Discussiones Mathematicae
General Algebra and Applications 41 (2021) 343–360
https://doi.org/10.7151/dmgaa.1364

GENERALIZED ROUGH SETS APPLIED TO BCK/BCI-ALGEBRAS

Young Bae Jun

Department of Mathematics Education
Gyeongsang National University, Jinju 52828, Korea
e-mail: skywine@gmail.com

Seok-Zun Song [1]

Department of Mathematics
Jeju National University, Jeju 63243, Korea
e-mail: szsong@jejunu.ac.kr

and

Eun Hwan Roh

Department of Mathematics Education
Chinju National University of Education, Jinju 52673, Korea
e-mail: ehroh9988@gmail.com

Abstract

The concept of a (strong) set-valued BCK/BCI-morphism in BCK/BCI-algebras is considered, and several properties are investigated. Conditions for a set-valued mapping to be a set-valued BCK/BCI-morphism are given. Using the concept of generalized approximation space, generalized rough subalgebra (ideal) in BCK/BCI-algebras are introduced, and investigate their properties. Using the concept of generalized approximation space and ideal of BCK/bCI-algebra, another type of generalized lower and upper approximations based on the ideal is considered, and then several properties are investigated.

Keywords: generalized approximation space, generalized rough set, (strong) set-valued BCK/BCI-morphism, generalized lower rough subalgebra (ideal), generalized upper rough subalgebra (ideal).

2010 Mathematics Subject Classification: 06F35, 03G25.

---

[1] Corresponding author.

This research is financially supported by the 2021 scientific promotion program funded by Jeju National University.

1. Introduction

The theory of rough sets (see [17–19] and [20]) is a new mathematical tool to deal with uncertain, vague and inexact knowledge. It has been applied to many areas such as data and conflict analysis, approximate classification, machine learning, knowledge discovery, etc. (see [9, 12, 14, 15, 21, 22, 34, 35, 38]). Rough sets are applied to several algebraic structures such as semigroup [10,25,32], hypergroup [11], ring [1,16], BCI-algebra [4], etc. by using a congruence relation.

Recently, one of the main research aspects/directions in rough set theory is to consider the generalization of the Pawlak's rough set approximations (see [3,23,29,30,36,37,39–43]).

The initiation and majority of studies on rough sets for algebraic structures such as semigroups, groups, rings, and modules etc. have been concentrated on a congruence relation. However, the congruence relation seems to be restrict the application of the generalized rough set model for algebraic sets. To solve this problem, Davvaz [2] introduced the concept of a set-valued homomorphism for groups. From this point of view, Yamak *et al.* [33] considered the concept of a set-valued homomorphism for rings. They introduced the concepts of set-valued homomorphism and strong set-valued homomorphism of a ring, and investigated related properties.

The primary goal of this paper is to consider the (strong) set-valued mapping in other algebraic structure so called BCK/BCI-algebras. In this paper, we introduce the concept of a (strong) set-valued BCK/BCI-morphism in BCK/BCI-algebras, and investigate several properties. We provide conditions for a set-valued mapping to be a set-valued BCK/BCI-morphism. Using the concept of generalized approximation space, we introduce generalized rough subalgebra (ideal) in BCK/BCI-algebras, and investigate their properties. Using the concept of generalized approximation space and ideal of BCK/BCI-algebra, we consider another type of generalized lower and upper approximations based on the ideal, and then several properties are investigated.

Rough set theory is intended to deal with uncertainty in addition to fuzzy set theory, which is based on concepts such as approximation, dependence and reduction of attributes, decision tables, decision rules, etc. Rough set theory is applied in hybrid ways in many areas, such as pattern recognition, information processing, business and finance, industry and environmental engineering, medical diagnosis and medical data analysis, and system defects and monitoring. Therefore, there is a limit to dealing with uncertain issues in a segmentation environment that requires accurate harm. Fuzzy set theory and Rough set theory are emerging as alternatives to overcoming this limitation. Since most real-world problems pose uncertainty, the Rough Set theory can be used in many applications of the real world. The Rough Set theory solves the problem in an integrated way with the various areas of the computer, so it is expected that there will be many advances along with the development of fuzzy theory, neural circuit network theory and decision theory. So, as a secondary goal, we hope that many (applied) scientists dealing with uncertainty can use the results of this paper in their research.

2. Preliminaries

A BCK/BCI-algebra, which is an important class of logical algebras, is introduced by K. Iséki (see [6] and [7]).

A *BCI-algebra* is defined to be the structure $(X, *, 0)$ which satisfies the following conditions (see [13]):

(I) $(\forall x, y, z \in X)\ ((x * y) * (x * z) \leq z * y)$,

(II) $(\forall x, y \in X)\ ((x * (x * y) \leq y)$,

(III) $(\forall x \in X)\ (x \leq x)$,

(IV) $(\forall x, y \in X)\ (x \leq y,\ y \leq x\ \Rightarrow\ x = y)$

where $x \leq y$ means $x * y = 0$ for all $x, y \in X$. If a BCI-algebra X has the following identity:

(V) $(\forall x \in X)\ (0 \leq x)$,

then X is called a *BCK-algebra.* In any BCK/BCI-algebra X, the following conditions are valid (see [13]).

$$(\forall x \in X)\, (x * 0 = x)\,, \tag{2.1}$$

$$(\forall x, y, z \in X)\, (x \leq y\ \Rightarrow\ x * z \leq y * z,\ z * y \leq z * x)\,, \tag{2.2}$$

$$(\forall x, y, z \in X)\, ((x * y) * z = (x * z) * y)\,, \tag{2.3}$$

$$(\forall x, y, z \in X)\, ((x * z) * (y * z) \leq x * y)\,. \tag{2.4}$$

We say that a non-empty subset S of a BCK/BCI-algebra X is a *subalgebra* of X (see [13]) if $x*y \in S$ for all $x, y \in S$. We say that a subset I of a BCK/BCI-algebra X is an *ideal* of X (see [13]) if it satisfies

$$0 \in I, \tag{2.5}$$

$$(\forall x \in X)\, (\forall y \in I)\, (x * y \in I\ \Rightarrow\ x \in I)\,. \tag{2.6}$$

Let A be an ideal of a BCK/BCI-algebra X. Define a binary relation θ on X as follows

$$(\forall x, y \in X)\, ((x, y) \in \theta\ \Leftrightarrow\ x * y \in A,\ y * x \in A)\,. \tag{2.7}$$

Then θ is a congruence on X, and the quotient structure $(X/A, *, A_0)$ is a BCK/BCI-algebra (see [5,13]).

We now display basic definitions on generalized rough sets (see [24, 26–28] and [33]).

Let X and Y be non-empty universes and consider the mapping $t : X \to 2^Y$ which is called a *set-valued mapping*. Then the triple (X, Y, t) is called a *generalized approximation space*.

Any set-valued mapping $t : X \to 2^Y$ define a binary relation from X to Y by setting

$$\rho_t := \{(x, y) \in X \times Y \mid y \in t(x)\}. \tag{2.8}$$

Obviously, if ρ is an arbitrary relation from X to Y, then the mapping

$$t_\rho : X \to 2^Y,\ x \mapsto \{y \in Y \mid (x, y) \in \rho\} \tag{2.9}$$

is a set-valued mapping. For any subset A of Y, the generalized lower and upper approximations, $\underline{t}(A)$ and $\overline{t}(A)$, are defined by

$$\underline{t}(A) = \{x \in X \mid t(x) \subseteq A\} \text{ and } \overline{t}(A) = \{x \in X \mid t(x) \cap A \neq \emptyset\}.$$

We say that the pair $\big(\underline{t}(A), \overline{t}(A)\big)$ is a *generalized rough set*.

3. Set-valued morphisms in BCK/BCI-algebras

In classical approximation spaces, a set is approximated by its lower and upper approximations that are made by classes of elements that are indistinguishable. In this section, in order to make generalized approximation space on BCK/BCI-algebras, we define (strong) set-valued morphism.

In what follows, let X and Y denote BCK/BCI-algebras unless otherwise specified.

For any non-empty subsets A and B of X, we define

$$A * B := \{a * b \mid a \in A,\ b \in B\}. \tag{3.1}$$

Definition 3.1. A set-valued mapping $t : X \to 2^Y$ is called a *set-valued $BCK/$ BCI-morphism* if it satisfies

$$(\forall x, y \in X)\,(t(x) * t(y) \subseteq t(x * y))\,. \tag{3.2}$$

A set-valued mapping $t : X \to 2^Y$ is called a *strong set-valued BCK/BCI-morphism* if it satisfies:

$$(\forall x, y \in X)\,(t(x) * t(y) = t(x * y))\,. \tag{3.3}$$

Example 3.2. Consider a set $X = \{0, a, b, c, d\}$ with the binary operation $*$ which is given in Table 1.

Table 1. Cayley table for the binary operation "$*$".

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	0
c	c	c	c	0	0
d	d	d	d	c	0

Then $(X; *, 0)$ is a BCK-algebra (see [13]). Define a mapping t as follows

$$t : X \to 2^X, \quad x \mapsto \begin{cases} \{0, a\} & \text{if } x \in \{0, a\}, \\ \{b\} & \text{if } x = b, \\ \{c\} & \text{if } x = c, \\ \{d\} & \text{if } x = d. \end{cases}$$

Then (X, X, t) is a generalized approximation space. It is routine to verify that t is a set-valued BCK-morphism. But it is not strong since

$$t(b) * t(c) = \{b\} * \{c\} = \{0\} \neq \{0, a\} = t(b * c).$$

Example 3.3. Let $X = \{0, a, b, c\}$ be set the binary operations $*$ which are given in Table 2.

Table 2. Cayley table for the binary operation "$*$".

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Define a mapping t as follows

$$t : X \to 2^X, \quad x \mapsto \begin{cases} \{0, c\} & \text{if } x \in \{0, c\}, \\ \{a, b\} & \text{if } x \in \{a, b\}. \end{cases}$$

Then (X, X, t) is a generalized approximation space. It is easy to check that t is a strong set-valued BCK/BCI-morphism.

Note that if ρ is an arbitrary relation from X to Y, then the mapping

$$t_\rho : X \to 2^Y,\ x \mapsto \{y \in Y \mid (x,y) \in \rho\} \tag{3.4}$$

is a set-valued mapping. But it is not a set-valued BCK/BCI-morphism as seen in the following example.

Example 3.4. Let $X = \{0, a, b, c\}$ and $Y = \{0, 1, 2, 3\}$ be sets with the binary operations $*_X$ and $*_Y$ which are given in Table 3 and Table 4, respectively.

Table 3. Cayley table for the binary operation "$*_X$".

$*_X$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

Table 4. Cayley table for the binary operation "$*_Y$".

$*_Y$	0	1	2	3
0	0	0	3	2
1	1	0	3	2
2	2	2	0	3
3	3	3	2	0

Let $\rho := \{(0,1), (0,3), (a,1), (b,2), (b,3), (c,1), (c,2)\}$ be a relation from X to Y. Then (X, Y, t) is a generalized approximation space in which the mapping t defined by

$$t : X \to 2^Y,\quad x \mapsto \begin{cases} \{1,3\} & \text{if } x = 0, \\ \{1\} & \text{if } x = a, \\ \{2,3\} & \text{if } x = b, \\ \{1,2\} & \text{if } x = c. \end{cases}$$

is a set-valued mapping which is not a set-valued BCK/BCI-morphism since

$$t(a) * t(b) = \{1\} * \{2,3\} = \{2,3\} \nsubseteq \{1\} = t(a * b).$$

We provide conditions for a set-valued mapping to be a set-valued BCK/BCI-morphism.

Theorem 3.5. *Let θ be a congruence relation on X. Then the mapping*

$$t_\theta : X \to 2^X,\ x \mapsto [x]_\theta \tag{3.5}$$

is a set-valued BCK/BCI-morphism.

Proof. For any $x, y \in X$, we have

$$t(x) * t(y) = [x]_\theta * [y]_\theta \subseteq [x * y]_\theta = t(x * y).$$

Hence t_θ is a set-valued BCK/BCI-morphism. ■

Theorem 3.6. *Given a generalized approximation space (X, Y, t) in which t is a set-valued BCK/BCI-morphism, if A is an ideal of Y, then the set-valued mapping*

$$t_A : X \to 2^{Y/A},\ x \mapsto \{A_a \mid a \in t(x)\} \tag{3.6}$$

is a set-valued BCK/BCI-morphism.

Proof. Assume that t is a set-valued BCK/BCI-morphism. For any $x, y \in X$, we have

$$\begin{aligned} t_A(x) * t_A(y) &= \{A_a \mid a \in t(x)\} * \{A_b \mid b \in t(y)\} \\ &= \{A_c \mid c = a * b, a \in t(x), b \in t(y)\} \\ &= \{A_c \mid c \in t(x) * t(y)\} \\ &\subseteq \{A_c \mid c \in t(x * y)\} = t_A(x * y). \end{aligned}$$

Therefore t_A is a set-valued BCK/BCI-morphism. ■

The following example illustrates Theorem 3.6.

Example 3.7. In Example 3.2, we know that $t : X \to 2^X$ is a set-valued BCK-morphism. Let $A := \{0, a, b\}$. Then A is an ideal of X [13] and

$$X/A = \{A_0 = A_a = A_b = \{0, a, b\}, A_c = \{c\}, A_d = \{d\}\}.$$

If follows that

$$t_A : X \to 2^{X/A},\quad x \mapsto \begin{cases} \{A_0\} & \text{if } x \in A, \\ \{A_c\} & \text{if } x = c, \\ \{A_d\} & \text{if } x = d \end{cases}$$

which is a set-valued BCK-morphism.

Question 3.8. *Given a strong set-valued BCK/BCI-morphism t and an ideal A of Y, is the set-valued mapping t_A in* (3.6) *a strong set-valued BCK/BCI-morphism?*

The following example shows that the answer to Question 3.8 is negative.

Example 3.9. In Example 3.3, we know that $t : X \to 2^X$ is a strong set-valued BCK-morphism. Let $A := \{0, a, b\}$. Then A is an ideal of X [13] and

$$X/A = \{A_0 = A_a = A_b = \{0, a, b\}, A_c = \{c\}\}.$$

It follows that

$$t_A : X \to 2^{X/A}, \quad x \mapsto \begin{cases} \{A_0, A_c\} & \text{if } x \in \{0, c\}, \\ \{A_0\} & \text{if } x \in \{a, b\} \end{cases}$$

and it is not a strong set-valued BCK-morphism since

$$t_A(a) * t_A(a) = \{A_0\} * \{A_0\} = \{A_0\} \neq \{A_0, A_c\} = t_A(0) = t_A(a * a).$$

Proposition 3.10. *Let (X, Y, t) be a generalized approximation space in which t is a set-valued BCK/BCI-morphism. Then*

(1) $0 \in t(0)$.

(2) *If A and B are non-empty subsets of Y, then $\overline{t}(A) * \overline{t}(B) \subseteq \overline{t}(A * B)$.*

(3) *If A and B are non-empty subsets of Y and t is strong, then $\underline{t}(A) * \underline{t}(B) \subseteq \underline{t}(A * B)$.*

Proof. (1) We have $0 \in t(x) * t(x) \subseteq t(x * x) = t(0)$ for all $x \in X$.

(2) Let $a \in \overline{t}(A) * \overline{t}(B)$. Then $a = b * c$ for some $b \in \overline{t}(A)$ and $c \in \overline{t}(B)$. It follows that $t(b) \cap A \neq \emptyset$ and $t(c) \cap B \neq \emptyset$, which imply that there exist $u, v \in Y$ such that $u \in t(b) \cap A$ and $v \in t(c) \cap B$. Hence $u * v \in A * B$ and $u * v \in t(b) * t(c) \subseteq t(b * c) = t(a)$. Thus $a \in \overline{t}(A * B)$, and so $\overline{t}(A) * \overline{t}(B) \subseteq \overline{t}(A * B)$.

(3) Let $a \in \underline{t}(A) * \underline{t}(B)$. Then $a = b * c$ for some $b \in \underline{t}(A)$ and $c \in \underline{t}(B)$. It follows that $t(b) \subseteq A$ and $t(c) \subseteq B$. Since t is strong, we have $t(a) = t(b * c) = t(b) * t(c) \subseteq A * B$, and so $a \in \underline{t}(A * B)$. Thus $\underline{t}(A) * \underline{t}(B) \subseteq \underline{t}(A * B)$. ∎

If we omit the condition "t is strong" in Proposition 3.10(3), then the inclusion

$$\underline{t}(A) * \underline{t}(B) \subseteq \underline{t}(A * B)$$

does not hold as seen in the following example.

Example 3.11. In Example 3.2, the set-valued BCK/BCI-morphism is not strong. Let $A := \{b, c\}$ and $B := \{a, b\}$. Then we have

$$\underline{t}(A) * \underline{t}(B) = \{b, c\} * \{b\} = \{0, c\} \nsubseteq \{b, c\} = \underline{t}(A * B)$$

Question 3.12. *Let (X, Y, t) be a generalized approximation space in which t is a set-valued BCK/BCI-morphism and A be a subset of Y. Then are the generalized lower and upper approximations $\underline{t}(A)$ and $\overline{t}(A)$ subalgebras (resp., ideals) of X?*

The answer to the question above is negative as seen in the following example.

Example 3.13. In Example 3.2, (X, X, t) is a generalized approximation space in which t is a set-valued BCK/BCI-morphism. If $A := \{b, c\}$, then $\underline{t}(A) = \{b, c\} = \overline{t}(A)$ which is neither a subalgebra nor an ideal of X.

Definition 3.14. Let (X, Y, t) be a generalized approximation space. Then a subset A of Y is called

- a *generalized lower rough subalgebra* (resp., *ideal*) if the generalized lower approximation $\underline{t}(A)$ is a subalgebra (resp., ideal) of X.
- a *generalized upper rough subalgebra* (resp., *ideal*) if the generalized upper approximation $\overline{t}(A)$ is a subalgebra (resp., ideal) of X.

If A is both a generalized lower rough subalgebra (resp., ideal) and a generalized upper rough subalgebra (resp., ideal) of X, we say that A is a *generalized rough subalgebra* (resp., *ideal*) of Y.

Example 3.15. Consider two sets $X = \{0, 1, 2, 3, 4\}$ and $Y = \{0, 1, 2, a, b\}$ with binary operations $*_X$ and $*_Y$ given by Table 5 and Table 6, respectively.

Table 5. Cayley table for the binary operation "$*_X$".

$*_X$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	2	0
3	3	1	1	0	1
4	4	4	4	4	0

Then $(X, *_X, 0)$ is a BCK-algebra and $(Y, *_Y, 0)$ is a BCI-algebra (see [5]). Define a mapping t_1 as follows

$$t_1 : X \to 2^Y, \quad x \mapsto \begin{cases} \{0, a\} & \text{if } x \in \{0, 2\}, \\ \{b\} & \text{if } x \in \{1, 4\}, \\ \{2\} & \text{if } x = 3. \end{cases}$$

Table 6. Cayley table for the binary operation "$*_Y$".

$*_Y$	0	1	2	a	b
0	0	0	0	a	a
1	1	0	1	a	a
2	2	2	0	a	a
a	a	a	a	0	0
b	b	a	b	1	0

Then (X, Y, t_1) is a generalized approximation space. For a subset $A = \{0, 1, 2, a\}$ of Y, we have $\underline{t_1}(A) = \{0, 2, 3\} = \overline{t_1}(A)$ which is neither a subalgebra nor an ideal of X. Hence A is neither a generalized lower rough subalgebra nor a generalized lower rough ideal, and A is neither a generalized upper rough subalgebra nor a generalized upper rough ideal.

Consider a generalized approximation space (X, Y, t_2) in which t_2 is given as follows

$$t_2 : X \to 2^Y, \quad x \mapsto \begin{cases} \{0, 2, a\} & \text{if } x = 0, \\ \{1, 2\} & \text{if } x = 1, \\ \{b\} & \text{if } x = 2, \\ \{1, a\} & \text{if } x = 3, \\ \{2, b\} & \text{if } x = 4. \end{cases}$$

For a subset $A = \{0, 1, 2, a\}$ of Y, we have $\underline{t_2}(A) = \{0, 1, 3\}$ which is both a subalgebra and an ideal of X. Hence A is a generalized lower rough subalgebra and a generalized lower rough ideal. But $\overline{t_2}(A) = \{0, 1, 3, 4\}$ which is neither a subalgebra nor an ideal of X. Thus A is neither a generalized upper rough subalgebra nor a generalized upper rough ideal.

Consider a generalized approximation space (X, Y, t_3) in which t_3 is given as follows

$$t_3 : X \to 2^Y, \quad x \mapsto \begin{cases} \{0, a\} & \text{if } x = 0, \\ \{a, b\} & \text{if } x = 1, \\ \{1\} & \text{if } x = 2, \\ \{2, b\} & \text{if } x = 3, \\ \{1, a\} & \text{if } x = 4. \end{cases}$$

For a subset $B = \{0, 2, b\}$ of Y, we have $\underline{t_3}(B) = \{3\}$ which is neither a subalgebra nor an ideal of X. Hence B is neither a generalized lower rough subalgebra nor a generalized lower rough ideal. But $\overline{t_3}(B) = \{0, 1, 3\}$ which is a subalgebra and an ideal of X. Hence B is a generalized upper rough subalgebra and a generalized upper rough ideal.

Consider a generalized approximation space (X, Y, t_4) in which t_4 is given as follows

$$t_4 : X \to 2^Y, \quad x \mapsto \begin{cases} \{0,2\} & \text{if } x = 0, \\ \{1,2\} & \text{if } x = 1, \\ \{2,a,b\} & \text{if } x = 2, \\ \{0,1\} & \text{if } x = 3, \\ \{a,b\} & \text{if } x = 4. \end{cases}$$

For a subset $C = \{0, 1, 2\}$ of Y, we have $\underline{t_4}(C) = \{0, 1, 3\}$ which is a subalgebra and an ideal of X. Hence C is a generalized lower rough subalgebra and a generalized lower rough ideal. Also $\overline{t_4}(C) = \{0, 1, 2, 3\}$ which is a subalgebra and an ideal of X. Hence C is a generalized upper rough subalgebra and a generalized upper rough ideal. Therefore C is a generalized rough subalgebra and ideal.

Theorem 3.16. *Let (X, Y, t) be a generalized approximation space in which t is a set-valued BCK/BCI-morphism. Then*

(1) *If A is a subalgebra of Y, then the generalized upper approximation $\overline{t}(A)$ is a subalgebra of X whenever it is non-empty, that is, every subalgebra is a generalized lower rough subalgebra.*

(2) *If A is a subalgebra of Y and t is strong, then the generalized lower approximation $\underline{t}(A)$ is a subalgebra of X whenever it is non-empty, that is, every subalgebra is a generalized upper rough subalgebra.*

Proof. (1) Let A be a subalgebra of Y such that $\overline{t}(A)$ is non-empty. Let $x, y \in \overline{t}(A)$. Then $t(x) \cap A \neq \emptyset$ and $t(y) \cap A \neq \emptyset$. Hence there exist $a \in t(x) \cap A$ and $b \in t(y) \cap A$, which imply that $a * b \in A$ and $a * b \in t(x) * t(y) \subseteq t(x * y)$. Thus $a * b \in t(x * y) \cap A$, i.e., $t(x * y) \cap A$ is non-empty. Hence $x * y \in \overline{t}(A)$, and therefore $\overline{t}(A)$ is a subalgebra of X.

(2) Suppose that A is a subalgebra of Y and t is strong. Let $x, y \in \underline{t}(A)$. Then $t(x) \subseteq A$ and $t(y) \subseteq A$. Hence $t(x * y) = t(x) * t(y) \subseteq A * A = A$, and so $x * y \in \underline{t}(A)$. Therefore $\underline{t}(A)$ is a subalgebra of X. ∎

Corollary 3.17. *If (X, Y, t) is a generalized approximation space in which t is a strong set-valued BCK/BCI-morphism, then every subalgebra of Y is a generalized rough subalgebra of Y.*

Theorem 3.18. *Let (X, Y, t) be a generalized approximation space in which t is a set-valued BCK/BCI-morphism. If A is an ideal of Y, then the generalized lower approximation $\underline{t}(A)$ is an ideal of X.*

Proof. Let A be an ideal of Y. Since $0 \in t(0) \subseteq t(A)$, we get $0 \in \underline{t}(A)$. Let $x, y \in X$ be such that $x * y \in \underline{t}(A)$ and $y \in \underline{t}(A)$. Then $t(x) * t(y) \subseteq t(x * y) \subseteq A$ and $t(y) \subseteq A$. Since A is an ideal of Y, it follows that $t(x) \subseteq A$. Hence $x \in \underline{t}(A)$ and therefore $\underline{t}(A)$ is an ideal of X. ∎

Theorem 3.19. *Let (X, Y, t) be a generalized approximation space in which t is a (strong) set-valued BCK/BCI-morphism. Then*

(1) *If $f : Z \to X$ is a homomorphism of BCK/BCI-algebras, then the composition $t \circ f$ is a (strong) set-valued BCK/BCI-morphism, and*

$$(\forall A \in 2^Y)\left(\overline{t \circ f}(A) = f^{-1}(\overline{t}(A)),\ \underline{t \circ f}(A) = f^{-1}(\underline{t}(A))\right). \tag{3.7}$$

(2) *Given a homomorphism $g : Y \to Z$ of BCK/BCI-algebras, define a mapping*

$$t_g : X \to 2^Z,\ x \mapsto g(t(x)). \tag{3.8}$$

Then t_g is a set-valued BCK/BCI-morphism and

$$(\forall B \in 2^Z)\left(\overline{t_g}(B) = \overline{t}(g^{-1}(B)),\ \underline{t_g}(B) = \underline{t}(g^{-1}(B))\right). \tag{3.9}$$

Proof. (1) For any $a, b \in Z$, we have

$$\begin{aligned}(t \circ f)(a) * (t \circ f)(b) &= t(f(a)) * t(f(b)) \subseteq t(f(a) * f(b))\\ &= t(f(a * b)) = (t \circ f)(a * b),\end{aligned}$$

and so $t \circ f$ is a set-valued BCK/BCI-morphism. Similarly, $t \circ f$ is a strong set-valued BCK/BCI-morphism. Let $a \in f^{-1}(\overline{t}(A))$. Then $f(a) = x$ for some $x \in X$ with $t(x) \cap A \neq \emptyset$. It follows that

$$(t \circ f)(a) \cap A = t(f(a)) \cap A \neq \emptyset,$$

which implies that $a \in \overline{t \circ f}(A)$, that is, $f^{-1}(\overline{t}(A)) \subseteq \overline{t \circ f}(A)$. Similarly, we show that $\overline{t \circ f}(A) \subseteq f^{-1}(\overline{t}(A))$. If $a \in \underline{t \circ f}(A)$, then $t(f(a)) = (t \circ f)(a) \subseteq A$. Hence $f(a) \in \underline{t}(A)$, and so $a \in f^{-1}(\underline{t}(A))$. Similarly, we can prove that $f^{-1}(\underline{t}(A)) \subseteq \underline{t \circ f}(A)$.

(2) For any $x, y \in X$, we get

$$t_g(x) * t_g(y) = g(t(x)) * g(t(y)) = g(t(x) * t(y)) \subseteq g(t(x * y)) = t_g(x * y).$$

Thus t_g is a set-valued BCK/BCI-morphism. If $x \in \overline{t_g}(B)$, then $g(t(x)) \cap B = t_g(x) \cap B \neq \emptyset$. It follows that

$$t(x) \cap g^{-1}(B) = g^{-1}(g(t(x)) \cap B) \neq \emptyset,$$

that is, $x \in \overline{t}(g^{-1}(B))$. Similarly, we can check $\overline{t}(g^{-1}(B)) \subseteq \overline{t_g}(B)$. If $x \in \underline{t_g}(B)$, then $g(t(x)) = t_g(x) \subseteq B$, which implies that $t(x) \in g^{-1}(B)$. Hence $x \in \underline{t}(g^{-1}(B))$. Similarly, we have $\underline{t}(g^{-1}(B)) \subseteq \underline{t_g}(B)$. ∎

Definition 3.20. Let (X, Y, t) be a generalized approximation space. Let A be an ideal of Y and S be a non-empty subset of Y. Then the sets

$$\underline{t}_A(S) := \{x \in X \mid t(x) * A \subseteq S\} \text{ and}$$
$$\overline{t}_A(S) := \{x \in X \mid (t(x) * A) \cap S \neq \emptyset\} \tag{3.10}$$

are called *generalized lower* and *upper approximations* of S, respectively, based on the ideal A.

Definition 3.20 is illustrated in the following example.

Example 3.21. Consider the generalized approximation space (X, Y, t) in Example 3.4. Let $A = \{0, 1\}$ be an ideal of Y. Given a subset $S = \{0, 1, 3\}$ of Y, we have $\underline{t}_A(S) = \{0, a\}$ and $\overline{t}_A(S) = X$. If $S = \{2, 3\}$, then $\underline{t}_A(S) = \{b\}$ and $\overline{t}_A(S) = \{0, b, c\}$.

Proposition 3.22. *Let (X, Y, t) be a generalized approximation space. If A and B are ideals of Y and S is a non-empty subset of Y, then*

$$A \subseteq B \;\Rightarrow\; \underline{t}_A(S) \supseteq \underline{t}_B(S) \text{ and } \overline{t}_A(S) \subseteq \overline{t}_B(S), \tag{3.11}$$
$$\underline{t}_A(S) \cap \underline{t}_B(S) \subseteq \underline{t}_{A\cap B}(S), \tag{3.12}$$
$$\overline{t}_{A\cap B}(S) \subseteq \overline{t}_A(S) \cap \overline{t}_B(S). \tag{3.13}$$

Proof. Assume that $A \subseteq B$. If $x \in \underline{t}_B(S)$, then $t(x) * A \subseteq t(x) * B \subseteq S$, and so $x \in \underline{t}_A(S)$. If $x \in \overline{t}_A(S)$, then $\emptyset \neq (t(x) * A) \cap S \subseteq (t(x) * B) \cap S$. Thus $x \in \overline{t}_B(S)$. If $x \in \underline{t}_A(S) \cap \underline{t}_B(S)$, then $t(x) * (A \cap B) \subseteq t(x) * A \subseteq S$ and thus $x \in \underline{t}_{A\cap B}(S)$. If $x \in \overline{t}_{A\cap B}(S)$, then $(t(x) * (A \cap B)) \cap S \neq \emptyset$. Since A and B are super sets of $A \cap B$, it follows that $(t(x) * A) \cap S \neq \emptyset$ and $(t(x) * B) \cap S \neq \emptyset$. Thus $x \in \overline{t}_A(S)$ and $x \in \overline{t}_B(S)$, which implies that $x \in \overline{t}_A(S) \cap \overline{t}_B(S)$. ■

Theorem 3.23. *Let (X, Y, t) be a generalized approximation space in which t is a set-valued BCK/BCI-morphism and 2^Y satisfies the right self-distributive law. Let S be a non-empty subset of Y and let A be an ideal of Y.*

(1) *If S is a subalgebra of Y, then $\overline{t}_A(S)$ is a subalgebra of X whenever it is non-empty.*
(2) *If t is strong and S is a subalgebra of Y, then $\underline{t}_A(S)$ is a subalgebra of X whenever it is non-empty.*
(3) *If B is an ideal of Y and 2^Y satisfies the right $\cap$-distributive law, that is, $(P * Q) \cap R = (P \cap R) * (Q \cap R)$ for all $P, Q, R \in 2^Y$, then $\overline{t}_A(B)$ is an ideal of X whenever it is non-empty.*

Proof. (1) Assume that $\overline{t}_A(S)$ is non-empty and let $x, y \in \overline{t}_A(S)$. Then there exist $a \in (t(x) * A) \cap S$ and $b \in (t(y) * A) \cap S$, which imply that $a \in t(x) * A$,

$b \in t(y) * A$ and $a * b \in S$. Since Y satisfies the right self-distributive law and t is a set-valued BCK/BCI-morphism, it follows that

$$a * b \in (t(x) * A) * (t(y) * A) = (t(x) * t(y)) * A \subseteq t(x * y) * A.$$

Thus $(t(x * y) * A) \cap S \neq \emptyset$, and so $x * y \in \overline{t}_A(S)$. Therefore $\overline{t}_A(S)$ is a subalgebra of X.

(2) Let $x, y \in \underline{t}_A(S)$. Then $t(x) * A \subseteq S$ and $t(y) * A \subseteq S$. Since t is strong and Y satisfies the right self-distributive law, we have

$$t(x * y) * A = (t(x) * t(y)) * A = (t(x) * A) * (t(y) * A) \subseteq S.$$

Hence $x * y \in \underline{t}_A(S)$, and therefore $\underline{t}_A(S)$ is a subalgebra of X.

(3) It is clear that $0 \in \overline{t}_A(B)$. Let $x * y \in \overline{t}_A(B)$ and $y \in \overline{t}_A(B)$. Then $(t(y) * A) \cap B \neq \emptyset$ and

$$\begin{aligned}((t(x) * A) \cap B) * ((t(y) * A) \cap B) &= ((t(x) * A) * (t(y) * A)) \cap B \\ &= ((t(x) * t(y)) * A) \cap B \neq \emptyset.\end{aligned}$$

Since $(t(y) * A) \cap B \neq \emptyset$, it follows that $(t(x) * A) \cap B \neq \emptyset$. Hence $x \in \overline{t}_A(B)$, and therefore $\overline{t}_A(B)$ is an ideal of X. ■

Let I be an ideal of a BCK/BCI-algebra Y and let $\rho(I)$ be a relation on Y related to I defined by

$$(x, y) \in \rho(I) \;\Leftrightarrow\; x * y \in I \text{ and } y * x \in I. \tag{3.14}$$

Then $\rho(I)$ is an equivalence relation on Y. Denote by $[x]_{\rho(I)}$ the equivalence class of x with respect to $\rho(I)$. Let (X, Y, t) be a generalized approximation space. Define a mapping

$$t_{\rho(I)} : X \to 2^Y,\; x \mapsto \{b \in [a]_{\rho(I)} \mid a \in t(x)\}. \tag{3.15}$$

Then $t_{\rho(I)}$ is a set-valued mapping and $t(x) \subseteq t_{\rho(I)}(x)$ for all $x \in X$.

Theorem 3.24. *If (X, Y, t) is a generalized approximation space in which t is a set-valued BCK/BCI-morphism, then the mapping $t_{\rho(I)}$ in* (3.15) *is a set-valued BCK/BCI-morphism and*

$$(\forall S, T \in 2^Y)\left(\overline{t}_{\rho(I)}(S) * \overline{t}_{\rho(I)}(T) \subseteq \overline{t}_{\rho(I)}(S * T)\right). \tag{3.16}$$

If $t_{\rho(I)}$ is strong, then

$$(\forall S, T \in 2^Y)\left(\underline{t}_{\rho(I)}(S) * \underline{t}_{\rho(I)}(T) \subseteq \underline{t}_{\rho(I)}(S * T)\right). \tag{3.17}$$

Proof. Let $x, y \in X$ and $z \in t_{\rho(I)}(x) * t_{\rho(I)}(y)$. Then there exist $z_x \in t_{\rho(I)}(x)$ and $z_y \in t_{\rho(I)}(y)$ such that $z = z_x * z_y$. It follows that $z_x \in [a]_{\rho(I)}$ and $z_y \in [b]_{\rho(I)}$ for some $a \in t(x)$ and $b \in t(y)$. Hence

$$z_x * z_y \in [a]_{\rho(I)} * [b]_{\rho(I)} \subseteq [a * b]_{\rho(I)}.$$

Since t is a set-valued BCK/BCI-morphism, we have $a * b \in t(x) * t(y) \subseteq t(x * y)$. Thus $z = z_x * z_y \in t_{\rho(I)}(x * y)$, which shows that $t_{\rho(I)}(x) * t_{\rho(I)}(y) \subseteq t_{\rho(I)}(x * y)$. Therefore $t_{\rho(I)}$ in (3.15) is a set-valued BCK/BCI-morphism. Let $z \in \overline{t}_{\rho(I)}(S) * \overline{t}_{\rho(I)}(T)$. Then $z = x * y$ for some $x \in \overline{t}_{\rho(I)}(S)$ and $y \in \overline{t}_{\rho(I)}(T)$. Hence $t_{\rho(I)}(x) \cap S \neq \emptyset$ and $t_{\rho(I)}(y) \cap T \neq \emptyset$. Taking $z_x \in t_{\rho(I)}(x) \cap S$ and $z_y \in t_{\rho(I)}(y) \cap T$ imply that $z_x * z_y \in S * T$, $z_x \in [a]_{\rho(I)}$ and $z_y \in [b]_{\rho(I)}$ for some $a \in t(x)$ and $b \in t(y)$. It follows that $z_x * z_y \in [a]_{\rho(I)} * [b]_{\rho(I)} \subseteq [a * b]_{\rho(I)}$ and $a * b \in t(x) * t(y) \subseteq t(x * y)$. Hence $z = x * y \in \overline{t}_{\rho(I)}(S * T)$. Therefore (3.16) is valid. Assume that $t_{\rho(I)}$ is strong and let $z \in \underline{t}_{\rho(I)}(S) * \underline{t}_{\rho(I)}(T)$. Then $z = x * y$ for some $x \in \underline{t}_{\rho(I)}(S)$ and $y \in \underline{t}_{\rho(I)}(T)$. Hence $t_{\rho(I)}(x) \subseteq S$ and $t_{\rho(I)}(y) \subseteq T$, and so

$$t_{\rho(I)}(x * y) = t_{\rho(I)}(x) * t_{\rho(I)}(y) \subseteq S * T.$$

Thus $z = x * y \in \underline{t}_{\rho(I)}(S * T)$. ■

Theorem 3.25. *Let (X, Y, t) be a generalized approximation space in which Y is a BCK-algebra and $t(x) \neq \emptyset$ for all $x \in X$. If I is an ideal of Y, then*

$$(\forall x \in X)\left(x \in \underline{t}(I) \iff t_{\rho(I)}(x) = I\right). \tag{3.18}$$

Proof. Let $x \in X$ be such that $x \in \underline{t}(I)$. If $z \in t_{\rho(I)}(x)$, then $z \in [a]_{\rho(I)}$ for some $a \in t(x) \subseteq I$. It follows that $z * a \in I$. Since I is an ideal of Y, we have $z \in I$ and so $t_{\rho(I)}(x) \subseteq I$. Let $y \in I$. Since $t(x) \neq \emptyset$, there exists $a \in t(x) \subseteq I$. Since I is an ideal and hence a subalgebra of X, we have $y * a \in I$ and $a * y \in I$, and so $y \in [a]_{\rho(I)}$. Hence $y \in t_{\rho(I)}(x)$. Conversely, let $y \in t(x)$. Since $(y, y) \in \rho(I)$, we get $y \in t_{\rho(I)}(x) = I$. Hence $t(x) \subseteq I$, that is, $x \in \underline{t}(I)$. ■

Before ending this article, we pose a question.

Question 3.26. *Let (X, Y, t) be a generalized approximation space in which t is a set-valued BCK/BCI-morphism. If A is an ideal of Y, then is the generalized upper approximation $\overline{t}(A)$ an ideal of X?*

Acknowledgements

The authors wish to thank the anonymous reviewers for their valuable suggestions.

REFERENCES

[1] B. Davvaz, *Roughness in rings*, Inform. Sci. **164** (2004) 147–163. https://doi.org/10.1016/j.ins.2003.10.001

[2] B. Davvaz, *A short note on algebraic T-rough sets*, Inform. Sci. 178 (2008) 3247–3252. https://doi.org/10.1016/j.ins.2008.03.014

[3] B. Davvaz and M. Mahdavipour, *Rough approximations in a general approximation space and their fundamental properties*, Int. J. Gen. Syst. **37** (2008), 373–386. https://doi.org/10.1080/03081070701250994

[4] W.A. Dudek, Y.B. Jun and H.S. Kim, *Rough set theory applied to BCI-algebras*, Quasigroups and Related Systems **9** (2002) 45–54.

[5] Y. Huang, BCI-algebra (Science Press, Beijing, China, 2006).

[6] K. Iséki, *On BCI-algebras*, Math. Seminar Notes **8** (1980) 125–130.

[7] K. Iséki and S. Tanaka, *An introduction to the theory of BCK-algebras*, Math. Japon. **23** (1978) 1–26.

[8] M. Kondo, *On the structure of generalized rough sets*, Inform. Sci. **176** (2006) 589–600. https://doi.org/10.1016/j.ins.2005.01.001

[9] M. Kryszkiewicz, *Rough set approach to incomplete information systems*, Inform. Sci. **112** (1998) 39–49. https://doi.org/10.1016/S0020-0255(98)10019-1

[10] N. Kuroki, *Rough ideals in semigroups*, Inform. Sci. **100** (1997) 139–163. https://doi.org/10.1016/S0020-0255(96)00274-5

[11] V. Leoreanu-Fotea and B. Davvaz, *Roughness in n-ary hypergroups*, Inform. Sci. **178** (2008) 4114–4124. https://doi.org/10.1016/j.ins.2008.06.019

[12] J. Liang, J. Wang and Y. Qian, *A new measure of uncertainty based on knowledge granulation for rough sets*, Inform. Sci. **179** (2009) 45–470. https://doi.org/10.1016/j.ins.2008.10.010

[13] J. Meng and Y.B. Jun, BCK-Algebras, Kyungmoon Sa Co. (Seoul, Korea, 1994).

[14] Z. Meng and Z. Shi, *A fast approach to attribute reduction in incomplete decision systems with tolerance relation-based rough sets*, Inform. Sci. **179** (2009) 2774–2793. https://doi.org/10.1016/j.ins.2009.04.002

[15] D.Q. Miao, Y. Zhao, Y.Y. Yao, H.X. Li and F.F. Xu, *Relative reducts in consistent and inconsistent decision tables of the Pawlak rough set model*, Inform. Sci. **179** (2009) 4140–4150. https://doi.org/10.1016/j.ins.2009.08.020

[16] J.N. Mordeson, *Rough set theory applied to (fuzzy) ideal theory*, Fuzzy Sets and Systems **121** (2001) 315–324. https://doi.org/10.1016/S0165-0114(00)00023-3

[17] Z. Pawlak, *Rough sets*, Int. J. Inform. Comp. Sci. **11** (1982) 341–356. https://doi.org/10.1007/BF01001956

[18] Z. Pawlak and A. Skowron, *Rudiments of rough sets*, Inform. Sci. **177** (2007) 3–27. https://doi.org/10.1016/j.ins.2006.06.003

[19] Z. Pawlak and A. Skowron, *Rough sets: some extensions*, Inform. Sci. **177** (2007) 28–40. https://doi.org/10.1016/j.ins.2006.06.006

[20] Z. Pawlak and A. Skowron, *Rough sets and boolean reasoning*, Inform. Sci. **177** (2007) 41–73. https://doi.org/10.1016/j.ins.2006.06.007

[21] D. Pei, *On definable concepts of rough set models*, Inform. Sci. **177** (2007) 4230–4239. https://doi.org/10.1016/j.ins.2007.01.020

[22] Y. Qian, J. Liang, D. Li, H. Zhang and C. Dang, *Measures for evaluating the decision performance of a decision table in rough set theory*, Inform. Sci. **178** (2008) 181–202. https://doi.org/10.1016/j.ins.2007.08.010

[23] A. Skowron and J. Stepaniuk, *Generalized approximation spaces*, in: T.Y. Lin, A.M. Wildberger (Eds.), Soft Computing, The Society for Computer Simulation (San Diego, 1995) 18–21.

[24] J. Stepaniuk, *Approximation spaces in extensions of rough set theory*, in: L. Polkowski, A. Skowron (Eds.), RSCTC98, LNAI **1424** (1998) 290–297. https://doi.org/10.1007/3-540-69115-4_40

[25] Q. Wang and J. Zhan, *Rough semigroups and rough fuzzy semigroups based on fuzzy ideals*, Open Math. 14 (2016) 1114–1121. https://doi.org/10.1515/math-2016-0102

[26] W.Z. Wu and W.X. Zhang, *Constructive and axiomatic approaches of fuzzy approximation operators*, Inform. Sci. **159** (2004) 233–254. https://doi.org/10.1016/j.ins.2003.08.005

[27] W.Z. Wu, J.S. Mi and W.X. Zhang, *Generalized fuzzy rough sets*, Inform. Sci. **151** (2003) 263–282. https://doi.org/10.1016/S0020-0255(02)00379-1

[28] [54] W.Z. Wu, Y. Leung and J.-S. Mi, *On characterizations of $(\mathcal{I},\mathcal{T})$-fuzzy rough approximation operators*, Fuzzy Sets and Systems **154** (2005) 76–102. https://doi.org/10.1016/j.fss.2005.02.011

[29] W.-Z. Wu and W.-X. Zhang, *Neighborhood operator systems and approximations*, Inform. Sci. **144** (2002) 201–217. https://doi.org/10.1016/S0020-0255(02)00180-9

[30] U. Wybraniec-Skardowska, *On a generalization of approximation space*, Bull. Polish Acad. Sci. Math. **37** (1989) 51–65.

[31] O.G. Xi, *Fuzzy BCK-algebra*, Math. Japon. **36** (1991) 935–942.

[32] Q.-M. Xiao and Z.-L. Zhang, *Rough prime ideals and rough fuzzy prime ideals in semigroups*, Inform. Sci. **176** (2006) 725–733.
https://doi.org/10.1016/j.ins.2004.12.010

[33] S. Yamak, O. Kazanci and B. Davvaz, *Generalized lower and upper approximations in a ring*, Inform. Sci. **180** (2010) 1759–1768.
https://doi.org/10.1016/j.ins.2009.12.026

[34] Y. Yao, *Three-way decisions with probabilistic rough sets*, Inform. Sci. **180** (2010) 341–353.
https://doi.org/10.1016/j.ins.2009.09.021

[35] Y.Y. Yao and T.Y. Lin, *Generalization of rough sets using modal logic*, Intel. Automat. Soft Comp., Int. J. **2** (1996) 103–120.
https://doi.org/10.1080/10798587.1996.10750660

[36] Y. Yao, *On generalizing Pawlak approximation operators*, LNAI **1424** (1998) 298–307.
https://doi.org/10.1007/3-540-69115-4_41

[37] Y.Y. Yao, *Constructive and algebraic methods of the theory of rough sets*, Inform. Sci. **109** (1998) 2–47.
https://doi.org/10.1016/S0020-0255(98)00012-7

[38] L.A. Zadeh, *Toward a generalized theory of uncertainty (GTU) – an outline*, Inform. Sci. **172** (2005) 1–40.
https://doi.org/10.1109/GRC.2005.1547227

[39] W. Zhu, *Relationship between generalized rough sets based on binary relation and covering*, Inform. Sci. **179** (2009) 210–225.
https://doi.org/10.1016/j.ins.2008.09.015

[40] W. Zhu, *Generalized rough sets based on relations*, Inform. Sci. **177** (22) (2007) 4997–5011.
https://doi.org/10.1016/j.ins.2007.05.037

[41] W. Zhu and Fei-Yue Wang, *On three types of covering rough sets*, IEEE Transactions On Knowledge and Data Engineering **19** (8) (2007) 1131–1144.
https://doi.org/10.1109/TKDE.2007.1044

[42] W. Zhu, *Topological approaches to covering rough sets*, Inform. Sci. **177** (2007) 1499–1508.
https://doi.org/10.1016/j.ins.2006.06.009

[43] W. Zhu, *Relationship among basic concepts in covering-based rough sets*, Inform. Sci. **179** (2009) 2478–2486.
https://doi.org/10.1016/j.ins.2009.02.013

Received 24 March 2019
First Revised 27 May 2019
Second Revised 3 June 2019
Accepted 23 September 2020