Discussiones Mathematicae
General Algebra and Applications 41 (2021) 265–281
https://doi.org/10.7151/dmgaa.1365

QUASI-COMPLEMENTED BE-ALGEBRAS

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Abstract

The concept of O-filters is introduced in commutative BE-algebras. An equivalent condition is derived for every strong regular filter of a BE-algebra to become an O-filter. The concept of quasi-complemented BE-algebras is introduced and also characterized these classes of BE-algebras in terms of dual annihilators. The concept of strong regular filter is introduced and then quasi-complemented BE-algebras and strong BE-algebras are characterized in terms of strong regular filters and O-filters.

Keywords: commutative BE-algebra, O-filter, quasi-complemented BE-algebra, strong BE-algebra, strong regular filter.

2010 Mathematics Subject Classification: 03G25.

1. Introduction

The notion of BE-algebras was introduced and extensively studied by H.S. Kim and Y.H. Kim in [7]. These classes of BE-algebras were introduced as a generalization of the class of BCK-algebras of Iseki and Tanaka [6]. Some properties of filters of BE-algebras were studied by Ahn and Kim in [1] and by Meng in [8]. In [14], Walendziak discussed some properties of commutative BE-algebras. He also investigated the relationship between BE-algebras, implicative algebras and J-algebras. In 2012, Rezaei, and Borumand Saeid [9], stated and proved the first, second and third isomorphism theorems in self distributive BE-algebras. Later, these authors [10] introduced the notion of commutative ideals in a BE-algebra. In 2013, Borumand Saeid, Rezaei and Borzooei [2] extensively studied the properties of some types of filters in BE-algebras. In [3], Chajda *et al.*, Characterized the complements and relative complements of the set of all deductive systems as the so-called annihilators of Hilbert algebras. Later, Halaš[5] introduced the concepts of an annihilator and a relative annihilator of a given subset of a BCK-algebra. Cornish [4] introduced the concept of quasi-complements in distributive lattices. In [12], some properties of dual annihilator filters of commutative BE-algebras are studied. It is proved that the class of all dual annihilator filters of a BE-algebra is a complete Boolean algebra. A set of equivalent conditions is derived for every prime filter of a commutative BE-algebra to become a maximal filter.

In this paper, the notion of O-filters is introduced in commutative BE-algebras. Some properties of prime O-filters are studied. A relation between the O-filters and minimal prime filters of a commutative BE-algebra is observed. An equivalent condition is derived for every strong regular filter of a BE-algebra to become an O-filter. The notion of quasi-complemented BE-algebras is introduced and also characterized these classes of BE-algebras in terms of dual annihilators. The concept of strong regular filter is introduced and then quasi-complemented BE-algebras and strong BE-algebras are characterized in terms of strong regular filters and O-filters.

2. Preliminaries

In this section, we present certain definitions and results which are taken mostly from the papers [1, 7, 8, 11, 12], and [13] for the ready reference of the reader.

Definition [7]**.** An algebra $(X, *, 1)$ of type $(2, 0)$ is called a BE-algebra if it satisfies the following properties:

(1) $x * x = 1$,

(2) $x * 1 = 1$,

(3) $1 * x = x$,

(4) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$.

A BE-algebra X is called self-distributive if $x * (y * z) = (x * y) * (x * z)$ for all $x, y, z \in X$. A BE-algebra X is called commutative if $(x * y) * y = (y * x) * x$ for all $x, y \in X$. Every commutative BE-algebra is transitive. For any $x, y \in X$, define $x \vee y = (y * x) * x$. If X is commutative then $(X, \vee)$ is a semilattice [14]. We introduce a relation $\leq$ on a BE-algebra X by $x \leq y$ if and only if $x * y = 1$ for all $x, y \in X$. Clearly $\leq$ is reflexive. If X is commutative, then $\leq$ is transitive, anti-symmetric and hence a partial order on X.

Theorem 1 [8]. *Let X be a transitive BE-algebra and $x, y, z \in X$. Then*

(1) $1 \leq x$ *implies* $x = 1$,

(2) $y \leq z$ *implies* $x * y \leq x * z$ *and* $z * x \leq y * x$.

Definition [1]. A non-empty subset F of a BE-algebra X is called a filter of X if, for all $x, y \in X$, it satisfies the following properties:

(1) $1 \in F$,

(2) $x \in F$ and $x * y \in F$ imply that $y \in F$.

Let $\mathcal{F}(X)$ denote the set of all filters of a self-distributive BE-algebra X, then $\mathcal{F}(X)$ forms a complete lattice with respect to the operations $F \wedge G = F \cap G$ and $F \vee G = \{x \in X \mid a * (b * x) = 1 \text{ for some } a \in F, b \in G\}$. For any non-empty subset A of a transitive BE-algebra X, the set $\langle A \rangle = \{x \in X \mid a_1 * (a_2 * (\cdots * (a_n * x) \cdots)) = 1 \text{ for some } a_1, a_2, \ldots a_n \in A\}$ is the smallest filter containing A. For any $a \in X, \langle a \rangle = \{x \in X \mid a^n * x = 1 \text{ for some } n \in \mathbb{N}\}$, where $a^n * x = a * (a * (\cdots * (a * x) \cdots))$ with the repetition of a is n times, is called the *principal filter generated by* a. If X is self-distributive, then $\langle a \rangle = \{x \in X \mid a * x = 1\}$. A proper filter P of a BE-algebra is called prime [11] if $F \cap G \subseteq P$ then $F \subseteq P$ or $G \subseteq P$ for any two filters F and G of X.

Theorem 2 [11]. *Let X be a self-distributive and commutative BE-algebra and P be a proper filter of X. Then the following conditions are equivalent:*

(1) *P is prime;*

(2) *For any $x, y \in X, \langle x \rangle \cap \langle y \rangle \subseteq P$ implies $x \in P$ or $y \in P$;*

(3) *For any $x, y \in X, x \vee y \in P$ implies $x \in P$ or $y \in P$.*

For any non-empty subset A of a commutative BE-algebra X, the dual annihilator [12] of A is defined as $A^+ = \{x \in X \mid x \vee a = 1 \text{ for all } a \in A\}$. Clearly A^+ is a filter of X. Obviously $X^+ = \{1\}$ and $\{1\}^+ = X$. For $A = \{a\}$, we simply denote $\{a\}^+$ by $(a)^+$.

Proposition 3 [12]. *For any two filters F, G of a commutative BE-algebra X, we have*

(1) $F \cap F^+ = \emptyset$,

(2) $F \subseteq F^{++}$,

(3) $F^{+++} = F^+$,

(4) $F \subseteq G$ *implies* $G^+ \subseteq F^+$,

(5) $(F \vee G)^+ = F^+ \cap G^+$,

(6) $(F \cap G)^{++} = F^{++} \cap G^{++}$.

Lemma 4 [12]. *For any two elements a, b of a commutative BE-algebra X, we have*

(1) $(\langle a \rangle)^+ = (a)^+$,

(2) $\langle a \rangle \subseteq (a)^{++}$,

(3) $a \leq b$ *implies* $(a)^+ \subseteq (b)^+$,

(4) $(a \vee b)^{++} = (a)^{++} \cap (b)^{++}$.

Definition [12]. A filter F of a commutative BE-algebra X is called a *dual annihilator filter* if $F = F^{++}$.

Proposition 5 [13]. *For any prime filter P of a commutative BE-algebra X, the set $O(P) = \{x \in X \mid x \vee s = 1$ for some $s \notin P\}$ is a filter of X.*

A prime filter P of a commutative BE-algebra X is called *minimal* [13] if there exists no prime filter Q of X such that $Q \subseteq P$.

Theorem 6 [13]. *A prime filter P of a self-distributive and commutative BE-algebra X is minimal if and only if for each $x \in P$ there exists $y \notin P$ such that $x \vee y = 1$.*

Definition [13]. A filter F of a commutative BE-algebra X is called a *regular filter* if $(x)^{++} \subseteq F$ for all $x \in F$.

Theorem 7 [13]. *Let X be a self-distributive and commutative BE-algebra X. Then the following conditions are equivalent:*

(1) *every filter is a regular filter;*

(2) *every principal filter is a regular filter;*

(3) *every prime filter is a regular filter;*

(4) *for $a, b \in X, (a)^+ = (b)^+$ implies $\langle a \rangle = \langle b \rangle$.*

Every minimal prime filter of a commutative BE-algebra is a regular filter and every dual annihilator filter of a commutative BE-algebra is a regular filter.

3. O-FILTERS OF BE-ALGEBRAS

In this section, the notion of O-filters is introduced in commutative BE-algebras. Some properties of prime O-filters are studied. A relation between the O-filters and minimal prime filters of a commutative BE-algebra is observed.

Definition. Let X be a commutative BE-algebra. A subset A of X is called a $\vee$-closed subset of X if $a, b \in A$ implies $a \vee b \in A$.

Example 8. Let $X = \{1, a, b, c\}$ be a set. Define a binary operation $*$ on X as follows:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	c
c	1	a	b	1

$\vee$	1	a	b	c
1	1	1	1	1
a	1	a	1	1
b	1	1	b	1
c	1	1	1	c

Then clearly $(X, *, \vee, 1)$ is a commutative BE-algebra. Consider the subset $A = \{1, a, b\}$ of X. It can be easily verified that A is a $\vee$-closed subset of X.

Clearly $\{1\}$ is a $\vee$-closed subset of a commutative BE-algebra X and in fact every single-ton subset of a commutative BE-algebra is $\vee$-closed. It can be easily observed the class of all $\vee$-closed subsets of a commutative BE-algebra is closed under set-intersection. For any commutative BE-algebra X, in what follows, $\mathcal{J}(X)$ denote the set of all $\vee$-closed subsets of a commutative BE-algebra unless otherwise mentioned.

Proposition 9. *Let X and Y be two commutative BE-algebras and $f : X \to Y$ be a homomorphism. If $S \in \mathcal{J}(Y)$, then $f^{-1}(S) \in \mathcal{J}(X)$.*

Proposition 10. *Let X and Y be two commutative BE-algebras. If R and S are subsets of X and Y respectively such that $R \in \mathcal{J}(X)$ and $S \in \mathcal{J}(Y)$, then $R \times S \in \mathcal{J}(X \times Y)$.*

Proposition 11. *If P is a prime filter of a self-distributive and commutative BE-algebra X, then $X - P \in \mathcal{J}(X)$.*

Proof. Let P be a prime filter of X. Suppose $a, b \in X - P$. Then $a \notin P$ and $b \notin P$. Since P is prime, we get $a \vee b \notin P$. Hence $a \vee b \in X - P$. Therefore $X - P \in \mathcal{J}(X)$. ■

Definition. An element x of a BE-algebra X is called *dual-dense* if $(x)^+ = \{1\}$.

Example 12. Let $X = \{1, a, b, c\}$ be a set. Define a binary operation $*$ on X as follows:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	b	1	1
c	1	c	c	1

$\vee$	1	a	b	c
1	1	1	1	1
a	1	a	b	c
b	1	b	b	c
c	1	c	c	c

Then $(X, *, \vee, 1)$ is a commutative BE-algebra. Clearly $(c)^+ = \{1\}$.

Let us denote the class of all dual-dense elements of a BE-algebra X by $\bar{D}_X$.

Proposition 13. *If X is a self-distributive and commutative BE-algebra, then $\bar{D}_X \in \mathcal{J}(X)$.*

Proof. Let $a, b \in \bar{D}_X$. Then $(a)^+ = \{1\}$ and $(b)^+ = \{1\}$. By Lemma 4, we get $(a \vee b)^{++} = (a)^{++} \cap (b)^{++} = \{1\}^+ \cap \{1\}^+ = X \cap X = X$. Hence $(a \vee b)^+ = (a \vee b)^{+++} = X^+ = \{1\}$. Thus $a \vee b \in \bar{D}_X$. Therefore $\bar{D}_X$ is a $\vee$-closed subset of X. ■

Definition. Let X be a commutative BE-algebra. For any $A \in \mathcal{J}(X)$, define the set $O(A)$ as $O(A) = \{x \in X \mid x \vee a = 1 \text{ for some } a \in A\}$.

Proposition 14. *Let X be a commutative BE-algebra such that $\vee$ is right distributive over $*$. Then for any $A \in \mathcal{J}(X)$, the set $O(A)$ is a filter of X.*

Lemma 15. *The following conditions hold in a commutative BE-algebra X:*

(a) *for $A \in \mathcal{J}(X)$, $O(A) = \bigcup_{x \in A} (x)^+$,*
(b) *for $x \in X$, $O(\{x\}) = (x)^+$,*
(c) *for $A \in \mathcal{J}(X)$, $A \cap O(A) \neq \emptyset$ implies $1 \in A$ and $O(A) = X$,*
(d) *for $A, B \in \mathcal{J}(X)$, $A \subseteq B$ implies $O(A) \subseteq O(B)$,*
(e) *for $A, B \in \mathcal{J}(X)$, $O(A \cap B) \subseteq O(A) \cap O(B)$,*
(f) *for $F, G \in \mathcal{F}(X)$, $O(F \cap G) = O(F) \cap O(G)$ where $\mathcal{F}(X)$ is the set of all filters of X.*

Proof. (a) and (b) are obvious.

(c) Suppose $A \cap O(A) \neq \emptyset$. Choose $x \in A \cap O(A)$. Then $x \in A$ and $x \vee a = 1$ for some $a \in A$. Hence it yields $1 = x \vee x = x \in A$. Since $(1)^+ = X$, we get $O(A) = \bigcup_{x \in A} (x)^+ = X$.

(d) Suppose $A \subseteq B$. Let $x \in O(A)$. Then $x \vee a = 1$ for some $a \in A \subseteq B$. Hence it concludes that $x \in O(B)$. Therefore we get $O(A) \subseteq O(B)$.

(e) Since $A \cap B \subseteq A, B$, we get $O(A \cap B) \subseteq O(A), O(B)$ and hence $O(A \cap B) \subseteq O(A) \cap O(B)$.

(f) Since $F \cap G \subseteq F, G$, we get $O(F \cap G) \subseteq O(F) \cap O(G)$. Conversely, let $x \in O(F) \cap O(G)$. Then $x \vee a = 1$ and $x \vee b = 1$ for some $a \in F$ and $b \in G$. Clearly $a \vee b \in F \cap G$. Hence $x \vee (a \vee b) = (x \vee a) \vee (x \vee b) = 1$. Thus $x \in O(F \cap G)$. Therefore $O(F) \cap O(G) \subseteq O(F \cap G)$. ■

Since $O(A)$ is a filter for all $A \in \mathcal{J}(X)$, the following is clear.

Proposition 16. *Let $A, B \in \mathcal{J}(X)$ be such that $A \cap O(B) = \emptyset$, then there exists a prime filter P containing $O(B)$ and disjoint from A.*

The concept of *O-filters* is introduced in commutative BE-algebras.

Definition. A filter F of a commutative BE-algebra X is called an O-filter if $F = O(A)$ for some $A \in \mathcal{J}(X)$.

Example 17. Let $X = \{1, a, b, c, d\}$ and define a binary operation $*$ on X as follows:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	b
b	1	a	1	b	a
c	1	a	1	1	a
d	1	1	1	b	1

$\vee$	1	a	b	c	d
1	1	1	1	1	1
a	1	a	1	1	a
b	1	1	b	d	b
c	1	1	d	c	b
d	1	a	b	b	d

Clearly $(X, *, \vee, 1)$ is a commutative BE-algebra. It is easy to check that $F = \{b, c, 1\}$ is a filter of X. Now, consider $A = \{a, d\}$. Clearly A is $\vee$-closed and observe that $O(A) = F$. Therefore F is an O-filter of X. It can also be observed that $A \cap O(A) = \emptyset$.

Since $O(X) = X$, it is clear that X is an O-filter. Since $\{x\} \in \mathcal{J}(X)$, from Lemma 15(b), it is observed that each $(x)^+$ is an O-filter. It is still to be observed that the filter $\{1\}$ is an O-filter or not. However, in the following theorem, a necessary and sufficient condition is derived for $\{1\}$ of a commutative BE-algebra to become an O-filter.

Theorem 18. *Let X be a self-distributive and commutative BE-algebra. Then the smallest filter $\{1\}$ of X is an O-filter of X if and only if X has a dual-dense element.*

Proof. Assume that $\{1\}$ is an O-filter of X. Then $\{1\} = O(A)$ for some $A \in \mathcal{J}(X)$. Hence $\{1\} = \bigcup_{x \in A} (x)^+$, which implies that $(x)^+ = \{1\}$ for all $x \in A$. Thus, it yields $\emptyset \neq A = \bar{D}_X$. Conversely, assume that X has an element x of the form $(x)^+ = \{1\}$. Then $\bar{D}_X \neq \emptyset$. By Proposition 13, we get $\bar{D}_X \in \mathcal{J}(X)$. Thus $O(\bar{D}_X) = \bigcup_{x \in \bar{D}_X} (x)^+ = \{1\}$. Therefore $\{1\}$ is an O-filter of X. ■

Proposition 19. *Let X be a commutative BE-algebra. A proper O-filter of X contains no dual-dense element.*

Proof. Let F be an O-filter and $d \in X$ such that $(d)^+ = \{1\}$. Since F is an O-filter, we get that $F = O(A)$ for some $A \in \mathcal{J}(X)$. Suppose $d \in F = O(A)$. Then $d \vee a = 1$ for some $1 \neq a \in A$. Thus $a \in (d)^+$, which is a contradiction to that $(d)^+ = \{1\}$. ■

Proposition 20. *Every O-filter of a commutative BE-algebra is a regular filter.*

Proof. Let F be an O-filter of a commutative BE-algebra X. Then $F = O(A)$ for some $A \in \mathcal{J}(X)$. Let $x \in F = O(A)$. Then we get $s \vee x = 1$ for some $s \in A$. Hence $x \in (s)^+$. Thus it yields $(x)^{++} \subseteq (s)^+$. Suppose $t \in (x)^{++} \subseteq (s)^+$. Then, we get $s \vee t = 1$ and $s \in A$. Hence $t \in O(A) = F$. Thus $(x)^{++} \subseteq F$. Therefore F is a regular filter of X. ■

Theorem 21. *Let X be a self-distributive and commutative BE-algebra. Then every minimal prime filter of X is an O-filter.*

Proof. Let P be a minimal prime filter of X. Let $x \in P$. Since P is minimal, by Theorem 6, there exists $y \in X - P$ such that $x \vee y = 1$. Since $X - P \in \mathcal{J}(X)$, we get $x \in O(X - P)$. Therefore $P \subseteq O(X - P)$. Conversely, let $x \in O(X - P)$. Then $x \vee a = 1 \in P$ for some $a \in X - P$. Since P is prime and $a \notin P$, we must have $x \in P$. Hence $P = O(X - P)$. Therefore P is an O-filter of X. ■

We now turn our intension towards the converse of above theorem. In general, every O-filter of a commutative BE-algebra need not be a minimal prime filter. In fact it need not even be a prime filter. Though every O-filter need not be a prime filter, we derive a necessary and sufficient condition for an O-filter of a commutative BE-algebra to become a prime filter of X.

Theorem 22. *Let F be a proper O-filter of a self-distributive and commutative BE-algebra X. Then F is prime if and only if F contains a prime filter.*

Proof. The necessary part is clear. For sufficiency, assume that F contains a prime filter, say P. Since F is an O-filter, we get $F = O(A)$ for some $A \in \mathcal{J}(X)$. Choose $a, b \in X$ such that $a \notin F$ and $b \notin F$. Then $a \notin P$ and $b \notin P$. Since P is prime, we get $a \vee b \notin P$. Thus $(a \vee b)^+ \subseteq P \subseteq F = O(A)$. Suppose $a \vee b \in F = O(A)$. Then $a \vee b \vee t = 1$ for some $t \in A$. Hence $t \in (a \vee b)^+ \subseteq O(A)$. Thus $t \in A \cap O(A)$. Therefore $A \cap O(A) \neq \emptyset$. By Lemma 15(c), we get $1 \in A$ and $F = O(A) = X$, which is a contradiction. Therefore F is a prime filter of X. ■

It is observed in Theorem 21 that every minimal prime filter of a commutative BE-algebra is a prime O-filter. Now, in the following theorem, the equivalency between prime O-filters and minimal prime filters of BE-algebras is derived.

Theorem 23. *Every prime O-filter of a commutative BE-algebra is a minimal prime filter.*

Proof. Let P be a prime O-filter of a commutative BE-algebra X. Then $P = O(A)$ for some $A \in \mathcal{J}(X)$. Let $x \in P = O(A)$. Then $x \vee y = 1$ for some $y \in A$. Suppose $y \in P$. Then $y \in A \cap P = A \cap O(A)$. Hence $A \cap O(A) \neq \emptyset$. By Lemma 15(c), we get $P = O(A) = X$, which is a contradiction. Hence $y \notin P$. Therefore P is minimal. ∎

Proposition 24. *Every proper O-filter of a commutative BE-algebra is contained in a minimal prime filter.*

Proof. Let F be a proper O-filter of a commutative BE-algebra X. Then $F = O(A)$ for some $A \in \mathcal{J}(X)$. Clearly $F \cap A = O(A) \cap A = \emptyset$. Otherwise $F = O(A) = X$, which is a contradiction. Let $\Im = \{B \in \mathcal{J}(X) \mid A \subseteq B \text{ and } F \cap B = \emptyset\}$. Clearly $A \in \Im$ and $\Im$ satisfies the Zorn's lemma. Let M be a maximal element of $\Im$. Hence $M \in \mathcal{J}(X)$ is maximal with respect to the properties $A \subseteq M$ and $F \cap M = \emptyset$. Since $F \cap M = \emptyset$, we get $F \subseteq X - M$. We claim that $X - M$ is a minimal prime filter of X. Suppose P is a prime filter of X such that $P \subset X - M$. Then $X - P \in \mathcal{J}(X)$ with $A \subseteq M \subset X - P$. By the maximality of M, we get $F \cap (X - P) \neq \emptyset$. Choose $x \in F \cap (X - P)$. Then $x \in F = O(A)$ and $x \notin P$. Hence $x \vee a = 1$ for some $a \in A \subset X - P$. Thus $x \vee a = 1 \in P$, which is a contradiction to $a \notin P$ and $x \notin P$. Therefore $X - M$ is a minimal prime filter such that $F \subseteq X - M$. ∎

4. Quasi-complemented BE-algebras

In this section, the concept of quasi-complemented BE-algebras is introduced and also characterized these classes of BE-algebras in terms of dual annihilators. The concept of strong regular filters is introduced and then quasi-complemented BE-algebras and strong BE-algebras are characterized in terms of strong regular filters and O-filters.

Definition. A commutative BE-algebra X is called a *quasi-complemented BE-algebra* if for each $x \in X$, there exists $y \in X$ such that $x \vee y = 1$ and $(x)^+ \cap (y)^+ = \{1\}$.

In this case, the element y is called the quasi-complement of x and vice versa.

Example 25. Let $X = \{1, a, b, c\}$ be a set. Define a binary operation $*$ on X as follows:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	b
b	1	a	1	a
c	1	1	1	1

$\vee$	1	a	b	c
1	1	1	1	1
a	1	a	1	a
b	1	1	b	b
c	1	a	b	c

Then $(X, *, \vee, 1)$ is a commutative BE-algebra. Clearly 1 and c are quasi-complements are each other. Also a and b are quasi-complements are each other. Hence X is a quasi-complemented BE-algebra.

Proposition 26. *A quasi-complemented BE-algebra possesses a dual-dense element.*

Proof. Let X be a quasi-complemented BE-algebra. Let $1 \neq x \in X$. Since X is quasi-complemented, there exists $y \in X$ such that $x \vee y = 1$ and $(x)^+ \cap (y)^+ = \{1\}$. Since $x \vee y = 1$, we get $y \in (x)^+$. Hence $(y)^+ \subseteq (x)^+$. Thus $(y)^+ = (x)^+ \cap (y)^+ = \{1\}$. Therefore y is a dual-dense element. ■

In the following theorem, quasi-complemented BE-algebras are characterized with the help of dual annihilators.

Theorem 27. *A commutative BE-algebra X is quasi-complemented if and only if for each $x \in X$, there exists $y \in X$ such that $(x)^{++} = (y)^+$.*

Proof. Assume that X is quasi-complemented. Let $x \in X$. Then there exists $y \in X$ such that $x \vee y = 1$ and $(x)^+ \cap (y)^+ = \{1\}$. Since $x \vee y = 1$, we get $y \in (x)^+$ and hence $(x)^{++} \subseteq (y)^+$. Again, since $(x)^+ \cap (y)^+ = \{1\}$, we get $(y)^+ \subseteq (x)^{++}$. Hence $(x)^{++} = (y)^+$.

Conversely, assume the condition. Let $x \in X$. Then there exists $y \in X$ such that $(x)^{++} = (y)^+$. Hence $x \in (x)^{++} = (y)^+$. Thus $x \vee y = 1$. For $x, y \in X$, we have $(x)^+ \cap (y)^+ = (x)^+ \cap (x)^{++} = \{1\}$. Therefore X is a quasi-complemented BE-algebra. ■

For any self-distributive and commutative BE-algebra X, define $X^{++} = \{(x)^{++} \mid x \in X\}$. We can observe that X^{++} is closed under set-intersection but it need not be a sublattice of the lattice $\langle \mathcal{F}(X), \vee, \cap \rangle$ of all filters of the BE-algebra X. However, in the following theorem, a set of equivalent conditions is derived for X^{++} of a quasi-complemented BE-algebra X to become a sublattice of the lattice $\mathcal{F}(X)$ of all filters of X.

Theorem 28. *The following are equivalent in a quasi-complemented BE-algebra.*

(1) *For each $x \in X, (x)^+ \vee (x)^{++} = X$;*

(2) *for all $x, y \in X, (x)^+ \vee (y)^+ = (x \vee y)^+$;*

(3) *X^{++} is a sublattice of $\mathcal{F}(X)$.*

Proof. (1)⇒(2) Assume the condition (1). Let $x, y \in X$. It is clear that $(x)^+ \vee (y)^+ \subseteq (x \vee y)^+$. Let $a \in (x \vee y)^+$. Then $a \vee x \vee y = 1$. Then we have

$$\begin{aligned} a \vee x \vee y = 1 &\Rightarrow \langle a \vee x \vee y \rangle = \{1\} \\ &\Rightarrow \langle x \rangle \cap \langle a \vee y \rangle = \{1\} \\ &\Rightarrow \langle a \vee y \rangle \subseteq (x)^+ \\ &\Rightarrow (x)^{++} \subseteq (a \vee y)^+ \\ &\Rightarrow (x)^{++} \cap \langle a \vee y \rangle = \{1\} \\ &\Rightarrow (x)^{++} \cap \{\langle a \rangle \cap \langle y \rangle\} = \{1\} \\ &\Rightarrow \{(x)^{++} \cap \langle a \rangle\} \cap \langle y \rangle = \{1\} \\ &\Rightarrow (x)^{++} \cap \langle a \rangle \subseteq (y)^+. \end{aligned}$$

It is clear that $(x)^+ \cap \langle a \rangle \subseteq (x)^+$. By the condition (1), we get $a \in \langle a \rangle = X \cap \langle a \rangle = \{(x)^+ \vee (x)^{++}\} \cap \langle a \rangle = \{(x)^+ \cap \langle a \rangle\} \vee \{(x)^{++} \cap \langle a \rangle\} \subseteq (x)^+ \vee (y)^+$. Hence $(x \vee y)^+ \subseteq (x)^+ \vee (y)^+$. Therefore $(x \vee y)^+ = (x)^+ \vee (y)^+$ for all $x, y \in X$.

(2)⇒(3) Assume that the condition (2) holds. Let $x, y \in X$. Clearly $(x)^{++} \cap (y)^{++} = (x \vee y)^{++}$. Since X is quasi-complemented, there exists $x_0, y_0 \in X$ such that $(x)^{++} = (x_0)^+$ and $(y)^{++} = (y_0)^+$. Since $x_0 \vee y_0 \in X$, there exists $c \in X$ such that $(x_0 \vee y_0)^{++} = (c)^+$. Hence $(x)^{++} \vee (y)^{++} = (x_0)^+ \vee (y_0)^+ = (x_0 \vee y_0)^+ = (c)^{++}$. Therefore X^{++} is a sublattice of $\mathcal{F}(X)$.

(3)⇒(1) Assume that X^{++} is a sublattice of $\mathcal{F}(X)$. Let $x \in X$. Since X is quasi-complemented, there exists $x_0 \in X$ such that $(x)^{++} = (x_0)^+$. Since X^{++} is a sublattice of $\mathcal{F}(X)$, we get $(x)^{++} \vee (x_0)^{++} = (t)^{++}$ for some $t \in X$. Hence

$$\begin{aligned} (t)^+ = (t)^{+++} &= \{(x_0)^{++} \vee (x)^{++}\}^+ \\ &= (x_0)^{+++} \cap (x)^{+++} \\ &= (x_0)^+ \cap (x)^+ \\ &= \{1\} \end{aligned}$$

which concludes that $(t)^+ = \{1\}$. Suppose $(x)^+ \vee (x)^{++} \neq X$. Then there exists a prime filter P such that $(x)^+ \vee (x)^{++} \subseteq P$. Hence $X = \{1\}^+ = (t)^{++} = (x_0)^{++} \vee (x)^{++} = (x)^+ \vee (x)^{++} \subseteq P$, which is a contradiction. Therefore $(x)^+ \vee (x)^{++} = X$. ■

Theorem 29. *Let X be a quasi-complemented BE-algebra such that $\{1\}$ is a prime filter. Then the following conditions are equivalent:*

(1) *every filter is a regular filter;*

(2) *for every proper filter F of $X, F \cap \bar{D}_X = \emptyset$;*

(3) *for every prime filter P of $X, P \cap \bar{D}_X = \emptyset$;*

(4) *every prime filter is a minimal prime filter;*

(5) *every prime filter is a regular filter;*

(6) *for any* $x, y \in X, (x)^+ = (y)^+$ *implies* $\langle x \rangle = \langle y \rangle$.

Proof. (1)⇒(2) Let X be a quasi-complemented BE-algebra. Assume that every proper filter of X is a regular filter. Let F be a proper filter of X. Suppose $x \in F \cap \bar{D}_X$. Then $(x)^+ = \{1\}$ and $(x)^{++} \subseteq F$. Hence $X = \{1\}^+ = (x)^{++} \subseteq F$, which is a contradiction. Therefore $F \cap \bar{D}_X = \emptyset$.

(2)⇒(3) It is obvious.

(3)⇒(4) Assume that the condition (3) holds. Let P be a prime filter of X. Let $x \in P$. Since X is quasi-complemented, there exists $y \in X$ such that $x \vee y = 1 \in P$ and $(x)^+ \cap (y)^+ = \{1\}$. Since $\{1\}$ is prime and $P \cap \bar{D}_X = \emptyset$, we get

$$\begin{aligned}(x)^+ \cap (y)^+ = \{1\} \quad &\Rightarrow \quad (x)^+ = \{1\} \text{ or } (y)^+ = \{1\} \\ &\Rightarrow \quad x \in \bar{D}_X \text{ or } y \in \bar{D}_X \\ &\Rightarrow \quad y \in \bar{D}_X \qquad \text{since } x \in P \\ &\Rightarrow \quad y \notin P \qquad \text{since } P \cap \bar{D}_X = \emptyset\end{aligned}$$

Thus to each $x \in P$, there exists $y \notin P$ such that $x \vee y = 1$. Therefore P is minimal.

(4)⇒(5) Assume that every prime filter is a minimal prime filter. Since every minimal prime filter is a regular filter, it is clear.

The equivalence of the conditions (5), (6) and (1) is proved in Theorem 7. ■

The notion of *strong regular filters* is now introduced in BE-algebras.

Definition. A filter F of a commutative BE-algebra X is called a strong regular filter if for all $a, b, c \in X$, $(a)^+ \cap (b)^+ = (c)^+$ and $a, b \in F$ imply that $c \in F$.

Example 30. Let $X = \{1, a, b, c, d\}$ be a set. Define a binary operation $*$ on X as follows:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	a	c	d
b	1	1	1	c	d
c	1	a	b	1	d
d	1	1	1	c	1

$\vee$	1	a	b	c	d
1	1	1	1	1	1
a	1	a	a	1	a
b	1	a	b	1	b
c	1	1	1	c	1
d	1	a	b	1	d

Then clearly $(X, *, \vee, 1)$ is a commutative BE-algebra. Consider the subset $F = \{1, a, b, d\}$ of X. It can be easily verified that F is a strong regular filter of X.

Proposition 31. *Every strong regular filter of a commutative* BE*-algebra is a regular filter.*

Proof. Let F be a strong regular filter of a commutative BE-algebra X. Let $x, y \in X$ be such that $(x)^+ = (y)^+$ and $x \in F$. Since $y \leq x \vee y$, we get

$$(x)^+ \cap (x \vee y)^+ = (y)^+ \cap (x \vee y)^+ = (y)^+.$$

Since $x, x \vee y \in F$ and F is strong regular, we get $y \in F$. Hence F is regular. ∎

Proposition 32. *Every dual annihilator filter of a commutative BE-algebra is a strong regular filter.*

Proof. Let F be a dual annihilator filter of X. Let $a, b, c \in X$ be such that $(a)^+ \cap (b)^+ = (c)^+$ and suppose $a, b \in F$ then $\langle a\rangle, \langle b\rangle \subseteq F$. Hence $(a)^{++}, (b)^{++} \subseteq F^{++} = F$. Thus

$$\begin{aligned}(a)^{++} \subseteq F, (b)^{++} \subseteq F &\Rightarrow (a)^{++} \vee (b)^{++} \subseteq F \\ &\Rightarrow [(a)^{++} \vee (b)^{++}]^{++} \subseteq F \\ &\Rightarrow [(a)^{+++} \cap (b)^{+++}]^{+} \subseteq F \\ &\Rightarrow [(a)^{+} \cap (b)^{+}]^{+} \subseteq F.\end{aligned}$$

Hence $c \in (c)^{++} \subseteq F$. Therefore F is a strong regular filter of X. ∎

The converse of the above proposition is not true. For consider a proper strong regular filter $F \neq X$ satisfying the property $F^+ = \{1\}$ is not a dual annihilator because of $F^{++} = (F^+)^+ = (\{1\})^+ = X \neq F$. Hence F is not a dual annihilator. However, in the following, we derive a sufficient condition for a strong regular filter to become a dual annihilator filter.

Definition [13]**.** A filter F of a commutative BE-algebra is said to satisfy s-condition if to each $x \notin F$, there exists $y \in F$ such that $(x)^{++} = (y)^+$ for $x, y \in X$.

Theorem 33 [13]**.** *Let F be a non-dense ($F^+ \neq \{1\}$) regular filter of a commutative BE-algebra X. If F satisfies the s-condition, then F is a dual annihilator filter of X.*

Theorem 34. *A non-dense strong regular filter of a commutative BE-algebra is a dual annihilator filter if it satisfies the s-condition.*

Proof. Let F be a non-dense strong regular filter of a commutative BE-algebra X. Then by Proposition 31, F is a regular filter. By Theorem 33, F is a dual annihilator filter. ∎

Theorem 35. *A non-dense regular filter of a commutative BE-algebra X is a strong regular filter if it satisfies the s-condition.*

Proof. Let F be a non-dense regular filter of X. Then by Theorem 33, F is a dual annihilator filter. By Proposition 32, F is a strong regular filter. ■

Definition. A commutative BE-algebra X is called a *strong BE-algebra* if for each $a, b \in X$, there exists $c \in X$ such that $(a)^+ \cap (b)^+ = (c)^+$.

Example 36. Let $X = \{1, a, b, c, d\}$ be a set. Define a binary operation $*$ on X as follows:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	a	c	c
b	1	1	1	c	c
c	1	a	b	1	a
d	1	1	a	1	1

$\vee$	1	a	b	c	d
1	1	1	1	1	1
a	1	a	a	1	a
b	1	a	b	1	a
c	1	1	1	c	c
d	1	a	a	c	d

Then clearly $(X, *, \vee, 1)$ is a commutative BE-algebra. It can be easily verified that X is a strong BE-algebra.

Every commutative BE-algebra X satisfying the property that $(x)^+ = \{1\}$ for all $x \in X$ is a strong BE-algebra.

Proposition 37. *Every quasi-complemented BE-algebra is a strong BE-algebra.*

Proof. Assume that X is a quasi-complemented BE-algebra. Let $a, b \in X$. Since X is quasi-complemented, there exist $a_0, b_0 \in X$ such that $(a)^{++} = (a_0)^+$ and $(b)^{++} = (b_0)^+$. Since X is quasi-complemented and $a_0 \vee b_0 \in X$, there exits some $c \in X$ such that $(a_0 \vee b_0)^{++} = (c)^+$. Hence $(a)^+ \cap (b)^+ = (a_0)^{++} \cap (b_0)^{++} = (a_0 \vee b_0)^{++} = (c)^+$. Therefore X is a strong BE-algebra. ■

In general, the converse of the above proposition is not true. In the following, we derive a set of equivalent conditions for a strong BE-algebra to become quasi-complemented.

Theorem 38. *Let X be a commutative BE-algebra. Then the following are equivalent:*

(1) *X is quasi-complemented;*

(2) *X is strong;*

(3) *every strong regular filter is an O-filter;*

(4) *every dual annihilator filter is an O-filter.*

Proof. (1)⇒(2) By Proposition 37, it is clear.

(2)⇒(3) Assume that X is a strong BE-algebra. Let F be a strong regular filter of X. Consider the set $F^0 = \{\ x \in X \mid (a)^+ \subseteq (x)^{++} \text{ for some } a \in F\ \}$.

We first prove that $F^0 \in \mathcal{J}(X)$. Clearly $\emptyset \neq \bar{D}_X \subseteq F^0$. Let $x, y \in F^0$. Then we get $(a)^+ \subseteq (x)^{++}$ and $(b)^+ \subseteq (y)^{++}$ for some $a, b \in F$. Since X is strong, there exists $c \in X$ such that $(a)^+ \cap (b)^+ = (c)^+$. Now $(c)^+ = (a)^+ \cap (b)^+ \subseteq (x)^{++} \cap (y)^{++} = (x \vee y)^{++}$. Since $a, b \in F$ and F is a strong regular filter, we get $c \in F$. Hence $x \vee y \in F^0$. Therefore $F^0 \in \mathcal{J}(X)$. We now show that $F = O(F^0)$. Let $x \in O(F^0)$. Then $x \vee s = 1$ for some $s \in F^0$. Hence $x \in (s)^+$. Now

$$\begin{aligned} s \in F^0 &\Rightarrow (a)^+ \subseteq (s)^{++} \text{ for some } a \in F \\ &\Rightarrow (s)^+ \subseteq (a)^{++} \subseteq F \qquad \text{since } F \text{ is a regular filter and } a \in F \\ &\Rightarrow x \in F. \end{aligned}$$

Therefore $O(F^0) \subseteq F$. Conversely, let $x \in F$. Since X is a quasi-complemented, there exists $y \in X$ such that $(x)^+ = (y)^{++}$. Since $x \in F$, we get that $y \in F^0$. Also $x \in (x)^{++} = (y)^+$ and $y \in F^0$. Hence $x \in O(F^0)$. Thus $F \subseteq O(F^0)$. Therefore F is an O-filter.

(3)⇒(4) Since every dual annihilator filter is a regular filter, it is clear.

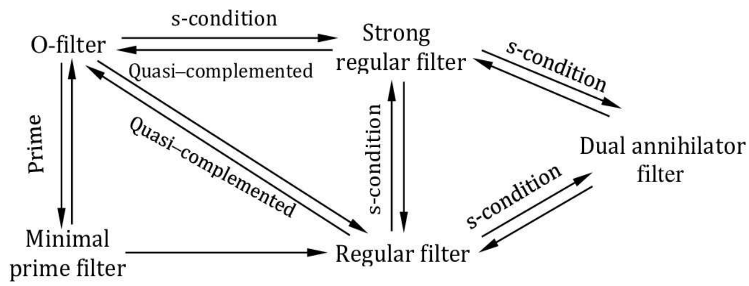
(4)⇒(1) Let $x \in X$. Since $(x)^{++}$ is a dual annihilator filter, by condition (4), we get $(x)^{++} = O(S)$ for some $S \in \mathcal{J}(X)$. Now let $t \in (x)^{++} = O(S)$. Then we get $t \in (y)^+$ for some $y \in S$. Hence $(x)^{++} \subseteq (y)^+$. On the other hand, we have $(y)^+ \subseteq \bigcup_{y \in S} (y)^+ = O(S) = (x)^{++}$. Hence $(x)^{++} = (y)^+$. Therefore X is quasi-complemented. ■

5. CONCLUSION

In this paper, we introduced the notion of O-filters in commutative BE-algebras and obtained some sufficient conditions for a prime filter to become an O-filter. The notion of quasi-complemented BE-algebras is introduced and then studied the relations among O-filters, dual annihilator filters and strong regular filters of BE-algebras. In addition, we established some interconnections among prime filters, regular filters, strong regular filters, dual annihilator filters and O-filters of a commutative BE-algebra. We think such results are very useful for the further characterization of prime O-filters in terms of congruences of this structure.

Now, in the following diagram we summarize the results of this paper and the past results in this field and we give the relations among prime filters, minimal prime filters, dual annihilator filters, regular filters, strong regular filters and O-filters. The mark $A \rightarrow B$ means that A implies B. A condition with the mark $A \rightarrow B$ indicates that A conclude B with the condition.

For the future research, we investigate some new filters of commutative BE-algebras with the help of dual annihilator filter and regular filters.



Acknowledgement

The authors would like to thank the referees for their valuable suggestions and comments that improved the presentation of this article.

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Received 21 July 2020
Revised 20 August 2020
Accepted 20 August 2020