. ■

Corollary 5.4. *Let L be a stone ADL. Then $Spec_{\mathcal{N}_F}(L)$ is a compact space.*

Theorem 5.5. *Let L be a stone ADL. Then the following are equivalent:*

(1) *$Spec_{\mathcal{N}_F}(L)$ is T_1-space*

(2) *every $\mathcal{N}$-prime filter is maximal*

(3) *every $\mathcal{N}$-prime filter is minimal*

(4) *$Spec_{\mathcal{N}_F}(L)$ is Haudorff space.*

Proof. (1)⇒(2) Assume that $Spec_{\mathcal{N}_F}(L)$ is T_1-space. Let P be an $\mathcal{N}$-prime filter of L. Suppose Q is any $\mathcal{N}$-prime filter of L with $P \subsetneq Q$. Since $Spec_{\mathcal{N}_F}(L)$ is T_1-space, there exist basic open sets $h(a)$ and $h(b)$ such that $P \in h(a) \setminus h(b)$ and $Q \in h(b) \setminus h(a)$. Since $P \notin h(b)$, we get that $b \in P \subsetneq Q$. Therefore $Q \notin h(b)$, which is a contradiction. Hence P is maximal.

(2)⇒(3) It is obvious.

(3)⇒(1) Assume that every $\mathcal{N}$-prime filter is minimal. Let $P, Q \in Spec_{\mathcal{N}_F}(L)$ with $P \neq Q$. Since P and Q are minimal, it is clear that $P \nsubseteq Q$ and $Q \nsubseteq P$. Then there exist $a, b \in L$ such that $a \in P \setminus Q$ and $b \in Q \setminus P$. That implies $P \in h(b) \setminus h(a)$ and $Q \in h(a) \setminus h(b)$. Therefore $Spec_{\mathcal{N}_F}(L)$ is T_1-space.

(2)⇒(4) Assume that every $\mathcal{N}$-prime filter is maximal. Let $P, Q \in Spec_{\mathcal{N}_F}(L)$ with $P \neq Q$. Choose an element $x \in P$ such that $x \notin Q$. By our assumption, P is maximal filter of L. Since $x \in P$, then there $y \notin P$ such that $x \vee y$ is maximal element. So that $Q \in h(x)$ and $P \in h(y)$. Now $h(x) \cap h(y) = h(x \vee y) = \emptyset$, since $x \vee y$ is maximal.

(4)⇒(1) Clear. ■

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