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GREEN'S RELATIONS ON SUBMONOIDS OF GENERALIZED HYPERSUBSTITUTIONS OF TYPE (n)

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Abstract

A generalized hypersubstitution of type $\tau = (n)$ is a function which takes the n-ary operation symbol f to the term of the same type $\sigma(f)$ which does not necessarily preserve the arity. Let $Hyp_G(n)$ be the set of all these generalized hypersubstitutions of type (n). The set $Hyp_G(n)$ with a binary operation and the identity generalized hypersubstitution forms a monoid. The objective of this paper is to study Green's relations on the set of all regular elements of $Hyp_G(n)$.

Keywords: generalized hypersubstitutions, green's relation, regular elements.

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1. Introduction

The concept of terms is one of the fundamental concepts of universal algebra. Terms may be considered as words formed by letters. To define terms, one needs variables and operation symbols. Let $(f_i)_{i \in I}$ be a sequence of n_i-ary operation

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symbols indexed by the set I where $n_i \in \mathbb{N}^+ := \mathbb{N} \setminus \{0\}$. We denote by $X := \{x_1, \ldots, x_n, \ldots\}$ a countably infinite set of symbols called variables and for each $n \geq 1$ let $X_n := \{x_1, \ldots, x_n\}$. We call the sequence $\tau := (n_i)_{i \in I}$ of arities of f_i, the type. An *n-ary term of type τ* is defined inductively as follows.

(i) Every variable $x_j \in X_n$ is an n-ary term of type τ.

(ii) If $t_1, \ldots, t_{n_i}$ are n-ary terms of type τ and f_i is an n_i-ary operation symbol, then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term of type τ.

Let $W_\tau(X_n)$ be the set of all n-ary terms of type τ which contains $x_1, \ldots, x_n$ and is closed under finite application of (ii) and let $W_\tau(X) := \bigcup_{n \in \mathbb{N}^+} W_\tau(X_n)$ be the set of all terms of type τ.

The application of terms in algebra is the defining of identities. We use identities to classify algebras into collections called varieties. Moreover, the knowledge of the identities valid in algebra could be useful for solving functional equations (see [1]). Not only the concept of identities is important in universal algebra but also the concept of hyperidentities is so. We can also use hyperidentities to classify varieties into collections called hypervarieties. The main tool used to study hyperidentities and hypervarieties is the concept of a hypersubstitution. The notion of a hypersubstitution originated by Denecke, Lau, Pöschel, Schweigert [2]. A hypersubstitution of type τ is a map which takes every n_i-ary operation symbol to an n_i-ary term of the same type. Such mapping can be uniquely extended to a map defined on the set of all terms of the same type, and then any two such hypersubstitutions can be composed in a natural way. They proved that the set of all hypersubstitutions of type τ together with the identity forms a monoid. In 2000, Leeratanavalee and Denecke generalized the concepts of a hypersubstitution and a hyperidentity to the concepts of a generalized hypersubstitution and a strong hyperidentity [4].

The concept of a regular subsemigroup plays an important role in the theory of semigroups. Puninagool and Leeratanavalee determined all regular elements in the monoid of all generalized hypersubstitutions of type $\tau = (n)$ [6]. In 2010, Puninagool and Leeratanavalee studied Green's relations on $Hyp_G(2)$ [5]. In this paper, we study Green's relations on some classes of elements of the monoid of generalized hypersubstitutions of type $\tau = (n)$.

2. PRELIMINARIES

In this section, we recall some basic concepts for the discussion in the next section. Let $\tau = (n_i)_{i \in I}$ be a type. A generalized hypersubstitution of type τ is a function which takes the n_i-ary operation symbol f_i to the term $\sigma(f_i)$ of the same type which does not necessarily preserve the arity. The set of all generalized hypersubstitutions of type τ is denoted by $Hyp_G(\tau)$.

To define a binary operation on the set of all generalized hypersubstitutions of type τ, we have to define the concept of a generalized superposition of terms.

Definition 1 [4]**.** Let $\tau = (n_i)_{i\in I}$ and $t, s_1, \ldots, s_n \in W_\tau(X)$. Then a generalized superposition of terms

$$S^n : W_\tau(X) \times W_\tau(X)^n \to W_\tau(X)$$

is inductively defined by the following steps:

(i) If $t = x_j$ for $1 \leq j \leq n$, then $S^n(x_j, s_1, \ldots, s_n) := s_j$.

(ii) If $t = x_j$ for $n < j$, then $S^n(x_j, s_1, \ldots, s_n) := x_j$.

(iii) If $t = f_i(t_1, \ldots, t_{n_i})$, then $S^n(f_i(t_1, \ldots, t_{n_i}), s_1, \ldots, s_n) := f_i(S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_{n_i}, s_1, \ldots, s_n))$.

Every generalized hypersubstitution σ can be extended to a mapping $\widehat{\sigma} : W_\tau(X) \to W_\tau(X)$ by the following steps.

(i) $\widehat{\sigma}[x] := x \in X$.

(ii) $\widehat{\sigma}[f_i(t_1 \ldots, t_{n_i})] := S^{n_i}(\sigma(f_i), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_i}])$, for any n_i-ary operation symbol f_i and supposed that $\widehat{\sigma}[t_j], 1 \leq j \leq n_i$ are already defined.

Then we define a binary operation $\circ_G$ on $Hyp_G(\tau)$ by $\sigma \circ_G \alpha := \hat{\sigma} \circ \alpha$ where $\circ$ is the usual composition of mappings and $\sigma, \alpha \in Hyp_G(\tau)$. Let σ_{id} be the hypersubstitution which maps each n_i-ary operation symbol f_i to the term $f_i(x_1, \ldots, x_{n_i})$. Leeratanavalee and Denecke proved the following proposition.

Proposition 1 [4]**.** For arbitrary terms $t, t_1, \ldots, t_n \in W_\tau(X)$ and for arbitrary generalized hypersubstitutions σ, α we have

(i) $S^n(\hat{\sigma}[t], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]) = \hat{\sigma}[S^n(t, t_1, \ldots, t_n)]$,

(ii) $(\hat{\sigma} \circ \alpha)^\wedge = \hat{\sigma} \circ \hat{\alpha}$.

By using the previous result, Leeratanavalee and Denecke proved that $\underline{Hyp_G(\tau)} := (Hyp_G(\tau), \circ_G, \sigma_{id})$ is a monoid, for more detail see [4].

2.1. Green's Relations

Let S be a semigroup with a binary operation $\cdot$ and $1 \notin S$. We extend the binary operation from S to $S \cup \{1\}$ by define $1 * 1 = 1, x * 1 = 1 * x = x$ and $x * y = x \cdot y$ for all $x, y \in S$. Then $(S \cup \{1\}, *)$ is a semigroup with the identity 1.

Let S be a semigroup. Then we define,

$$S^1 = \begin{cases} S, & \text{if } S \text{ has an identity} \\ S \cup \{1\}, & \text{otherwise.} \end{cases}$$

Let S be a semigroup and $\varnothing \neq A \subseteq S$. We call A a left (right) ideal of S if $SA \subseteq A(AS \subseteq A)$ and call$A$ an ideal of S if it is both left and right ideal of S.

Let S be a semigroup and $\varnothing \neq A \subseteq S$. We now set

$$(A)_l = \cap \{L|L \text{ is a left ideal of } S \text{ containing } A\},$$
$$(A)_r = \cap \{R|R \text{ is a right ideal of } S \text{ containing } A\},$$
$$(A)_i = \cap \{I|I \text{ is an ideal of } S \text{ containing } A\}.$$

We call (A_l) $((A_r),(A_i))$ the left ideal (right ideal, ideal) of S generated by A.

It is easy to see that

$$\begin{aligned} (A_l) &= S^1A = SA \cup A, \\ (A_r) &= AS^1 = A \cup SA, \\ (A_i) &= S^1AS^1 = SAS \cup SA \cup AS \cup A. \end{aligned}$$

For $a_1, \ldots, a_n \in S$, we write $(a_1, \ldots, a_n)_l$ instead of $(\{a_1, \ldots, a_n\})_l$ and call it a left ideal of S generated by $a_1, \ldots, a_n$. Similarly, we can define $(a_1, \ldots, a_n)_r$ and $(a_1, \ldots, a_n)_i$. If A is a left ideal of S and $A = (a)_l$ for some $a \in S$, we then call A the *principal left ideal generated by a.* We can define the principal right ideal and principal ideal in the same manner.

Let S be a semigroup. We define the relations $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}, \mathcal{J}$ on S as follow:

$$\begin{aligned} a\mathcal{L}b &\Leftrightarrow (a)_l = (b)_l, \\ a\mathcal{R}b &\Leftrightarrow (a)_r = (b)_r, \\ \mathcal{H} &\Leftrightarrow \mathcal{L} \cap \mathcal{R}, \\ \mathcal{D} &\Leftrightarrow \mathcal{L} \circ \mathcal{R}, \\ a\mathcal{L}b &\Leftrightarrow (a)_i = (b)i. \end{aligned}$$

Lemma 1 [3]**.** *Let S be a semigroup. Then for any two element $a, b \in S^1$*

$$\begin{aligned} a\mathcal{L}b &\Leftrightarrow a = xb \text{ and } b = ya \text{ for some } x, y \in S^1, \\ a\mathcal{R}b &\Leftrightarrow a = bx \text{ and } b = ay \text{ for some } x, y \in S^1, \\ a\mathcal{H}b &\Leftrightarrow a\mathcal{L}b \text{ and } a\mathcal{R}b, \\ a\mathcal{D}b &\Leftrightarrow (a,c) \in \mathcal{L} \text{ and } (c,b) \in \mathcal{R} \text{ for some } c \in S^1, \\ a\mathcal{J}b &\Leftrightarrow a = xby \text{ and } b = uav \text{ for some } x, y, u, v \in S^1. \end{aligned}$$

Remark. Let S be a semigroup. Then the following statements hold.

1. $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}$ and $\mathcal{J}$ are equivalence relations.
2. $\mathcal{H} \subseteq \mathcal{R} \subseteq \mathcal{D} \subseteq \mathcal{J}$ and $\mathcal{H} \subseteq \mathcal{L} \subseteq \mathcal{D} \subseteq \mathcal{J}$.

3. GREEN'S RELATIONS ON REGULAR SUBMONOIDS OF THE MONOID OF ALL GENERALIZED HYPERSUBSTITUTION OF TYPE (n)

Green's relations on $Hyp_G(2)$ have been studied by Puninagool and Leeratanavalee [5]. They studied Green's relations on some classes of elements of the monoid of generalized hypersubstitutions of type $\tau = (2)$. In this section, we study Green's relations on the monoid of all regular elements of $Hyp_G(n)$.

For a type $\tau = (2)$ with an n-ary operation symbol f and $t \in W_{(n)}(X)$, we denote:

σ_t:= the generalized hypersubstitution σ of type $\tau = (n)$ which maps f to the term t,

$var(t)$:= the set of all variables occurring in the term t.

Let $\sigma_t \in Hyp_G(n)$, we denote

$R_1 := \{\sigma_{x_i} \mid x_i \in X\}$;

$R_2 := \{\sigma_t \mid t \notin X \text{ and } var(t) \cap X_n = \varnothing\}$;

$R_3 := \{\sigma_t \mid t = f(t_1, \ldots, t_n)$ where $t_{i_1} = x_{j_1}, \ldots, t_{i_m} = x_{j_m}$ for some $i_1, \ldots, i_m \in \{1, \ldots, n\}$ and for distinct $j_1, \ldots, j_m \in \{1, \ldots, n\}$ and $var(t) \cap X_n = \{x_{j_1}, \ldots, x_{j_m}\}\}$.

In 2010, Puninagool and Leeratanavalee showed that $\bigcup_{i=1}^{3} R_i$ is the set of all regular elements in $Hyp_G(n)$.

Definition 2 [7]**.** For a type $\tau = (n)$ with an n-ary operation symbol f, $t \in W_{(n)}(X)$ and $1 \leq i \leq n$, an $i - most(t)$ is defined inductively by:

(i) if t is a variable, then $i - most(t) = t$,

(ii) if $t = f(t_1, \ldots, t_n)$ where $t_1, \ldots, t_n \in W_{(n)}(X)$, then $i - most(t) := i - most(t_i)$.

Example 1. Let $\tau = (3)$ be a type, $t = f(x_2, f(x_5, x_1, x_3), f(x_4, x_7, x_5))$. Consider $1 - most(t) = x_2$, $2 - most(t) = 2 - most(f(x_5, x_1, x_3)) = x_1$ and $3 - most(t) = 3 - most(f(x_4, x_7, x_5)) = x_5$.

Lemma 2 [7]**.** *Let $s, t \in W_{(n)}(X)$. If $j - most(t) = x_k \in X_n$ and $k - most(s) = x_i$, then $j - most(\hat{\sigma}_t[s]) = x_i$.*

Lemma 3 [6]**.** *Let $\sigma_s, \sigma_t \in Hyp_G(n)$. Then the following statements hold*

(i) *$var((\sigma_s \circ_G \sigma_t)(f)) \cap X_n \subseteq var(t) \cap X_n$.*

(ii) *If s uses only one variable , then the term $\sigma_s \circ_G \sigma_t$ uses only one variable.*

Theorem 1. *Every $\sigma_{x_i} \in R_1$ is $\mathcal{L}$-related and $\mathcal{H}$-related only to itself, but is $\mathcal{R}$-related, $\mathcal{D}$-related and $\mathcal{J}$-related to all elements of R_1 and not $\mathcal{R}$-,$\mathcal{L}$-,$\mathcal{H}$-,$\mathcal{D}$- and $\mathcal{J}$-related to any other generalized hypersubstitution.*

Proof. Let $\sigma_{x_i} \in R_1$. Since every element in R_1 is idempotent [6], i.e., $(\sigma_{x_i} \circ_G \sigma_{x_i})(f) = \widehat{\sigma}_{x_i}[x_i] = x_i = \sigma_{x_i}(f)$, we have that every $\sigma_{x_i} \in R_1$ can only be $\mathcal{L}$-related to itself. Since $\sigma_{x_i} \circ_G \sigma_{x_j} = \sigma_{x_j}$ for all σ_{x_i}, $\sigma_{x_j} \in R_1$, we have that any two elements in R_1 are $\mathcal{R}$-related and also $\mathcal{D}$-related and $\mathcal{J}$-related to each other because $\mathcal{R} \subseteq \mathcal{D} \subseteq \mathcal{J}$. Since $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$, we have σ_{x_i} is $\mathcal{H}$-related only to itself for any $\sigma_{x_i} \in R_1$. Moreover, if we asume that $\sigma_{x_i}\mathcal{J}\sigma_t$ where $t \notin X$ then there exist $\sigma_p, \sigma_q, \sigma_u, \sigma_v \in Hyp_G(n)$ such that

$$\sigma_{x_i} = \sigma_p \circ_G \sigma_t \circ_G \sigma_q, \tag{1}$$

$$\sigma_t = \sigma_u \circ_G \sigma_{x_i} \circ_G \sigma_v. \tag{2}$$

Then by generalized superposition of terms, we have $\sigma_u \circ_G \sigma_{x_i} \circ_G \sigma_v \in R_1$ which contradicts to $\sigma_t \notin R_1$. So $\sigma_t \in R_1$. Therefore $\sigma_{x_i} \in R_1$ is $\mathcal{R}$-, $\mathcal{L}$-,$\mathcal{H}$-, $\mathcal{D}$ and $\mathcal{J}$-related to any other generalized hypersubstitution. ■

Theorem 2. *Every $\sigma_t \in R_2$ is $\mathcal{R}$ and $\mathcal{H}$-related only to itself, but is $\mathcal{L}$-related, $\mathcal{D}$-related and $\mathcal{J}$-related to all elements of R_2 and not $\mathcal{R}$-,$\mathcal{L}$-,$\mathcal{H}$-,$\mathcal{D}$- and $\mathcal{J}$-related to any other generalized hypersubstitution.*

Proof. Let $\sigma_t \in R_2$. Since every element in R_2 is idempotent [6], i.e., $(\sigma_t \circ_G \sigma_t)(f) = \widehat{\sigma}_t[t] = t = \sigma_t(f)$, we have every $\sigma_t \in R_2$ can only be $\mathcal{R}$-related to itself. Let $\sigma_s, \sigma_t \in R_2$. Then $\sigma_s \circ_G \sigma_t = \sigma_t$ and $\sigma_t \circ_G \sigma_s = \sigma_s$. So $\sigma_s\mathcal{L}\sigma_t$. Hence every two elements in R_2 are $\mathcal{L}$-related and also $\mathcal{D}$-related and $\mathcal{J}$-related to each other because $\mathcal{L} \subseteq \mathcal{D} \subseteq \mathcal{J}$. Since $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$, σ_t is $\mathcal{H}$-related only to itself for every $\sigma_{x_t} \in R_2$. Next we show that each element of R_2 is not related to any generalized hypersubstitutions. Assume $\sigma_s \in Hyp_G(n), \sigma_t \in R_2$ where $\sigma_s\mathcal{J}\sigma_t$. By Theorem 1, we get $\sigma_s \notin R_1$ then there exist $\sigma_p, \sigma_q, \sigma_u, \sigma_v \in Hyp_G(n)$ such that

$$\sigma_s = \sigma_p \circ_G \sigma_t \circ_G \sigma_q, \tag{3}$$

$$\sigma_t = \sigma_u \circ_G \sigma_s \circ_G \sigma_v. \tag{4}$$

Since $s \notin X$ and by (3), $q \notin X$. Since $\sigma_t \in R_2$ and $q \notin X$, $\sigma_t \circ_G \sigma_q = \sigma_t$. Since $\sigma_t \in R_2$, $x_1, \ldots, x_n$ are not occuring in the term $\sigma_p \circ_G \sigma_t = \sigma_p \circ_G \sigma_t \circ_G \sigma_q = \sigma_s$. Hence $x_1, \ldots, x_n \notin var(s)$. So $\sigma_s \in R_2$. Therefore $\sigma_t \in R_2$ is $\mathcal{R}$-, $\mathcal{L}$-,$\mathcal{H}$-, $\mathcal{D}$- and $\mathcal{J}$-related to any other generalized hypersubstitution. ■

Theorem 3. *Let $\sigma_s, \sigma_t \in R_3$. Then $\sigma_s\mathcal{R}\sigma_t$ if and only if s is the term obtained from t such that if $i - most(t) = x_k \in X_n$, then $i - most(s) = x_{\pi(k)}$ where π is a permutation on $\{1, \ldots, n\}$.*

Proof. Assume that $\sigma_s\mathcal{R}\sigma_t$. Then there exist $\sigma_p, \sigma_q \in Hyp_G(n)$ such that

$$\sigma_t = \sigma_s \circ_G \sigma_p, \tag{5}$$

$$\sigma_s = \sigma_t \circ_G \sigma_q. \tag{6}$$

Next, we prove by contradiction that if $i-most(t) = x_k \in X_n$, then $i-most(s) = x_{\pi(k)}$ where π is a permutation on $\{1,\ldots,n\}$. Assume that $i - most(s) \neq x_{\pi(k)}$.

Case I. $i-most(s) = x_l \in X\setminus X_n$. Then $i-most(\sigma_s \circ_G \sigma_p) = x_l \neq i-most(t)$ which contradicts to (5).

Case II. $i - most(t) = x_{k_1} \in X_n$ and $j - most(t) = x_{k_2} \in X_n$ where $i - most(s) = j-most(s) = x_{\pi(k_1)} \in X_n$. By Lemma 2, we get $j-most(\widehat{\sigma}_s[p]) = x_{k_1}$ which contradicts to (5). So $i - most(s) = x_{\pi(k)}$.

Conversely, assume the condition holds. We will prove that $\sigma_t = \sigma_s \circ_G \sigma_p$ and $\sigma_s = \sigma_t \circ_G \sigma_q$ for some $\sigma_p, \sigma_q \in Hyp_G(n)$. Choose $p = f(p_1,\ldots,p_n)$ where $\pi(k) - most(p) = x_k; \exists\ k \in \{1,\ldots,n\}$.

Case I. $i-most(t) = x_k \in X_n$. Since $i-most(s) = x_{\pi(k)}$ and $\pi(k)-most(p) = x_k$, by Lemma 2 we have, $i - most(t) = i - most(\widehat{\sigma}_s[p]) = x_k$.

Case II. $i - most(t) = x_k \in X \setminus X_n$. It is easy to see that $i - most(t) = i - most(\widehat{\sigma}_s[p])$. For $\sigma_s = \sigma_t \circ_G \sigma_q$, the proof is similar to the previous proof. ■

Theorem 4. *Let $\sigma_s, \sigma_t \in R_3$. Then $\sigma_s \mathcal{L} \sigma_t$ if and only if $var(t) \cap X_n = var(s) \cap X_n$.*

Proof. Assume that $\sigma_s \mathcal{L} \sigma_t$, then there exist $\sigma_p, \sigma_q \in Hyp_G(n)$ such that

$$\sigma_t = \sigma_p \circ_G \sigma_s, \tag{7}$$

$$\sigma_s = \sigma_q \circ_G \sigma_t. \tag{8}$$

By Lemma 3, we get $var(t) \cap X_n \subseteq var(s) \cap X_n$ and $var(s) \cap X_n \subseteq var(t) \cap X_n$. Hence $var(t) \cap X_n = var(s) \cap X_n$. Conversely, assume that the condition holds. Let $t = f(t_1,\ldots,t_n)$ where $t_{i_1} = x_{j_1},\ldots,t_{i_m} = x_{j_m}$ and $var(t) \cap X_n = \{x_{j_1},\ldots,x_{j_m}\}$, $s = f(s_1,\ldots,s_n)$ where $s_{l_1} = x_{j_1},\ldots,s_{l_m} = x_{j_m}$ and $var(s) \cap X_n = \{x_{j_1},\ldots,x_{j_m}\}$. We will prove that $\sigma_t = \sigma_p \circ_G \sigma_s$ and $\sigma_s = \sigma_q \circ_G \sigma_t; \exists \sigma_p, \sigma_q \in Hyp_G(n)$. Choose $p = f(p_1,\ldots,p_n)$ where p has the same pattern as the term t and $p_{i_1} = x_{l_1},\ldots,p_{i_m} = x_{l_m}$.

Case I. $i_k - most(t) = x_{j_k} \in X_n; \exists\ j_k \in \{j_1,\ldots,j_m\}$. Since $i_k - most(p) = x_{l_k}$ and $l_k - most(s) = x_{j_k}$, by Lemma 2 we have, $i_k - most(t) = i_k - most(\widehat{\sigma}_p[s]) = x_{j_k}$.

Case II. $f \in J \setminus \{j_1,\ldots,j_m\}$. Let $p_f = f(u_1,\ldots,u_n)$. If $i - most(p) = x_{l_k}$, then $i - most(t) = i - most(\widehat{\sigma}_p[s]) = x_{j_k}$. If $i - most(p) = x_f \in X \setminus X_n$, then $i - most(t) = i - most(\widehat{\sigma}_p[s]) = x_f$.

For $\sigma_s = \sigma_q \circ_G \sigma_t$, the proof is similar to the previous proof. ■

Theorem 5. *Let $\sigma_s, \sigma_t \in R_3$. Then $\sigma_s \mathcal{J} \sigma_t$ if and only if $|var(t) \cap X_n| = |var(s) \cap X_n|$.*

Proof. Let $\sigma_s, \sigma_t \in R_3$ where $s = f(s_1, \ldots, s_n)$ and $t = f(t_1, \ldots, t_n)$. Assume that $\sigma_s \mathcal{J} \sigma_t$, then there exist $\sigma_p, \sigma_q, \sigma_u, \sigma_v \in Hyp_G(n)$ where $p = f(p_1, \ldots, p_n)$ and $q = f(q_1, \ldots, q_n)$ such that

$$\sigma_t = \sigma_p \circ_G \sigma_s \circ_G \sigma_q, \tag{9}$$

$$\sigma_s = \sigma_u \circ_G \sigma_t \circ_G \sigma_v. \tag{10}$$

We prove by contradiction that $|var(t) \cap X_n| = |var(s) \cap X_n|$. Assume that $|var(t) \cap X_n| > |var(s) \cap X_n|$ such that $|var(t) \cap X_n| = c$ and $|var(s) \cap X_n| = b$ where $b < c$. Let $f(a_1, \ldots, a_n)$ be the term obtained from $\sigma_s \circ_G \sigma_q$. Since $|var(s) \cap X_n| = b$, $\sigma_s \circ_G \sigma_q(f) = f(a_1, \ldots, a_n)$ and $|var(f(a_1, \ldots, a_n)) \cap X_n| \leq b$. From (9), we get $S^n(f(p_1, \ldots, p_n), \widehat{\sigma}[a_1], \ldots, \widehat{\sigma}[a_n]) = f(t_1, \ldots, t_n)$. Since $c > b$ and $\sigma_s, \sigma_t \in R_3$, then $t_c = x_{k_c}$. So $p_c \in X$. If $p_c \notin X_n$, then $S^n(f(p_1, \ldots, p_n), \widehat{\sigma}[a_1], \ldots, \widehat{\sigma}[a_n]) \neq t$ which contradicts to (9). If $p_c \notin X_n \setminus \{x_{k_1}, \ldots, x_{k_b}\}$, then by $\sigma_t \in R_3$ we have $S^n(f(p_1, \ldots, p_n), \widehat{\sigma}[a_1], \ldots, \widehat{\sigma}[a_n]) \neq t$. If $p_c \in \{x_{k_1}, \ldots, x_{k_b}\}$, we get $p_c = x_{k_1}; i \in \{1, \ldots, b\}$ which contradicts to (9). Hence the number of distinct variables in X_n which occur in s and t are equal. So $|var(t) \cap X_n| = |var(s) \cap X_n|$. Conversely, assume that the condition holds. Let $t = f(t_1, \ldots, t_n)$ where $t_{i_1} = x_{j_1}, \ldots, t_{i_m} = x_{j_m}$ and $var(t) \cap X_n = \{x_{j_1}, \ldots, x_{j_m}\}$, $s = f(s_1, \ldots, s_n)$ where $s_{l_1} = x_{f_1}, \ldots, s_{f_m} = x_{j_m}$ and $var(s) \cap X_n = \{x_{f_1}, \ldots, x_{f_m}\}$. We will prove that $\sigma_t = \sigma_p \circ_G \sigma_s \circ_G \sigma_q$ and $\sigma_s = \sigma_u \circ_G \sigma_t \circ_G \sigma_v; \forall \sigma_p, \sigma_q, \sigma_u, \sigma_v \in Hyp_G(n)$. Choose $p = f(p_1, \ldots, p_n)$ with the same pattern as term t and $p_{i_1} = x_{l_1}, \ldots, p_{i_m} = x_{l_m}$ and $q = f(q_1, \ldots, q_n)$ where $q_{f_1} = x_{j_1}, \ldots, q_{f_m} = x_{j_m}$.

Case I. $i_k - most(t) = x_{j_k}; i_k \in \{i_1, \ldots, i_m\}$. Since $l_k - most(s) = x_{f_k}$ and $f_k - most(q) = x_{j_k}$, $l_k - most(\widehat{\sigma}_s[q]) = x_{j_k}$. Since $i_k - most(p) = x_{l_k}$ and $l_k - most(\widehat{\sigma}_s[q]) = x_{j_k}$, $i_k - most(t) = i_k - most(\widehat{\sigma}_p[\widehat{\sigma}_s[q]) = x_{j_k}$.

Case II. $f \in J \setminus \{j_1, \ldots, j_m\}$. Let $p_f = f(u_1, \ldots, u_n)$. If $i - most(p) = x_{l_k}$, then we can prove similarly to Case I that $i - most(t) = i - most(\widehat{\sigma}_p[\widehat{\sigma}_s[q]) = x_{j_k}$. If $i - most(p) = x_g \in X \setminus X_n$, then $i - most(t) = i - most(\widehat{\sigma}_p[\widehat{\sigma}_s[q]) = x_g$.

For $\sigma_s = \sigma_u \circ_G \sigma_t \circ_G \sigma_v$, the proof is similar to the previous proof. ∎

Theorem 6. *Let $\sigma_t, \sigma_s \in R_3$. Then $\sigma_s \mathcal{J} \sigma_t$ if and only if $\sigma_s \mathcal{D} \sigma_t$.*

Proof. Let $\sigma_t, \sigma_s \in R_3$. Then by Theorem 5, $\sigma_s \mathcal{J} \sigma_t$ if and only if $|var(t) \cap X_n| = |var(s) \cap X_n|$. Let $t = f(t_1, \ldots, t_n)$ where $t_{i_1} = x_{j_1}, \ldots, t_{i_m} = x_{j_m}$ and $var(t) \cap X_n = \{x_{j_1}, \ldots, x_{j_m}\}$, $s = f(s_1, \ldots, s_n)$ where $s_{l_1} = x_{k_1}, \ldots, s_{l_m} = x_{k_m}$ and $var(s) \cap X_n = \{x_{k_1}, \ldots, x_{k_m}\}$. Choose $r = f(r_1, \ldots, r_n)$ where $\sigma_r = \sigma_s \circ_G \sigma_{f(x_{\pi(k_1)}, \ldots, x_{\pi(k_n)})}$ such that $\pi = \begin{pmatrix} k_1 & \cdots & k_m & k_{m+1} & k_n \\ j_1 & \cdots & j_m & k_{m+1} & k_n \end{pmatrix}$. We will prove that $\sigma_r \mathcal{R} \sigma_s$, i.e., there exist $\sigma_p, \sigma_q \in Hyp_G(n)$ such that $\sigma_s = \sigma_r \circ_G \sigma_p$ and $\sigma_r = \sigma_s \circ_G \sigma_q$.

Choose $p = f(x_{\pi^{-1}(k_1)}, \ldots, x_{\pi^{-1}(k_n)})$ and $q = f(x_{\pi(k_1)}, \ldots, x_{\pi(k_n)})$. So $\sigma_r \circ_G \sigma_p = \sigma_s \circ_G \sigma_{f(x_{\pi(k_1)},\ldots,x_{\pi(k_n)})} \circ_G \sigma_{f(x_{\pi^{-1}(k_1)},\ldots,x_{\pi^{-1}(k_n)})} = \sigma_s \circ_G \sigma_{id} = \sigma_s$ and $\sigma_s \circ_G \sigma_q = \sigma_s \circ_G \sigma_{f(x_{\pi(k_1)},\ldots,x_{\pi(k_n)})} = \sigma_r$. Next we will prove that $\sigma_r \mathcal{L} \sigma_t$. Since $\sigma_r = \sigma_s \circ_G \sigma_{f(x_{\pi(k_1)},\ldots,x_{\pi(k_n)})}$, $var(t) \cap X_n = var(r) \cap X_n$. By Theorem 4, we get $\sigma_r \mathcal{L} \sigma_t$. Hence by Lemma 1 we get, $\sigma_s \mathcal{D} \sigma_t$. ■

Theorem 7. *Every $\sigma_t \in R_3$ is $\mathcal{H}$-related only to itself.*

Proof. Assume $\sigma_s \mathcal{H} \sigma_t$ and $t \neq s$. Then there exist $\sigma_p, \sigma_q, \sigma_u, \sigma_v \in Hyp_G(n)$ such that

$$\sigma_t = \sigma_s \circ_G \sigma_p, \tag{11}$$

$$\sigma_s = \sigma_t \circ_G \sigma_q, \tag{12}$$

$$\sigma_t = \sigma_u \circ_G \sigma_s, \tag{13}$$

$$\sigma_s = \sigma_v \circ_G \sigma_t. \tag{14}$$

By Theorem 3, there exist $\sigma_p, \sigma_q \in Hyp_G(n)$ such that $\sigma_t = \sigma_s \circ_G \sigma_p$ and $\sigma_s = \sigma_t \circ_G \sigma_q$ where $\sigma_s = \sigma_t \circ_G \sigma_{f(x_{\pi(1)},\ldots,x_{\pi(n)})}$, for some permutation π. Since $\sigma_s = \sigma_t \circ_G \sigma_{f(x_{\pi(1)},\ldots,x_{\pi(n)})}$ such that $\pi \neq (1)$, $var(t) \cap X_n \neq var(s) \cap X_n$. By Theorem 4, σ_s is not $\mathcal{L}$-related with σ_t which contradicts to (13), (14). By Theorem 4 again, there exist $\sigma_u, \sigma_v \in Hyp_G(n)$ such that $\sigma_t = \sigma_u \circ_G \sigma_s$ and $\sigma_s = \sigma_v \circ_G \sigma_t$ where $var(t) \cap X_n = var(s) \cap X_n$. Let $t = f(t_1, \ldots, t_n)$ where $t_{i_1} = x_{j_1}, \ldots, t_{i_m} = x_{j_m}$ and $s = f(s_1, \ldots, s_n)$ where $s_{l_1} = x_{j_1}, \ldots, s_{l_m} = x_{j_m}$. Moreover $\sigma_s \neq \sigma_t \circ_G \sigma_{f(x_{\pi(1)},\ldots,x_{\pi(n)})}$. By Theorem 3, σ_t is not $\mathcal{R}$-relate with σ_s which contradicts to (11), (12). So $t = s$. ■

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