Discussiones Mathematicae
General Algebra and Applications 41 (2021) 229–237
https://doi.org/10.7151/dmgaa.1369

FASCINATING NUMBER SEQUENCES FROM FOURTH ORDER DIFFERENCE EQUATION VIA QUATERNION ALGEBRAS

Asim Patra

Department of Mathematics
National Institute of Technology
Rourkela, Odisha, India

e-mail: asimp1993@gmail.com, 515ma3016@nitrkl.ac.in

Abstract

The balancing and Lucas-balancing numbers are solutions of second order recurrence relations. A linear combination of these numbers can also be obtained as solutions of a fourth order recurrence relation. This recurrence relation can be extended to generalized quaternion algebras. Also, the fourth order recurrence relation has application in coding theory.

Keywords: balancing numbers, Lucas-balancing numbers, quaternion algebra, Coding theory, Pell and Lucas-Pell numbers.

2010 Mathematics Subject Classification: 11B37, 11B39, 11D09.

1. Introduction

Flaut and Savin [6] studied a special number sequence satisfying a recurrence relation of order three. They obtained a linear combination of the Pell and Lucas-Pell numbers as solution of the the recurrence relation, which they named as the generalized Pell-Fibonacci-Lucas numbers. Motivated by their work, we consider a fourth-order difference equation and obtain some special number sequences as a particular case of this equation. We use this recurrence relation in quaternion algebra.

We start with a fourth order recurrence relation

$$Y_n = aY_{n-1} + bY_{n-2} + cY_{n-3} + dY_{n-4}, \tag{1.1}$$

which is defined on the finite field of integers modulo p denoted by Z_p, where p is a positive prime number and the initial values for (1.1) are given by $Y_0 = i_0$, $Y_1 = i_1$,

$Y_2 = i_2$ and $Y_3 = i_3$. If $a = 6$, $b = -1$, $c = d = 0$, $i_0 = 0$, $i_1 = 1$, then (1.1) reduces to the recurrence relation for balancing numbers and if $a = 6$, $b = -1$, $c = d = 0$, $i_0 = 1$, $i_1 = 3$, (1.1) reduces to the recurrence relation of Lucas-balancing numbers. In this paper, we study the properties of a sequence arising out of the recurrence relation (1.1). Apart from that, we use the sequence $\{Y_n\}$ to study some properties of quaternion algebra. In particular, we obtain certain formulas relating the product of balancing and Lucas-balancing numbers which are used in proving certain results involving quaternions. Finally, we conclude this paper by providing some applications related to a sequence arising out of (1.1).

The coding theory has been an emerging field in recent time for the applications point of view [2, 4, 7, 11, 12]. The utilization of coding-theoretic concepts have been an interesting area and reinterpretation of some earlier ideas on complexity emerged in the past few years. The concept of error-correcting codes came into existence for providing safety and security of sending information over noisy channels.

In [1], Elhameed *et al.* studied the generalized Fibonacci and Pell numbers. Furthermore, Voight [13] considered the arithmetic of quaternion algebras. In a different work, Flaut and Savin [6] dealt with some special sequences from a third degree difference equation. In [2], Basu and Prasad studied the generalized relation of the code elements for the Fibonacci coding theory. Motivated by these works, we introduce certain special numbers and exhibit their applications in coding theory. In addition, the concept of coding theory will be used to find certain polynomials which are used to prove an identity.

2. Coding theory

A code say C is a mapping from finite set say R_1^j to another finite set R_2^j which possess a characteristic that while a string s_1 is given, near to a valid encoding say $C(s_2)$, it is possible to obtain the message s_2 from the corrupted encoding s_1. There are a lot of applications of this property of the error-correcting codes in recent times. Let $C \subset \mathbb{Z}_p^n$ be a linear code of length n and $(c_0, c_1, c_2, \ldots, c_{n-1})$ be a codeword. Moreover, a linear code is said to be a cyclic code if

$$(c_0, c_1, c_2, \ldots, c_{n-1}) \in C \Rightarrow (c_{n-1}, c_0, c_1, c_2, \ldots, c_{n-2}) \in C.$$

It is very beneficial and effective if each codeword in a cyclic code is illustrated via polynomials. Hence, the main task is to associate a polynomial, known as the associated code polynomial, to the codeword which has numerous applications. Now, a code polynomial $g(x)$ generates the cyclic code of length n if and only if $g(x)|x^n - 1$. Furthermore, the polynomial $\frac{x^n-1}{g(x)}$ is said to be the check polynomial of C. Apart from that, the number of positions in which the symbols of the corresponding codeword are varied is termed as the Hamming distance d' between two codewords of equal length n. Similarly, the number of symbols which differ from the zero-symbol of the used alphabet is termed as the Hamming weight of a code, denoted by w. Moreover the minimum Hamming distance say d'_h and the minimum Hamming weight say w_h of a particular code is given by

$$d'_h = \min\{d'(c_1, c_2) : c_1 \neq c_2, c_1, c_2 \in C\} \tag{2.1}$$

and

$$w_h = \min\{w(c_1, c_2) : c_1 \neq c_2, c_1, c_2 \in C\}, \tag{2.2}$$

respectively. Now, we apply the above definitions and concepts for constructing an associated code polynomial which is associated with certain codeword.

In [6], authors have taken a **D**-polynomial associated with the sequence **D**, as defined in their paper. Furthermore, they have established a relation involving the **D**-polynomial (as defined in [6]), a cubic equation and the coefficient of the sequence **D**. Hence, in a similar way, generalizing their result by considering a fourth order difference equation, we construct our polynomial and name it the **Y**-polynomial associated with the sequence (1.1) and state our result in the following theorem.

Theorem 2.1. *Let $l_{d'}(p)$ be the period and $\beta_{d'}(p)$ be the number of zeros in a single period of the sequence in* (1.1)*. Furthermore, let us consider a polynomial $\delta(x) = \sum_{i=0}^{l_{d'}(p)} Y_i x^i \in \mathbb{Z}_p[x]$ which is associated with the sequence $\mathbf{Y} = \{Y_n\} = \{Y_0, Y_1, \ldots, Y_n, \ldots\}$, which behaves as a codeword and we name the polynomial $\delta(x)$ as the $\mathbf{Y}$-polynomial. Then, the following relation holds*

$$\begin{aligned}
&\delta(x)(dx^4 + cx^3 + bx^2 + ax - 1)\\
&= dY_{l-1}x^{l+3} + (dY_{l-2} + cY_{l-3})x^{l+2} + (dY_{l-3} + cY_{l-2}\\
&+ bY_{l-1})x^{l+1} + (aY_{l-1} + bY_{l-2} + cY_{l-3} + dY_{l-4})x^l\\
&+ (-Y_3 + aY_2 + bY_1 + cY_0)x^3 + (-Y_2 + aY_1 + bY_0)x^2 + (aY_0 - Y_1)x - Y_0.
\end{aligned}$$

Proof. Let us denote $l_{d'}(p) = l$ and let $Y_n = aY_{n-1} + bY_{n-2} + cY_{n-3} + dY_{n-4}, Y_0 = x_0, Y_1 = x_1, Y_2 = x_2, Y_3 = x_3$, be a difference equation of order four. Now, by definition of a $\mathbf{Y}$-polynomial $\delta(x)$,

$$\delta(x)(dx^4 + cx^3 + bx^2 + ax - 1) = (dx^4 + cx^3 + bx^2 + ax - 1)\sum_{i=0}^{l-1} Y_i x^i. \tag{2.3}$$

It can be verified that the power of x, in the expansion of (2.3), varies from 0 to $l+3$. Now, the coefficient of all x^m, $4 \leq m \leq l-1$, is obtained as $-Y_m + aY_{m-1} +$

$bY_{m-2}+cY_{m-3}+dY_{m-4}$. Using (1.1), it can be easily verified that the coefficient of x^m vanishes out for $4 \le m \le l-1$. So, the task is to calculate the coefficient of x^m for other possible values of m except $4 \le m \le l-1$. Furthermore,

$$l+3 = 4+(l-1),$$

$$l+2 = 4+(l-2) = 3+(l-1),$$

$$l+1 = 4+(l-3) = 3+(l-2) = 2+(l-1),$$

$$l = 4+(l-4) = 3+(l-3) = 2+(l-2) = 1+(l-1),$$

$$3 = 0+3 = 1+2 = 2+1,$$

$$2 = 0+2 = 1+1 = 2+0,$$

$$1 = 0+1 = 1+0,$$

$$0 = 0+0.$$

So, the coefficients of x^m are given by

m	Coefficient
$l+3$	dY_{l-1}
$l+2$	$dY_{l-2}+cY_{l-3}$
$l+1$	$dY_{l-3}+cY_{l-2}+bY_{l-1}$
l	$aY_{l-1}+bY_{l-2}+cY_{l-3}+dY_{l-4}$
3	$-Y_3+aY_2+bY_1+cY_0$
2	$-Y_2+aY_1+bY_0$
1	aY_0-Y_1
0	Y_0

Hence,

$$\begin{aligned}
&\delta(x)(dx^4+cx^3+bx^2+ax-1)\\
&= dY_{l-1}x^{l+3}+cY_{l-1}x^{l+2}+bY_{l-1}x^{l+1}+aY_{l-1}x^{l}-Y_{l-1}x^{l-1}\\
&+dY_{l-2}x^{l+2}+cY_{l-2}x^{l+1}+bY_{l-2}x^{l}+aY_{l-2}x^{l-1}-Y_{l-2}x^{l-2}\\
&+dY_{l-3}x^{l+1}+cY_{l-3}x^{l}+bY_{l-3}x^{l-1}+aY_{l-3}x^{l-2}-Y_{l-3}x^{l-3}\\
&+dY_{l-4}x^{l}+cY_{l-4}x^{l-1}+bY_{l-4}x^{l-2}+aY_{l-4}x^{l-3}-Y_{l-4}x^{l-4}\\
&+dY_{l-5}x^{l-1}+cY_{l-5}x^{l-2}+bY_{l-5}x^{l-3}+aY_{l-5}x^{l-4}-Y_{l-5}x^{l-5}\\
&+\cdots+dY_5x^9+cY_5x^8+bY_5x^7+aY_5x^6-Y_5x^5+dY_4x^8\\
&+cY_4x^7+bY_4x^6+aY_4x^5-Y_4x^4+dY_3x^7+cY_3x^6+bY_3x^5
\end{aligned}$$

$$+ aY_3x^4 - Y_3x^3 + dY_2x^6 + cY_2x^5 + bY_2x^4 + aY_2x^3 - Y_2x^2$$
$$+ dY_1x^5 + cY_1x^4 + bY_1x^3 + aY_1x^2 - Y_1x + dY_0x^4 + cY_0x^3$$
$$+ bY_0x^2 + aY_0x - Y_0.$$

Now, by virtue of (1.1), we get the desired result. ■

2.1. Maximum distance separable code

Let us consider a linear code C, with dimension e, and length of the codewords be n. If the code satisfies

$$e + d' = n + 1, \tag{2.4}$$

then C is called the Maximum distance separable(MDS) code.

2.1.1. Example–1

Here we take the example of tetranacci sequence $\{Y_n\}$ defined by $Y_n = Y_{n-1} + Y_{n-2} + Y_{n-3} + Y_{n-4}$, $Y_0 = 0,\ Y_1 = 1,\ Y_2 = 1,\ Y_3 = 2$ and taking $\{Y_n\}$ modulo 3, renders us the values $0, 1, 1, 2, 1, 2, 0, 2, 2, 0, 1, 2, 2, 2, 1, 1, 0, 1, 0, 2, 0, 0, 2, 1, 0, 0$ and hence, $l_{B_n}(5) = 26$ and $\beta_{B_n}(p) = 9$. Also, the corresponding balancing polynomial is given by $b(x) = x + x^2 + 2x^3 + x^4 + 2x^5 + 2x^7 + 2x^8 + x^{10} + 2x^{11} + 2x^{12} + 2x^{13} + x^{14} + x^{15} + x^{17} + 2x^{19} + 2x^{22} + x^{23}$. Furthermore, the Hamming distance generated by the polynomial $b'(x)$, of C, given by $d' = 17$. Since $n + 1 = 27 \neq d' + e$, hence C is not an MDS code.

3. THE GENERALIZED QUATERNION ALGEBRA RELATED TO THE PROPOSED SEQUENCES

In [5, 10], Flaut and Savin introduced the generalized Fibonacci-Lucas numbers and the generalized Fibonacci-Lucas quaternions and, in addition to it, several properties of these quaternion elements were also obtained. Furthermore, as mentioned in the introduction section, the generalized Pell-Fibonacci-Lucas numbers and the generalized Pell-Fibonacci-Lucas quaternions were introduced in [6]. In a similar manner, we introduce the L-C sequence and the L-C quaternions in this section.

A balancing number n is a positive integer that satisfies the Diophantine equation $1 + 2 + \cdots + (n-1) = (n+1) + \cdots + (n+r)$ for some positive integer r, called the balancer corresponding to n [3]. The n-th balancing number is denoted by B_n, for each n, $8B_n^2 + 1$ is a perfect square and its positive square root C_n is called a Lucas-balancing number [9]. The balancing and Lucas-balancing numbers are solutions of the Pell's equation $y^2-8x^2=1$ and their Binet formulas are $B_n=\frac{\alpha^{2n}-\beta^{2n}}{\alpha^2-\beta^2}$ and $C_n=\frac{\alpha^{2n}+\beta^{2n}}{2}$ respectively, where $\alpha=1+\sqrt{2}$ and $\beta=1-\sqrt{2}$. These numbers satisfy the recurrence relations $B_{n+1}=6B_n-B_{n-1}$, $C_{n+1}=6C_n-C_{n-1}$ with initial terms $B_0=0$, $B_1=1$, $C_0=1$ and $C_1=3$ [8, 9] respectively.

Lemma 3.1. *If n, l are positive integers, $(B_n)_{n\geq 0}$ is the sequence of balancing numbers and $(C_n)_{n\geq 0}$ is the sequence of Lucas-balancing numbers, then the following statements hold:*

(a) $C_nC_{n+l}=\frac{1}{2}(C_{2n+l}+C_l)$;

(b) $B_nC_{n+l}=\frac{1}{2}(B_{2n+l}-B_l)$;

(c) $C_nB_{n+l}=\frac{1}{2}(B_{2n+l}+B_l)$;

(d) $B_nB_{n+l}=\frac{1}{16}(C_{2n+l}-C_l)$.

Proof. The proof of the lemma is trivial by virtue of Binet formulas for balancing and Lucas-balancing numbers. ■

Let us define a sequence which is a linear combination of balancing numbers and Lucas-balancing numbers

$$l_{n+1}^{s,t}=sB_n+tC_{n+1}, \tag{3.1}$$

where s and t are arbitrary integers and $n\geq 0\in\mathbb{Z}^+$ and we call $(l_n^{s,t})_{n\geq 1}$ as the L-C sequence. Let us denote the rational generalized quaternion algebra as $\mathbb{H}(z_1,z_2)$, where z_1, $z_2\in\mathbb{Q}-\{0\}$ in which $\{1,g_1,g_2,g_3\}$ is the corresponding basis, where 1 is the identity element and, in which, the multiplication of the remaining elements possess the property given by

$g_1\cdot g_1=z_1$	$g_1\cdot g_2=g_3$	$g_1\cdot g_3=z_1\cdot g_2$
$g_2\cdot g_1=-g_3$	$g_2\cdot g_2=z_1$	$g_2\cdot g_3=-z_2\cdot g_1$
$g_3\cdot g_1=-z_1\cdot g_2$	$g_3\cdot g_2=z_2\cdot g_1$	$g_3\cdot g_3=z_1\cdot z_2$

3.1. The L–C quaternion

Here, we define the n^{th} L–C quaternion to be the quaternion element

$$L_n^{s,t}=l_n^{s,t}\cdot 1+l_{n+1}^{s,t}i+l_{n+2}^{s,t}j+l_{n+3}^{s,t}k.$$

Lemma 3.2.

$$L_n^{s,t}=0\iff s=t=0.$$

Proof. The converse part of the theorem is trivial, i.e., if $s = t = 0$, then

$$L_n^{0,0} = l_n^{0,0} \cdot 1 + l_{n+1}^{0,0} i + l_{n+2}^{0,0} j + l_{n+3}^{0,0} k = 0.$$

Now, since $\{1, g_1, g_2, g_3\}$ is the basis of the generalized quaternion algebra $H_{\mathbb{Q}}(z_1, z_2)$, $l_n^{s,t} = 0 = l_{n+1}^{s,t} = l_{n+2}^{s,t} = l_{n+3}^{s,t}$, from which we get $l_{n-1}^{s,t} = 0, \ldots, l_2^{s,t} = l_1^{s,t} = 0$. Now, by (3.1),

$$l_1^{s,t} = sB_0 + tC_1 = 3t$$

and since $l_1^{s,t} = 0$, we get

$$t = 0. \tag{3.2}$$

In a similar manner, $l_1^{s,t} = 0$ implies that $sB_1 + tC_2 = s + 17t = 0$ and consequently, we obtain $s = 0$ from (3.2). ■

Lemma 3.3. *The following result holds*

$$sB_{n+1} + tC_n = l_n^{-s,t} + l_{n+1}^{6s,0}, \forall n \in \mathbb{N} - \{0\},$$

where $(l_n^{-s,t})_{n \geq 1}$ *denotes the* L–C *sequence as in* (3.1).

Proof. By virtue of recurrence relation for balancing numbers,

$$sB_{n+1} + tC_n = s(6B_n - B_{n-1}) + tC_n = -sB_{n-1} + tC_n + 6sB_n = l_n^{-s,t} + l_{n+1}^{6s,0}.$$
■

Definition 3.4. *A subring $U \subseteq \mathbb{H}(z_1, z_2)$ is an order in $\mathbb{H}(z_1, z_2)$, if U is a finitely generated $\mathbb{Z}$-submodule of $\mathbb{H}(z_1, z_2)$.*

Theorem 3.5. *Let us consider the set defined by*

$$U = \left\{ \sum_{i=1}^{n} 16 L_{n_i}^{s_i, t_i} : n_i \in \mathbb{N}^*, s_i, t_i \in \mathbb{Z},\ i = 1, 2, \ldots, n \right\} \cup \{1\}.$$

Then, U is an order of the quaternion algebra $\mathbb{H}_{\mathbb{Q}}(z_1, z_2)$.

Proof. By virtue of Lemma 3.2, we have $0 \in U$. We need to claim that U is a $\mathbb{Z}$-submodule of the generalized quaternion algebra $\mathbb{H}_{\mathbb{Q}}(z_1, z_2)$. Furthermore, we have

$$a l_n^{s,t} + b l_m^{s',t'} = l_n^{as,at} + l_m^{bs',bt'}, \tag{3.3}$$

for $n, m \in \mathbb{N}^*$, and a, b, s, t, s', t' are arbitrary integers. Now, (3.3) can be simplified as

$$a L_n^{s,t} + b L_m^{s',t'} = L_n^{as,at} + L_m^{bs',bt'}.$$

Then, we obtain that U is a free $\mathbb{Z}$-submodule for the quaternion algebra $\mathbb{H}_{\mathbb{Q}}(z_1, z_2)$. Now we need to prove that U is subring of $\mathbb{H}_{\mathbb{Q}}(z_1, z_2)$, in order to find an order of the generalized quaternion algebra. Let m and n be two integers with $n < m$. Then,

$$\begin{aligned}16l_n^{s,t} \cdot 16l_m^{s',t'} &= 16(sB_{n-1} + tC_n) \cdot 16(s'B_{m-1} + t'C_m) \\ &= 256ss'B_{n-1}B_{m-1} + 256st'B_{n-1}C_m + 256s'tB_{m-1}C_n + 256tt'C_nC_m.\end{aligned}$$

By virtue of Lemma 3.1(a–d), we obtain

$$\begin{aligned}16l_n^{s,t} \cdot 16l_m^{s',t'} &= 16ss'\,(C_{m+n-2} - C_{m-n}) + 128st'\,(B_{m+n-1} - B_{m-n+1}) \\ &+ 128s't\,(B_{m+n-1} + B_{m-n-1}) + 128tt'(C_{m+n} + C_{m-n}) \\ &= 128(s'tB_{m-n-1} + tt'C_{m-n}) + 16(-8st'B_{m-n+1} - ss'C_{m-n}) \\ &+ 16(8s'tB_{m+n-1} + ss'C_{m+n-2}) + 128(st'B_{m+n-1} + tt'C_{m+n}) \\ &= 16l_{m-n}^{8s't,8tt'} + 16l_{m-n}^{8st',-ss'} + 16l_{m-n+1}^{-48st',0} + 16l_{m+n-2}^{-8s't,ss'} \\ &+ 16l_{m+n-1}^{48s't,0} + 16l_{m+n}^{8st',8tt'}.\end{aligned}$$

Hence, we obtain $16l_n^{s,t} \cdot 16l_m^{s',t'} \in U$, which implies that U is an order of $\mathbb{H}_{\mathbb{Q}}(z_1, z_2)$. ∎

Conclusion

In this paper, some special numbers, which we call the L–C numbers, were obtained as a particular case of a fourth order difference equation. Moreover, these numbers were represented by the linear combination of some known number sequences, and furthermore, their quarternion elements have also been defined and some properties related to them were proved. In addition to it, some applications of the special number sequences were shown in the field of coding theory which exhibits the importance of these sequences. One can study the properties and applications of certain other number sequences obatined from other higher order difference equations.

Acknowledgement

The author thanks the learned anonymous reviewers for their valuable suggestions which helped us to improve this paper. Furthermore, I provide my thanks to my supervisor for his valuable suggestions.

References

[1] W.M. Abd-Elhameed and N.A. Zeyada, *A generalization of generalized Fibonacci and generalized Pell numbers*, Int. J. Math. Educ. Sci. Technol. **48** (2017) 102–107. https://doi.org/10.1080/0020739X.2016.1170900

[2] M. Basu and B. Prasad, *The generalized relation among the code elements for Fibonacci coding theory*, Chaos, Solitons and Fractals **41** (2009) 2517–2525. https://doi.org/10.1016/j.chaos.2008.09.030

[3] A. Behera and G.K. Panda, *On the square roots of triangular numbers*, Fibonacci Quarterly **37** (1999) 98–105.

[4] T.M. Cover and J.A. Thomas, Elements of Information Theory (Wiley, New York, 1991).

[5] C. Flaut and D. Savin, *Quaternion algebras and generalized Fibonacci-Lucas quaternions*, Adv. Appl. Clifford Algebras **25** (2015) 853–62. https://doi.org/10.1007/s00006-015-0542-0

[6] C. Flaut and D. Savin, *Some special number sequences obtained from a difference equation of degree three*, Chaos, Solitons and Fractals **106** (2018) 67–71. https://doi.org/10.1016/j.chaos.2017.11.015

[7] M.E. Koroglu, I. Ozbek and I. Siap, *Optimal codes from Fibonacci polynomials and secret sharing schemes*, Arab J. Math. (2017) 1–12. https://doi.org/10.1007/s40065-017-0171-7

[8] G.K. Panda and P.K. Ray, *Some links of balancing and cobalancing numbers with Pell and associated Pell numbers*, Bull. Inst. Math. Acad. Sin. (New Series) **6** (2011) 41–72.

[9] P.K. Ray, *Some congruences for Balancing and Lucas-Balancing Numbers and their Applications*, Integers **14** (2014) A8.

[10] D. Savin, *A*bout special elements in quaternion algebras over finite fields, Adv. Appl. Clifford Algebras **27** (2017) 1801–13. https://doi.org/10.1007/s00006-016-0718-2

[11] A. Stakhov, V. Massingue and A. Sluchenkov, Introduction into Fibonacci Coding and Cryptography (Osnova, Kharkov, 1999).

[12] A.P. Stakhov, *Fibonacci matrices, a generalization of the Cassini formula and a new coding theory*, Chaos, Solitons and Fractals **30** (2006) 56–66. https://doi.org/10.1016/j.chaos.2005.12.054

[13] J. Voight, *The arithmetic of quaternion algebras*, http://www.math.dartmouth.edu/jvoight/crmquat/book/ quatmodforms-041310, (2015).

Received 12 December 2019
First Revised 20 February 2020
Second Revised 8 July 2020
Accepted 11 July 2020