Discussiones Mathematicae
General Algebra and Applications 41 (2021) 395–409
https://doi.org/10.7151/dmgaa.1362

ON eGE-ALGEBRAS

R.K. Bandaru[1]

Department of Mathematics
GITAM (Deemed to be University)
Hyderabad Campus, India-502 329

e-mail: ravimaths83@gmail.com

N. Rafi

Department of Mathematics
Bapatla Engineering College
Bapatla, Andhra Pradesh, India-522 101

e-mail: rafimaths@gmail.com

and

A. Rezaei

Department of Mathematics
Payame Noor University
P.O. Box 19395-3697, Tehran, Iran

e-mail: rezaei@pnu.ac.ir

Abstract

A new algebraic structure was introduced, called an eGE-algebra, which is a generalisation of a GE-algebra and investigated its properties. We explore the definition of filters and the quotient algebra associated with such filters.

Keywords: BE-algebra, GE-algebra, eGE-algebra, transitive, filter.

2010 Mathematics Subject Classification: 06D20, 06F35, 03G25.

[1]Corresponding author.

1. Introduction and preliminaries

L. Henkin and T. Skolem developed the idea of Hilbert algebra in the early 50-ties for some investigations of implication in intuitionistic and other nonclassical logics. In 60-ties, these algebras were particularly studied by Horn and Diego [6] from algebraic point of view. Hilbert algebras are a valuable tool for some algebraic logic investigations as they can be regarded as fragments of any propositional logic that contains a logical connective implication ($\to$) and the constant 1 that is assumed to be the logical meaning "true". Many researchers have done a significant amount of work on Hilbert algebras [3–5, 7–9, 13–16]. As a generalization of Hilbert algebras, Bandaru *et al.* [1] introduce the notion of GE-algebras. They studied the various properties and filter theory of GE-algebras. BCK-algebras and BCI-algebras were introduced by Imai and Iseki [10, 11]. H.S. Kim and Y.H. Kim [12] developed the concept of BE-algebra as a generalization of dual BCK-algebra. Many researchers developed theory of BE-algebras [2, 17–19]. Rezaei [20] has introduced the notion of eBE-algebra as a generalization of BE-algebra and has studied some of its properties. It is important to make clear the corresponding algebraic structures for the creation of many-valued logical system. As a generalization of GE-algebra, we are inspired to concentrate on a new algebraic structure, called eGE-algebra, and thus to investigate some properties.

Definition 1.1 [1]**.** Let X be a non-empty set with a constant 1 and $*$ a binary operation on X. Then an algebraic structure $(X, *, 1)$ of type $(2, 0)$ is said to be a GE-algebra if it satisfies the following axioms:

(GE1) $u * u = 1$,

(GE2) $1 * u = u$,

(GE3) $u * (v * w) = u * (v * (u * w))$

for all $u, v, w \in X$.

In a GE-algebra X, a binary relation "$\leq$" is defined by

$$(\forall u, v \in X)\,(u \leq v \;\Leftrightarrow\; u * v = 1)\,. \tag{1}$$

Proposition 1.2 [1]**.** Every GE-algebra X satisfies the following items.

$$(\forall u \in X)\,(u * 1 = 1)\,. \tag{2}$$

$$(\forall u, v \in X)\,(u * (u * v) = u * v)\,. \tag{3}$$

$$(\forall u, v \in X)\,(u \leq v * u)\,. \tag{4}$$

Definition 1.3 [1]**.** A GE-algebra X is said to be transitive, if it satisfies:

$$(\forall x, y, z \in X)\,(x * y \leq (z * x) * (z * y))\,. \tag{5}$$

Proposition 1.4 [1]**.** Every transitive GE-algebra X satisfies the following assertions.

$$(6)\qquad (\forall x, y, z \in X)\,(x * y \leq (y * z) * (x * z))\,.$$

$$(7)\qquad (\forall x, y, z \in X)\,(x \leq y \;\Rightarrow\; z * x \leq z * y,\; y * z \leq x * z)\,.$$

Definition 1.5 [1]**.** A subset F of a GE-algebra X is called a *filter* of X if it satisfies:

$$(8)\qquad 1 \in F,$$

$$(9)\qquad (\forall x, y \in X)(x * y \in F,\; x \in F \;\Rightarrow\; y \in F).$$

Lemma 1.6 [1]**.** *In a GE-algebra X, every filter F of X satisfies:*

$$(10)\qquad (\forall x, y \in X)\,(x \leq y,\; x \in F \;\Rightarrow\; y \in F)\,.$$

Definition 1.7 [20]**.** Let X be a non-empty set. By an eBE-algebra we shall mean an algebra $(X, *, A)$ such that "$*$" is a binary operation on X and A is a non-empty subset of X satisfying the following axioms:

(eBE1) $x * x \in A$,
(eBE2) $x * A \subseteq A$,
(eBE3) $A * x = \{x\}$,
(eBE4) $x * (y * z) = y * (x * z)$

for all $x, y, z \in X$.

Definition 1.8 [20]**.** An eBE-algebra X is said to be self distributive if it satisfies:

$$(11)\qquad (\forall x, y, z \in X) x * (y * z) = (x * y) * (x * z).$$

2. On eGE-algebras

In this section, we present the notion of eGE-algebra as a generalization of GE-algebra and study its properties.

Definition 2.1. An algebraic structure $(X, *, E)$, where $*$ is a binary operation on a non-empty set X and E is a non-empty subset of X, is said to be an extended GE-algebra (eGE-algebra for short) if it satisfies the following axioms:

(eGE1) $u * u \in E$,
(eGE2) $u * E \subseteq E$,
(eGE3) $E * u = \{u\}$,
(eGE4) $u * (v * w) = u * (v * (u * w)$

for all $u, v, w \in X$.

Throughout the paper, $E * u = \{e * u \mid e \in E\}$ and $u * E = \{u * e \mid e \in E\}$. If $a, b \in E$ then, by (eGE3), we have $a * b = b \in E$ and $b * a = a \in E$. Hence E is a closed subset of X.

We introduce a relation $\leq$ on X by $u \leq v$ if and only if $u * v \in E$. By (eGE1) the relation $\leq$ is reflexive.

Theorem 2.2. *Every GE-algebra is an eGE-algebra.*

Proof. Put $E = \{1\}$. Then $(X, *, E)$ is an eGE-algebra. ■

Every eGE-algebra need not be a GE-algebra which is shown in the following example.

Example 2.3. Let $X = \{a, b, c, d, e\}$ be a set and $*$ a binary operation given in the table:

$*$	a	b	c	d	e
a	c	b	c	c	b
b	d	d	d	d	d
c	a	b	c	d	e
d	a	b	c	d	e
e	a	c	c	d	c

Then $(X, *, E)$, where $E = \{c, d\}$, is an eGE-algebra, but not a GE-algebra. Since $b * b = d$ and $c * c = c$ and there is no $1 \in X$, such that $u * u = 1$, for all $u \in X$.

Note that the relation $\leq$ need not be transitive in an eGE-algebra. From Example 2.3, we can observe that $e * b = c \in E, b * a = d \in E$, but $e * a = a \notin E$.

In the following example, we show that the axioms (eGE1) to (eGE4) are independent.

Example 2.4. (i) Let $X = \{a, b, c, d\}$ be a set and $*$ a binary operation on X given in the following table:

$*$	a	b	c	d
a	b	b	d	d
b	a	b	c	c
c	a	b	b	b
d	a	b	b	b

Then $(X, *, E)$, where $E = \{a, b\}$, satisfies (eGE2), (eGE3) and (eGE4), but it does not satisfy (eGE1), since $E * u \neq \{u\}$ i.e., $a * c \neq c$ and $a \in E$.

(ii) Let $X = \{a, b, c, d\}$ be a set and $*$ a binary operation on X in the following table:

$*$	a	b	c	d
a	b	b	c	c
b	a	b	c	d
c	a	b	c	d
d	a	a	c	c

Then $(X, *, E)$, where $E = \{b, c\}$, satisfies (eGE1), (eGE3) ansd (eGE4), but it does not satisfy (eGE2), since $u * E \nsubseteq E$, i.e., $d * b = a \notin E$ and $b \in E$.

(iii) Let $X = \{a, b, c, d\}$ be a set and $*$ a binary operation on X given in the following table:

$*$	a	b	c	d
a	b	b	b	b
b	a	b	c	d
c	a	b	c	d
d	a	c	c	a

Then $(X, *, E)$, where $E = \{b, c\}$ satisfies (eGE1), (eGE2) and (eGE4), but it does not satisfy (eGE3), since $d * d = a \notin E$.

(iv) Let $X = \{a, b, c, d\}$ be a set and $*$ a binary operation on X given in the following table:

$*$	a	b	c	d
a	c	b	c	c
b	b	c	c	c
c	a	b	c	d
d	a	b	c	d

Then $(X, *, E)$, where $E = \{c, d\}$, satisfies (eGE1), (eGE2) and (eGE3), but it does not satisfy (eGE4), since

$$a * (b * a) = a * b = b \neq c = a * c = a * (b * c) = a * (b * (a * a)).$$

Theorem 2.5. *Let $(X, *, E)$ be an eGE-algebra. If E is a singleton set, then $(X, *, E)$ is a GE-algebra.*

Proof. Let $E = \{a\}$ be a singleton set. If we put $1 = a$, then $(X, *, 1)$ is a GE-algebra. ■

Theorem 2.6. *Let $(X, *, E_i)$, for $i = 1, 2$, be two eGE-algebras. Then $(X, *, E_1 \cap E_2)$ is also an eGE-algebra.*

Proof. Let $u \in X$. Since $u * u \in E_1$ and $u * u \in E_2$, we have $u * u \in E_1 \cap E_2$, and so (eGE1) holds. Let $a \in u * (E_1 \cap E_2)$. Then, we can find $b \in E_1 \cap E_2$ such that $a = u * b$. Since $b \in E_1,\ u * b \in E_1$ and $b \in E_2,\ u * b \in E_2$, we have $a = u * b \in E_1 \cap E_2$ and so $u * (E_1 \cap E_2) \subseteq E_1 \cap E_2$. Hence (eGE2) holds. Let $a \in (E_1 \cap E_2) * u$. Then, we can find $b \in E_1 \cap E_2$ such that $a = b * u$. Since $b * u = u$, we have $a = u$, and so $(E1 \cap E_2) * u = \{u\}$. Hence (eGE3) holds. (eGE4) is obvious. Thus $(X, *, E_1 \cap E_2)$ is also an eGE-algebra. ■

Corollary 2.7. *If $(X, *, E_i)$, for $i \in \Lambda$, is a family of eGE-algebras, then $\left(X, *, \bigcap_{i \in \Lambda} E_i\right)$ is an eGE-algebra.*

Theorem 2.8. *Let $(X, *, E_i)$, for $i = 1, 2$, be two eGE-algebras. Then $(X, *, E_1 \cup E_2)$ is also an eGE-algebra.*

Proof. Let $u \in X$. Since $u * u \in E_1$ and $u * u \in E_2$, we have $u * u \in E_1 \cup E_2$ and so (eGE1) holds. For (eGE2), let $a \in u * (E_1 \cup E_2)$. Then, we can find $b \in E_1 \cup E_2$ such that $a = u * b$. If $b \in E_1$, then $a \in E_1$. Also, if $b \in E_2$, then $a \in E_2$. Thus $a \in E_1 \cup E_2$ and so $u * (E_1 \cup E_2) \subseteq E_1 \cup E_2$. Let $a \in (E_1 \cup E_2) * u$. Then, we can find $b \in E_1 \cup E_2$ such that $a = b * u$. Since $b * u = u$, we have $a = u$ and so $(E_1 \cup E_2) * u = \{u\}$. Therefore (eGE3) holds. (eGE4) is obvious. Thus $(X, *, E_1 \cup E_2)$ is an eGE-algebra. ■

Corollary 2.9. *If $(X, *, E_i)$, for $i \in \Lambda$, is a family of eGE-algebras, then $\left(X, *, \bigcup_{i \in \Lambda} E_i\right)$ is also an eGE-algebra.*

Lemma 2.10. *Let $(X, *, E)$ be an eGE-algebra and $u, v \in X$. Then $u * (u * v) = u * v$.*

Proof. Let $u, v \in X$. Using $(eGE1), (eGE3)$ and $(eGE4)$, we get

$$u * (u * v) = u * ((u * u) * (u * v)) = u * ((u * u) * v) = u * v.$$

■

Theorem 2.11. *Every self-distributive eBE-algebra is an eGE-algebra.*

Proof. Let $(X, *, E)$ be a self-distributive eBE-algebra and $u, v, w \in X$. Then, by (eBE1), (eBE3), (eBE4), and self-distributivity,

$$u * (v * w) = (u * u) * (u * (v * w)) = u * (u * (v * w)) = u * (v * (u * w)).$$

Hence X is an eGE-algebra. ■

The converse of the Theorem 2.11 does not have to be true. From Example 2.3, we can observe that X is an eGE-algebra, but not a self-distributive eBE-algebra.

Theorem 2.12. *Let $(X, *, E)$ be an eBE-algebra having the property $u*(u*v) = u*v$, for all $u, v \in X$. Then X is an eGE-algebra.*

Proof. Let $u, v, w \in X$ and $u*(u*v) = u*v$. Then $u*(v*w) = u*(u*(v*w)) = u*(v*(u*w))$. Hence X is an eGE-algebra. ∎

Proposition 2.13. *Let $(X, *, E)$ be an eGE-algebra. Then*

(i) *$(X; *, X \setminus E)$ is not an eGE-algebra,*

(ii) *$v * w \in E$ implies $u * (v * w) \in E$,*

(iii) *$u * (v * u) \in E$,*

(iv) *$u \leq v * w$ implies $v \leq u * w$,*

(v) *$u \leq (u * v) * u$,*

(vi) *$u * (v * w) \in E$ implies $v * (u * w) \in E$ and $v * (u * (v * w)) \in E$,*

(vii) *$u * (v * w) \leq v * (u * w)$,*

(viii) *$u * (v * w) \notin E$ implies $u * w \notin E$ for all $u, v, w \in X$.*

Proof. (i) (eGE2) does not hold, since $u * E \nsubseteq X \setminus E$ and $u * E \subseteq E$.

(ii) By (eGE2), (ii) is obvious.

(iii) Using (eGE4), (eGE1) and (eGE2), we have

$$u * (v * u) = u * (v * (u * u)) \in u * E \subseteq E.$$

(iv) Let $u \leq v * w$. Hence $u * (v * w) \in E$. Then, by (eGE4) and (eGE2), we have $v * (u * w) = v * (u * (v * w)) \in v * E \subseteq E$. Therefore $v \leq u * w$.

(v) From (eGE4), (eGE1) and (eGE2) we have

$$u * ((u * v) * u) = u * ((u * v) * (u * u)) \in u * E \subseteq E.$$

Therefore $u \leq (u * v) * u$.

(vi) Applying (iv) and (eGE4), we can prove (vi).

(vii) By routine calculation we can see that

$$\begin{aligned}
&(u * (v * w)) * (v * (u * w)) \\
&= (u * (v * w)) * (v * ((u * (v * w)))) \\
&= (u * (v * w)) * (v * ((u * (v * w)) * (u * (v * w)))) \in E.
\end{aligned}$$

Thus $u * (v * w) \leq v * (u * w)$.

(viii) It is obvious by (ii). ∎

Theorem 2.14. *Let $(X, *, E)$ be an eGE-algebra. The following are equivalent.*

(i) *$u * v \leq (w * u) * (w * v)$,*

(ii) *$u * v \leq (v * w) * (u * w)$*

for all $u, v, w \in X$.

Proof. (i)⇒(ii) Let $u, v, w \in X$ and assume (i). Then

$$(u * v) * ((w * u) * (w * v)) \in E.$$

Hence, by (eGE4) and (eGE2), we get

$$(u * v) * ((v * w) * (u * w)) = (u * v) * ((v * w) * ((u * v) * (u * w))) \in E.$$

Therefore $u * v \leq (v * w) * (u * w)$.

(ii)⇒(i) Let $u, v, w \in X$ and assume (ii). Then $(u * v) * ((v * w) * (u * w)) \in E$. Hence, by (eGE4) and (eGE2), we get

$$(u * v) * ((w * u) * (w * v)) = (u * v) * ((w * u) * ((u * v) * (w * v))) \in E.$$

Therefore $u * v \leq (w * u) * (w * v)$. ■

Definition 2.15. An eGE-algebra $(X, *, E)$ is said to be transitive if it satisfies:

$$(\forall u, v, w \in X)\,(u * v \leq (w * u) * (w * v)). \tag{12}$$

Example 2.16. Let $X = \{a, b, c, d\}$ be a set and $*$ a binary operation given in the following table:

*	a	b	c	d
a	d	d	d	d
b	a	c	c	c
c	a	b	c	d
d	a	b	c	d

Then $(X, *, E)$, where $E = \{c, d\}$, is a transitive eGE-algebra but not an eBE-algebra, since $a * (b * c) = a * c = d \neq c = b * d = b * (a * c)$.

The following theorem can be proved easily.

Theorem 2.17. *Let $(X, *, E)$ be a transitive eGE-algebra. The following hold:*

(1) $u \leq v$ *implies* $w * u \leq w * v$,

(2) $u * v \leq (v * w) * (u * w)$,

(3) $u \leq v$ *implies* $v * w \leq u * w$,

(4) $((u * v) * v) * w \leq u * w$,

(5) $u \leq v$ *and* $v \leq w$ *imply* $u \leq w$,

(6) $u * (v * w) \leq (u * v) * (u * w)$

for all $u, v, w \in X$.

Theorem 2.18. *Let $(X,*,E)$ be an eGE-algebra. Consider $Y := (X \setminus E) \cup \{1\}$ and define the operation $\rhd$ on Y as follows:*

$$u \rhd v = \begin{cases} u * v & \text{if } u, v \neq 1 \text{ and } u * v \notin E, \\ 1 & \text{if } u, v \neq 1 \text{ and } u * v \in E, \\ v & \text{if } u = 1, \\ 1 & \text{if } v = 1. \end{cases}$$

Then $(Y, \rhd, 1)$ is a GE-algebra.

Proof. By (eGE1), $u * u \in E$, for all $u \in X$. Thus $u \rhd u = 1$, for all $u \in Y$, and so (GE1) holds. By definition of $\rhd$, (GE2) hold. To prove $(Y; \rhd, 1)$ is a GE-algebra it is sufficient to prove that $u \rhd (v \rhd w) = u \rhd (v \rhd (u \rhd w))$, for all $u, v, w \in Y$. If $u = 1$ or $v = 1$ or $w = 1$, then we have $u \rhd (v \rhd w) = u \rhd (v \rhd (u \rhd w))$. Now, let $u, v, w \neq 1$. If $v * w \in E$, then $v \rhd w = 1$, and so $u \rhd (v \rhd w) = 1$. On the other hand, if $u * w \in E$, then $u \rhd w = 1$ and $u \rhd (v \rhd (u \rhd w)) = u \rhd (v \rhd 1) = u \rhd 1 = 1 = u \rhd (v \rhd w)$. If $u * w \notin E$, then $u * w = u \rhd w$. By Proposition 2.13(ii), and $v * w \in E$, we have $v * (u * w) \in E$. Hence $u \rhd (v \rhd (u \rhd w)) = u \rhd 1 = 1 = u \rhd (v \rhd w)$. If $v * w \notin E$, then $v \rhd w = v * w$. We have two cases: $u * (v * w) \in E$ or $u * (v * w) \notin E$. If $u * (v \rhd w) = u * (v * w) \in E$, then $u \rhd (v \rhd w) = 1$. By Proposition 2.13(vi), $v * (u * w) \in E$, and so $u \rhd (v \rhd (u \rhd w)) = u \rhd (v \rhd (u * w)) = u \rhd 1 = 1$. Thus $u \rhd (v \rhd w) = u \rhd (v \rhd (u \rhd w)$, in this case. If $u * (v * w) \notin E$ then, by Proposition 2.13(ii & viii), $u * w \notin E, v * w \notin E$ and $v * (u * w) \notin E$. So that $u \rhd w = u * w, v \rhd w = v * w$ and $v \rhd (u * w) = v * (u * w)$. Hence $u \rhd (v \rhd w) = u * (v * w)$. Also, by (eGE4), $u * (v * (u * w)) = u * (v * w) \notin E$. Hence $u \rhd (v \rhd (u \rhd w)) = u * (v \rhd (u \rhd w)) = u * (v * (u * w)) = u * (v * w) = u \rhd (v \rhd w)$. Thus $u \rhd (v \rhd w) = u \rhd (v \rhd (u \rhd w))$. Therefore $(Y, \rhd, 1)$ is a GE-algebra. ∎

Example 2.19. Let $X = \{a, b, c, d\}$ and $E = \{c, d\}$. A binary operation $*$ on X is given in the following table:

$*$	a	b	c	d
a	c	b	c	c
b	d	d	d	d
c	a	b	c	d
d	a	b	c	d

Then $(X, *, E)$ is an eGE-algebra which is not an eBE-algebra.

Now, $Y = (X \setminus E) \cup \{1\} = \{1, a, b\}$. Define $\rhd$ on Y with the following table:

$\rhd$	1	a	b
1	1	a	b
a	1	1	b
b	1	1	1

Then $(Y, \triangleright, 1)$ is a GE-algebra.

We conclude this section with the following theorem whose proof is straightforward.

Theorem 2.20. *Let $(X, *, 1)$ be a GE-algebra and E_0 be a set such that $E_0 \cap X = \emptyset$. If we define $Y = X \cup E_0$, $E = E_0 \cup \{1\}$ and define the operation $\triangleleft$ on Y as follows:*

$$x \triangleleft y = \begin{cases} x * y & \text{if } x, y \notin A_0, \\ y & \text{otherwise.} \end{cases}$$

Then $(Y, \triangleleft, E)$ is an eGE-algebra.

3. Quotient eGE-algebras

In this section, we introduce the notion of a filter in an eGE-algebra and study its properties. We construct a quotient eGE-algebra via a filter of an eGE-algebra. Throughout this section, X means $(X, *, E)$ is an eGE-algebra, unless specified otherwise.

Definition 3.1. A subset F of X is called a filter of X if it satisfies:
(eGEF1) $E \subseteq F$,
(eGEF2) $u \in F$ and $u * v \in F$ imply $v \in F$.

The set of all filters of X will be denoted by $\mathcal{F}(X)$. Clearly, $\mathcal{F}(X) \neq \emptyset$, since $X \in \mathcal{F}(X)$.

Example 3.2. Let $X = \{a, b, c, d, e\}$ be a set and $*$ a binary operation on X given in the following table:

$*$	a	b	c	d	e
a	c	c	c	c	c
b	a	d	d	d	e
c	a	b	c	d	e
d	a	b	c	d	e
e	a	b	c	d	d

Then $(X, *, E)$, where $E = \{c, d\}$, is an eGE-algebra. Let $F = \{c, d, e\}$. Then $F \in \mathcal{F}(X)$.

Proposition 3.3. *Let $F \in \mathcal{F}(X)$. If $u \in F$ and $u \leq v$, then $v \in F$.*

Proof. Let $u \in F$ and $u \leq v$. Then $u * v \in E$ and $E \subseteq F$. So that $u * v \in F$. Hence $v \in F$, since $u \in F$ and F is a filter of X. ■

Theorem 3.4. *In X, $E \in \mathcal{F}(X)$.*

Proof. Clearly (eGEF1) holds, since $E \subseteq E$. Now we prove (eGEF2). Let $u, u * v \in E$. Now, by (eGE3), we have $v = u * v \in E$. Therefore $E \in \mathcal{F}(X)$. ∎

Proposition 3.5. *If $F_i \in \mathcal{F}(X)$, for $i \in \Lambda$, then $\bigcap_{i\in\Lambda} F_i \in \mathcal{F}(X)$.*

Theorem 3.6. *Let $F \in \mathcal{F}(X)$. Then $F_1 = (F \setminus E) \cup \{1\}$ is a filter of $(Y, \rhd, 1)$, which is defined in Theorem 2.18.*

Proof. Clearly $1 \in F_1$. Let $u \in F_1$ and $u \rhd v \in F_1$. If $u = 1$, then $v = 1 \rhd v \in F_1$. Let $u \neq 1$. If $v = 1$, then $v \in F_1$. If $v \neq 1$. Then $u \in F \setminus E$ and $v \in X \setminus E$. If $u \rhd v = 1$ by definition of $\rhd$ we get $u * v \in E$. Then $u * v \in F$, since F is a filter of X. Hence $v \in F$. Thus $v \in F_1$. If $u \rhd v \neq 1$, then by definition of $\rhd$, $u * v \notin E$ and $u \rhd v = u * v \in F_1$. Thus $u * v \in F$. Since $F \in \mathcal{F}(X)$, we have $v \in F$. Hence $v \in F_1$. Therefore $F_1 \in \mathcal{F}(Y)$. ∎

Example 3.7. From Theorem 2.18 and Example 2.19, we get $Y = \{1, a, b\}$ with the following table:

$\rhd$	1	a	b
1	1	a	b
a	1	1	b
b	1	1	1

which is a GE-algebra. We can observe that $F = \{a, c, d\}$ is a filter of $(X, *, E)$ and $F_1 = (F \setminus E) \cup \{1\} = \{1, a\}$ is a filter of $(Y, \rhd, 1)$.

The following theorem can be proved easily.

Theorem 3.8. *Let $(X, *, 1)$ be a GE-algebra, $F \in \mathcal{F}(X)$ and E_0 be a set such that $X \cap E_0 = \emptyset$. Then $F_0 = F \cup E_0$ is a filter of an eGE-algebra $(Y, \lhd, E)$, which is defined in Theorem 2.20.*

The following example decribes the above theorem

Example 3.9. Let $X = \{1, a, b\}$ and $E_0 = \{c, d\}$. According to Example 2.19, $(X, \rhd, 1)$ is a GE-algebra. We can observe that $F = \{1, a\}$ is a filter of X. By Theorem 2.20, we get $Y = \{1, a, b, c, d\}, E = \{1, c, d\}$ and $(Y, \lhd, E)$ is an eGE-algebra with the following table:

$\lhd$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	1	1	c	d
c	1	a	b	c	d
d	1	a	b	c	d

We can observe that $F_0 = F \cup E_0 = \{1, a, c, d\}$ is a filter of Y.

Proposition 3.10. *A non-empty subset F of an eGE-algebra X is a filter of X if and only if it satisfies:*

(i) $E \subseteq F$,

(ii) $u * (v * w) \in F, v \in F$ *implies* $u * w \in F$

for all $u, v, w \in X$.

Proof. Suppose $F \in \mathcal{F}(X)$. Then $E \subseteq F$. Let $u, v, w \in X$ be such that $u * (v * w) \in F$ and $v \in F$. Then, by Theorem 2.13(vii) and Proposition 3.3, we have $v * (u * w) \in F$. Then $u * w \in F$. Conversely, assume that the conditions hold. It is sufficient to prove (eGEF2). Let $x \in F$ and $x * y \in F$. Then $x * x \in E \subseteq F$ and $(x * x) * (x * y) = x * y \in F$. Hence $(x * x) * y = y \in F$. Thus $F \in \mathcal{F}(X)$. ■

Theorem 3.11. *Let F be a subset of X satisfying the following conditions:*

(eGEF1) $E \subseteq F$,

(eGEF3) $u \in X$ *and* $r \in F$ *imply* $u * r \in F$,

(eGEF4) $u \in X, r, s \in F$ *imply* $(r * (s * u)) * u \in F$.

Then $F \in \mathcal{F}(X)$.

Proof. It is sufficient to prove (eGEF2). Let $u \in F$ and $u * v \in F$. Then, by (eGE1), (eGE3) and (eGEF4), $v = [(u * v) * (u * v)] * v \in F$ and hence (eGEF2) holds. Therefore $F \in \mathcal{F}(X)$. ■

Theorem 3.12. *If X is an eGE-algebra and F is a filter of X, then F satisfies* (eGEF1), (eGEF3) *and* (eGEF4).

Proof. It is sufficient to prove (eGEF3) and (eGEF4). Let $F \in \mathcal{F}(X)$ and $r \in F, u \in X$. Then $r*(u*r) \in E \subseteq F$ and hence, by (GEF2), $u*r \in F$. Let $r, s \in F$. Since $r*((r*(s*u))*(s*u)) \in E \subseteq F$ and $r \in F$, we have $(r*(s*u))*(s*u) \in F$. Hence, by (eGE4) and (eGEF3), $s*((r*(s*u))*u) = s*((r*(s*u))*(s*u)) \in F$. Thus, by (eGEF2), $(r * (s * u)) * u \in F$. ■

Theorem 3.13. *Let $F \in \mathcal{F}(X)$. Then $(r * u) * u \in F$ for all $r \in F$ and $u \in X$.*

For a non-empty subset I of X, we define the binary relation $\sim_I$ in the following way:

$$u \sim_I v \text{ if and only if } u * v \in I \text{ and } v * u \in I.$$

The set $\{s \mid r \sim_I s\}$ will be denoted by $[r]_I$.

Lemma 3.14. *In the above relation $\sim_I$, if $E \subseteq I$ and $r \in E$, then $[r]_I = I$.*

Proof. Let $u \in I$ and $r \in E$. By (eGE3), we have $r * u \in E * u = \{u\} \subseteq I$ and so $r * u \in I$. From (eGE2), we have $u * r \in u * E \subseteq E \subseteq I$, then $u * r \in I$. Hence $r \sim_I u$. Therefore $I \subseteq [r]_I$. Conversely, let $r \in E$ and $u \in [r]_I$. Then $u \sim_I r$ and so $u * r \in I$ and $r * u = u \in I$. Hence $[r]_I \subseteq I$. Therefore $[r]_I = I$. ■

Theorem 3.15. *Let $(X, *, E)$ be a transitive eGE-algebra and $F \in \mathcal{F}(X)$. Then $\sim_F$ is a congruence relation on X.*

Proof. Since $u * u \in E \subseteq F$, we have $u * u \in F$, and so $u \sim_F u$. If $u \sim_F v$, then clearly $v \sim_F u$. Now, let $u \sim_F v$ and $v \sim_F w$. Then $u * v, v * u \in F$ and $v*w, w*v \in F$. By Proposition 2.17(1), we have $v*w \leq (u*v)*(u*w)$, and so by Proposition 3.3, we have $(u*v)*(u*w) \in F$. Since F is a filter and $u*v \in F$, we have $u * w \in F$. Similarly, we can prove that $w * u \in F$. Thus $u \sim_F w$. Therefore $\sim_F$ is an equivalent relation on X. If $r \sim_F s$ and $u \sim_F v$, then $r * s, s * r \in F$ and $u * v, v * u \in F$. By Proposition 2.17(1), we have $u * v \leq (r * u) * (r * v)$ and $v * u \leq (r * v) * (r * u)$, and so by Proposition 3.3, we have $(r * u) * (r * v) \in F$ and $(r * v) * (r * u) \in F$. Thus $r * u \sim_F r * v$. Similarly, we can prove that $r * v \sim_F s * v$. Since the relation $\sim_F$ is transitive, we have $r * u \sim_F s * v$ which proves that $\sim_F$ is a congruence relation on X. ■

Proposition 3.16. *Let $\sim_G$ be a congruence relation on X, $E \subseteq G$ and $r \in E$. Then $[r]_G \in \mathcal{F}(X)$.*

Proof. By Lemma 3.14, we have $[r]_G = G$. Let $u, u * v \in [r]_G$. Thus $u \sim_G r$ and $u * v \sim_G r$. Since $v \sim_G v$ and $\sim_G$ is a congruence relation, we can observe that $r \sim_G u * v \sim_G r * v = v$ (by (eGE3)). Thus $v \in [r]_G$. Therefore $[r]_G \in \mathcal{F}(X)$. ■

Denote $\frac{X}{\sim_G} = \{[u]_G \mid u \in X\}$. Define a binary operation $\bullet$ on $\frac{X}{\sim_G}$ by $[u]_G \bullet [v]_G := [u*v]_G$. Then by above theorem, $\bullet$ is well defined. The following theorem shows that for a transitive eGE-algebra $(X, *, E), r \in E$ and $F \in \mathcal{F}(X)$, the quotient algebra $\left(\frac{X}{\sim_{[r]_F}}, \bullet, [r]_F\right)$ is a GE-algebra.

Theorem 3.17. *Let $(X, *, E)$ be a transitive eGE-algebra, $F \in F(X)$ and $r \in E$. Then $\left(\frac{X}{\sim_{[r]_F}}; \bullet, [r]_F\right)$ is a GE-algebra.*

Proof. Since $E \subseteq F$, we can observe that $E \subseteq [r]_F$, for all $r \in E$. Hence $[r]_F$ is a filter by Proposition 3.16 and so $\sim_{[r]_F}$ is a congruence relation on X by Theorem 3.15. Now, we have

(GE1) $[u]_F \bullet [u]_F = [u * u]_F = [r]_F$, since $u * u \in E \subseteq [r]_F$,

(GE2) $[r]_F \bullet [u]_F = [r * u]_F = [u]_F$, since $E * u = \{u\}$ and so $r * u = u$,

(GE3) $[u]_F \bullet ([v]_F \bullet [w]_F) = [u]_F \bullet [v*w]_F = [u*(v*w)]_F = [u*(v*(u*w))]_F = [u]_f \bullet [(v * (u * w))]_F = [u]_F \bullet ([v]_F \bullet [u * v]_F) = [u]_F \bullet ([v]_F \bullet ([u]_F \bullet [w]_F))$.

Thus $\left(\frac{X}{\sim_{[r]_F}}, \bullet, [r]_F\right)$ is a GE-algebra. ■

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Received 15 September 2020
Revised 24 October 2020
Accepted 24 October 2020