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APPLYING THE CZÉDLI-SCHMIDT SEQUENCES TO CONGRUENCE PROPERTIES OF PLANAR SEMIMODULAR LATTICES

G. Grätzer

Department of Mathematics
University of Manitoba
Winnipeg, MB R3T 2N2, Canada

e-mail: gratzer@me.com

Abstract

Following Grätzer and Knapp, 2009, a planar semimodular lattice L is *rectangular*, if the left boundary chain has exactly one doubly-irreducible element, c_l, and the right boundary chain has exactly one doubly-irreducible element, c_r, and these elements are complementary.

The Czédli-Schmidt Sequences, introduced in 2012, construct rectangular lattices. We use them to prove some structure theorems. In particular, we prove that for a slim (no M_3 sublattice) rectangular lattice L, the congruence lattice $\operatorname{Con} L$ has exactly $\operatorname{length}[c_l, 1] + \operatorname{length}[c_r, 1]$ dual atoms and a dual atom in $\operatorname{Con} L$ is a congruence with exactly two classes. We also describe the prime ideals in a slim rectangular lattice.

Keywords: lattice, congruence, semimodular, planar, slim.

2010 Mathematics Subject Classification: Primary: 06C10, Secondary: 06B10.

1. Introduction

1.1. The Czédli-Schmidt Sequences

Czédli and Schmidt [10] proved the Structure Theorem for Slim Rectangular Lattices, according to which every slim rectangular lattice can be constructed from a planar distributive lattice, a grid, with the Czédli-Schmidt Sequences, see Section 2.2 for the definitions. In this paper, we find new applications for the Czédli-Schmidt Sequences.

1.2. Congruence lattices of SPS lattices

The topic of this paper started in Grätzer, Lakser and Schmidt [23], where we proved that every finite distributive lattice D can be represented as the congruence lattice of a PS (planar semimodular) lattice L. The sublattices M_3 played a crucial role in the construction of L, so we asked ([16, Problem 1] and [14, Problems 4.7–4.10]) what happens if we only consider SPS lattices (Slim PS, where "slim"means that there is no M_3 sublattice)?

1.3. The Two-cover Theorem

In [16, Theorem 5], I proved the Two-cover Theorem. The congruence lattice of an SPS lattice has the property

(2C) Every join-irreducible congruence has at most two join-irreducible covers (in the order of join-irreducible congruences).

See also [14, Theorem 25.2], Czédli [6, Theorem 1.1], and my paper [17].

Czédli [6, Theorem 1.1] proved that the converse is false by exhibiting an eight-element distributive lattice, D_8 (see Figure 1), satisfying (2C), which cannot be represented as the congruence lattice of an SPS lattice; see also my paper [15].

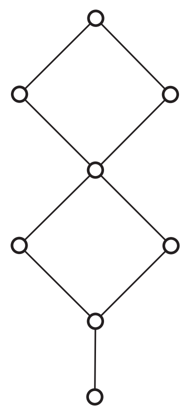

Figure 1. The distributive lattice D_8.

In [17], I observed that the three-element chain C_3 cannot be represented either as the congruence lattice of an SPS lattice. This paper is the start of the present one.

1.4. The main result

Following Grätzer and Knapp [21], a planar semimodular lattice (by definition, finite) L is *rectangular*, if the left boundary chain has exactly one doubly-irreducible element, c_l, and the right boundary chain has exactly one doubly-irreducible element, c_r, and these elements are complementary. Let B_n denote the Boolean lattice with n atoms.

In this paper, we use the Czédli-Schmidt Sequences to prove the following result.

Theorem 1. *Let L be a slim rectangular lattice L and let*

$$t = \text{length}[c_l, 1] + \text{length}[c_r, 1].$$

Then the congruence lattice $\operatorname{Con} L$ *has exactly t dual atoms and a dual atom in* $\operatorname{Con} L$ *is a congruence with exactly two classes.*

Since $\operatorname{Con} L$ is distributive, we obtain the following statement.

Corollary 2. *Let L be a slim rectangular lattice. Then* $\operatorname{Con} L$ *has a filter isomorphic to the Boolean lattice* B_t.

On the way to proving Theorem 1, we describe the prime ideals of a slim rectangular lattice L, following up an observation in [17]. We shall also discuss variants of Theorem 1 for rectangular lattices, PS lattices, and SPS lattices.

1.5. Notation

For the basic concepts and notation, see my books [12] and [14].

1.6. Outline

We recall some easy facts about slim rectangular lattices in Section 2 as well as we state the Structure Theorem and the Swing Lemma.

In Section 3, we prove some preliminary results on slim rectangular lattices. We describe the prime ideals of a slim rectangular lattice in Section 4. We investigate in Section 5 how adjacent congruence classes interface. A prime ideal P of a lattice L is naturally associated with a congruence $\boldsymbol{\pi}(P)$, which we call a prime congruence. In Section 6 we prove that a dual atom in $\operatorname{Con} L$ of a slim rectangular lattice L is a prime congruence. The main result of this paper follows.

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2. Background

2.1. Some known results

We will use the two results in the next lemma, implicitly or explicitly.

Lemma 3. *Let L be an SPS lattice. Then the following statements hold.*

(i) *An element of L has at most two covers.*

(ii) *Let $x \in L$ cover three distinct elements u, v, and w. Then the set $\{u, v, w\}$ generates an N_7 sublattice (see Figure 2).*

See my paper [15] and Czédli and Grätzer [8] for some proofs and references.

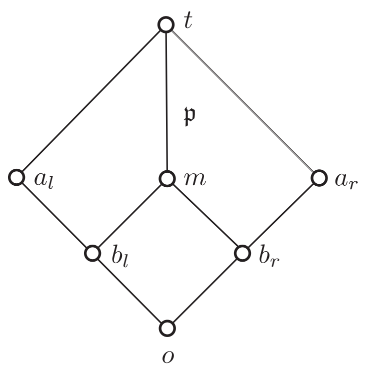

Figure 2. The lattice N_7.

As introduced in Ore [28], see also MacLane [27], a *cell A* in a planar lattice consists of two chains C (with zero 0_C and unit 1_C) and D (with zero 0_D and unit 1_D) such that the following conditions hold:

(i) $0_C = 0_D$ and $1_C = 1_D$;

(ii) C and D are maximal in $[0_C, 1_C] = [0_D, 1_D]$;

(iii) every $x \in C - \{0_C, 1_C\}$ is to the left of every $y \in D - \{0_D, 1_D\}$;

(iv) there are no elements inside the region bounded by C and D.

A 4-*cell* is a cell with $|C| = |D| = 3$. A 4-*cell lattice* is a lattice in which all cells are 4-cells.

For the following observation see Grätzer and Knapp [19, Section 4].

Lemma 4. *A PS lattice is a 4-cell lattice.*

The following statement, see Grätzer and Knapp [21, Lemma 4], plays an important role.

Lemma 5. *In a slim rectangular lattice, the bottom boundaries are ideals and the upper boundaries are filters.*

Corollary 6. *Let L be a slim rectangular lattice. Then for every $x \in L$, the element $x \vee c_r$ is in the upper right boundary of L, and symmetrically.*

Proof. Indeed, by Lemma 5, the upper right boundary of L is the filter generated by c_r. Since $x \vee c_r \in \mathrm{fil}\,(c_r)$, it follows that $x \vee c_r$ is in the upper right boundary. ■

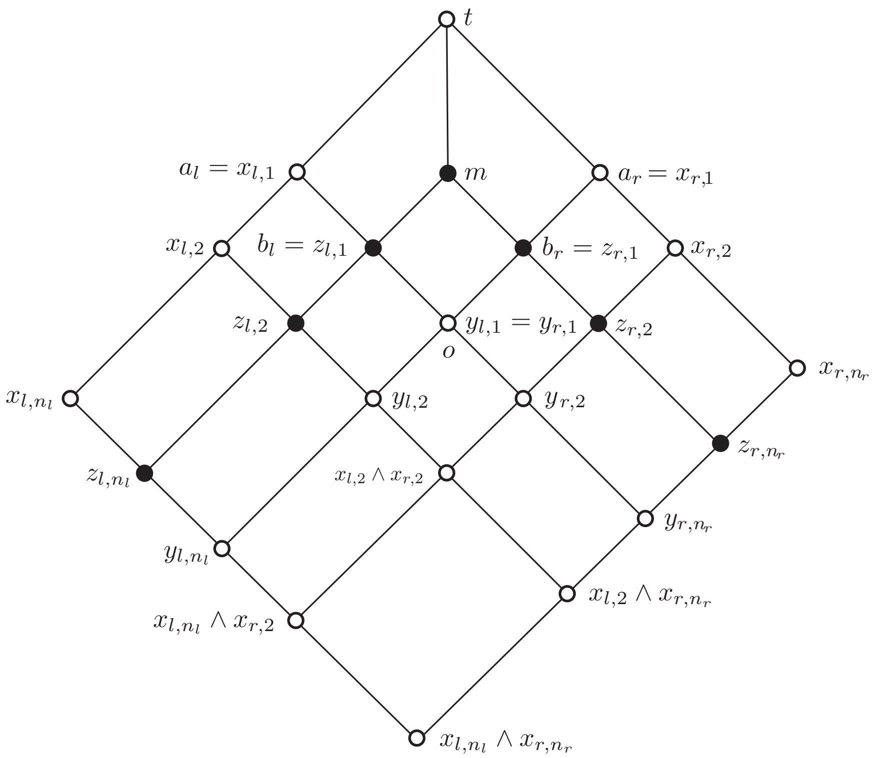

Figure 3. Notation for the fork insertion.

2.2. The Structure Theorem

Let L be a slim rectangular lattice. A *Czédli-Schmidt Sequence* for L is a sequence of slim rectangular lattices and a sequence of covering squares:

$$\begin{aligned} &D = L_1, L_2, \ldots, L_s = L, \\ (1) \qquad &S^1 = \{o^1, a_l^1, a_r^1, t^1\}, S^2 = \{o^2, a_l^2, a_r^2, t^2\}, \ldots, S^{s-1} \\ &\quad = \{o^{s-1}, a_l^{s-1}, a_r^{s-1}, t^{s-1}\}, \end{aligned}$$

where S^i is a covering square in L_i and we obtain L_{i+1} from L_i by inserting a fork at S^i (in formula, $L_{i+1} = L_i[S^i]$) for $i = 1, \ldots, s-1$.

For detailed descriptions of the fork extension, see Czédli and Schmidt [10], Grätzer [14], and other papers in the references.

We use the standard notation for fork insertions, see Figure 3 (where the black filled elements represent the inserted elements).

The following result is Czédli and Schmidt [10, Lemma 22].

Theorem (The Structure Theorem for Slim Rectangular Lattices)**.** *Let L be a* slim rectangular lattice*. Then there is a grid $D = \mathsf{C}_p \times \mathsf{C}_q$, where $p, q \geq 2$, and a Czédli-Schmidt Sequence from D to L.*

Note that the integer s in (1) is an invariant.

We call D the *grid* of L; it is isomorphic to a sublattice of L.

2.3. The Swing Lemma

For the prime intervals $\mathfrak{p}, \mathfrak{q}$ of an SPS lattice L, we define a binary relation: $\mathfrak{p}$ *swings* to $\mathfrak{q}$, if $1_\mathfrak{p} = 1_\mathfrak{q}$, this element covers at least three elements, and $0_\mathfrak{q}$ is neither the left-most nor the right-most element covered by $1_\mathfrak{p} = 1_\mathfrak{q}$, see Figure 4.

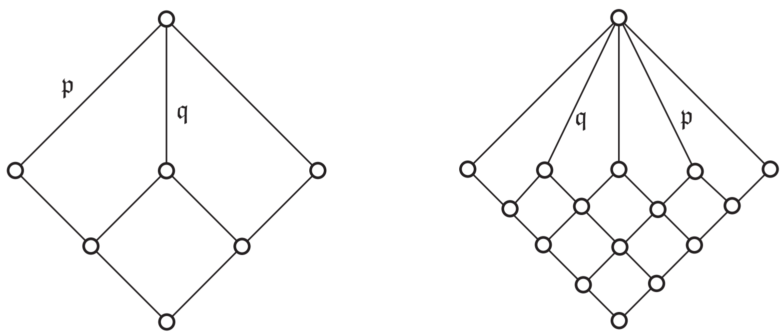

Figure 4. Swings, $\mathfrak{p} \curvearrowright \mathfrak{q}$.

The following result is from my paper [15].

Lemma 7 (Swing Lemma)**.** *Let L be an SPS lattice and let $\mathfrak{p}$ and $\mathfrak{q}$ be distinct prime intervals in L. If $\mathfrak{q}$ is collapsed by* $\mathrm{con}(\mathfrak{p})$*, then there exists a prime interval $\mathfrak{r}$ and sequence of pairwise distinct prime intervals*

$$\mathfrak{r} = \mathfrak{r}_0, \mathfrak{r}_1, \ldots, \mathfrak{r}_n = \mathfrak{q} \tag{2}$$

such that $\mathfrak{p}$ is up perspective to $\mathfrak{r}$, and $\mathfrak{r}_i$ is down perspective to or swings to $\mathfrak{r}_{i+1}$ for $i = 0, \ldots, n-1$. In addition, the sequence (2) *also satisfies*

$$1_{\mathfrak{r}_0} \geq 1_{\mathfrak{r}_1} \geq \cdots \geq 1_{\mathfrak{r}_n} = 1_\mathfrak{q}. \tag{3}$$

The Swing Lemma is easy to visualize. Perspectivity up is "climbing up", perspectivity down is "sliding". So we get from $\mathfrak{p}$ to $\mathfrak{q}$ by climbing up once and then alternating sliding and swinging.

3. Some preliminary results on slim rectangular lattices

In this section, we prove some elementary results about slim rectangular lattices. Let L be a slim rectangular lattice with the Czédli-Schmidt Sequence (1) and with the grid $D = \mathsf{C}_p \times \mathsf{C}_q$.

Let c_l and c_r be the corners of D, and let c_l^i and c_r^i be the corners of L_i for $i = 1, \ldots, s-1$.

We prove the next two lemmas utilizing the Czédli-Schmidt Sequences.

Lemma 8. *$c_l = c_l^i$ and $c_r = c_r^i$ for $i = 1, \ldots, s$.*

Proof. By induction on s as in (1). By definition, $c_l = c_l^1$ and $c_r = c_r^1$. Assume that the statement holds for $s-1$. We obtain L_s from L_{s-1} by adding a fork at S^{s-1}, see Figure 3, so there is only one new element on the left boundary, and it is a meet-reducible element below $c_l = c_l^{s-1}$. Therefore, c_l is the only doubly irreducible element on the left boundary of L_{s-1}, and so $c_l = c_l^s$. Similarly, $c_r = c_r^s$. ■

Corollary 9. *Let L be a slim rectangular lattice and let S be a covering square in L. Then the upper left boundaries of L and $L[S]$ are the same (and symmetrically). Therefore, the chains C_p and C_q are isomorphic to $[c_l, 1]$ and $[c_r, 1]$, respectively.*

Corollary 10. *For a slim rectangular lattice L, the grid is unique up to isomorphism.*

Lemma 11. *Let L be a slim rectangular lattice. Then*

$$\text{length}[0, c_l^s] = \text{length}[0, c_l] + s - 1, \tag{4}$$

$$\text{length}[0, c_r^s] = \text{length}[0_r, c_r] + s - 1. \tag{5}$$

Proof. Indeed, each step in (1) adds an element to the lower boundary chains. ■

It now follows that

$$\text{length}[0, c_l] - \text{length}[0, c_r] = \text{length}[c_r, 1] - \text{length}[c_l, 1]. \tag{6}$$

This immediately follows also from semimodularity.

4. Prime ideals

We describe the prime ideals of a slim rectangular lattice in this section.

The two lemmas of this section are proved using the Czédli-Schmidt Sequences.

Let L be a planar semimodular lattice. We call the element $m \in L$ a *middle* element of L if there is an N_7 sublattice such that m is the middle element of the N_7 sublattice.

Lemma 12. *Let L be a slim rectangular lattice. Let a be an element of L. Then one of the following statements holds:*

(i) *the element a is on the upper boundary of L;*

(ii) *the element a is meet-reducible;*

(iii) *the element a is a middle element.*

Proof. By induction on s as in (1). If $s = 1$, then $L = D$, and the statement holds for a grid. Let the statement hold for $s - 1$. The new elements of L_s, see Figure 3, form the set $[z_{l,n_l}, m] \cup [z_{r,n_r}, m]$, and they consist of the element m—satisfying (ii)—or an element in the set $[z_{l,n_l}, b_l] \cup [z_{r,n_r}, b_r]$, all of which are meet-reducible, so satisfying (ii). ∎

Lemma 13. *Let L be a slim rectangular lattice and let $p \in L$. If $p \neq 1$ and p is in the upper left boundary of L, then there exists an element q in the lower right boundary of L, so that $\{\mathrm{id}\,(p), \mathrm{fil}\,(q)\}$ is a partition of L.*

Proof. By induction on s as in (1). If $s = 1$, then $L = D$, and the statement holds for a grid with $q = p \wedge c_r$. Let the statement hold for $s-1$, and therefore, for L_{s-1}. So let $p \neq 1$, let p be on the upper left boundary of L_s (or symmetrically). Recall that by Corollary 9, the upper left boundaries of L and L_{s-1} are the same. Let q_{s-1} be the element in the lower right boundary of L_{s-1} that exits by the induction hypothesis and let $S = S^{s-1}$ be the covering square of L_{s-1}. We use the notation:

$$W = [m, z_{r,n_r}] \cup [m, y_{l,n_l}].$$

There are three cases to distinguish.

Case 1. $S \subseteq \mathrm{fil}_{L_{s-1}}(q_{s-1})$, as illustrated in Figure 5. Then

$$W \subseteq \mathrm{fil}_{L_s}(q_{s-1}) \cup \mathrm{id}_{L_s}(p),$$

therefore, $\{\mathrm{id}\,(p), \mathrm{fil}\,(q)\}$ is a partition of L with $q = q_{s-1}$.

Case 2. $S \subseteq \mathrm{id}_{L_{s-1}}(p)$, as illustrated in Figure 5. In this case,

$$W \subseteq \mathrm{id}_{L_s}(p),$$

so $\{\mathrm{id}\,(p), \mathrm{fil}\,(q)\}$ is a partition of L with $q = q_{s-1}$.

Case 3. $S \nsubseteq \mathrm{fil}_{L_{s-1}}(q_{s-1}), \mathrm{id}_{L_{s-1}}(q_{s-1})$, also illustrated in Figure 5. In this case, the two elements on the right upper boundary of S are in $\mathrm{fil}_{L_{s-1}}(q_{s-1})$ and the other two elements are in $\mathrm{id}_{L_{s-1}}(p)$. The newly inserted elements in $[m, y_{l,n_l}]$ are in $\mathrm{id}_{L_{s-1}}(p)$, and the rest of them, $[m, z_{r,n_r}]$, are in $\mathrm{fil}_L(z_{r,n_r})$, so $\{\mathrm{id}\,(p), \mathrm{fil}\,(q)\}$ is a partition of L with $q = z_{r,n_r}$. Note that $p \wedge c_r \prec q \prec q_{n-1}$. ∎

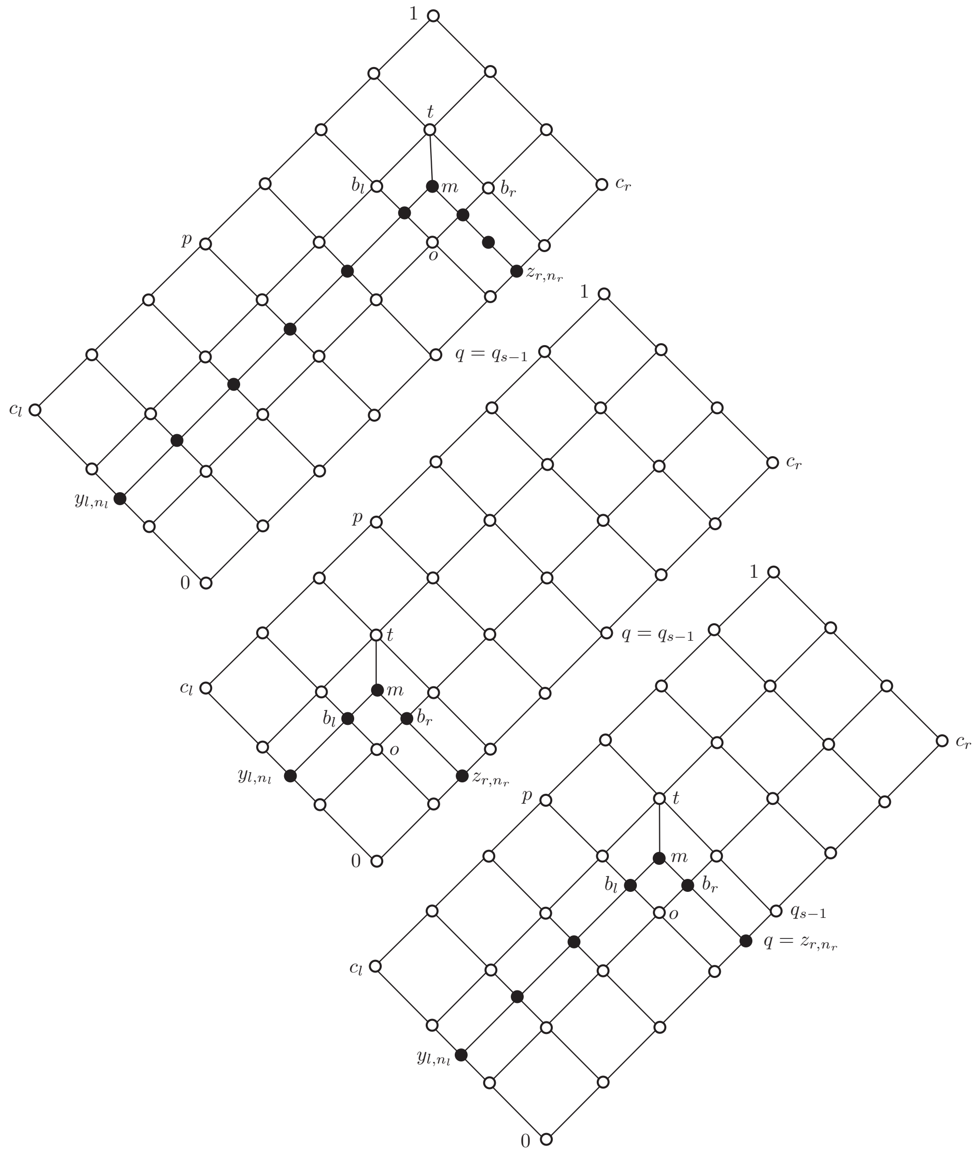

Figure 5. Proof of Lemma 13.

Corollary 14. *Let L be a slim rectangular lattice and let $p \in L$. If $p \neq 1$ and p is in the upper boundary of L, then the ideal $P = \mathrm{id}\,(p)$ of L is prime.*

Proof. By Lemma 13 and its symmetric counterpart. ■

A very special case of this result was found in Grätzer [17]. In a sense, this paper was the starting point of the present one.

Theorem 15. *Let L be a slim rectangular lattice and let $1 \neq a \in L$. Then $P = \mathrm{id}\,(a)$ is a prime ideal of L if and only if a is in the upper left or upper right boundary of L.*

Proof. Since $\mathrm{id}\,(p)$ is not a prime either for a meet reducible p or for a middle element $p = m$ (because $m > a_l \wedge a_r$) as in Figure 2, Lemma 12 applies. ■

5. The structure of congruence classes

I have known for a long time how adjacent congruence classes interface in a lattice. In this section, I prove two of these results, because they will be needed in Section 6. The first lemma is related to some discussions in Czédli [4] and [5].

Lemma 16. *Let $\boldsymbol{\alpha}$ be a congruence of a lattice L and let $A = [0_A, 1_A]$ and $B = [0_B, 1_B]$ be congruence classes of $\boldsymbol{\alpha}$ satisfying that $A \prec B$ in $L/\boldsymbol{\alpha}$. Then for every $x \in A$, there is a smallest $x^B \in B$ with $x \leq x^B$ and for every $x \in B$, there is a greatest $x_A \in A$ with $x \geq x_A$. Moreover, $(x^B)_A \prec x^B$ for every $x \in A$.*

Proof. Define $x^B = x \vee 0_B$ for $x \in A$ and $y_B = y \wedge 1_A$ for $y \in B$. This sets up a standard Galois connection, so only the last statement needs proof. Let us assume that $(x^B)_A < u < x^B$. By the definition of x_B, it follows that $u \notin B$ and similarly, $u \notin A$. Therefore, $A < u/\boldsymbol{\alpha} < B$ in $L/\boldsymbol{\alpha}$, contrary to the assumption that $A \prec B$ in $L/\boldsymbol{\alpha}$. ■

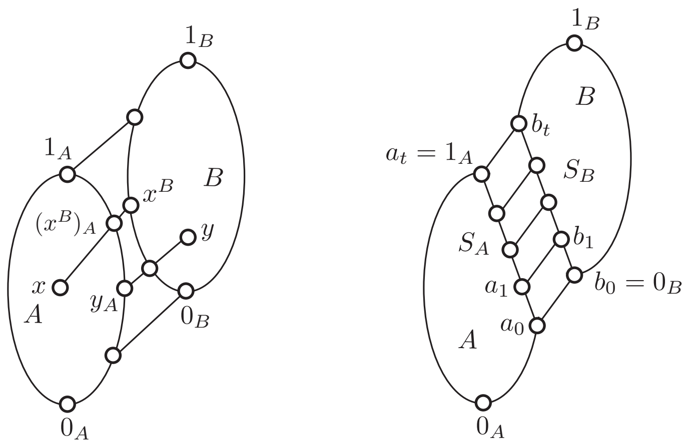

Figure 6. Two illustrations of $A \prec B$ in $L/\boldsymbol{\alpha}$.

Lemma 17. *Let L be a slim, planar, semimodular lattice. Let $\boldsymbol{\alpha}$ be a congruence of L and let A, B be congruence classes of $\boldsymbol{\alpha}$ satisfying that $A \prec B$ in $L/\boldsymbol{\alpha}$. Then there is a maximal chain*

$$S_A = \{1_A \wedge 0_B = a_0 \prec a_1 \prec \cdots \prec a_t = 1_A\}$$

on the right boundary of A and there is a maximal chain

$$S_B = \{0_B = b_0 \prec b_1 \prec \cdots \prec b_t = 1_A \vee 0_B\}$$

on the left boundary of B—or symmetrically. The chain S_A is isomorphic to S_B by the map $\varphi_A \colon x \mapsto x \vee 0_B$; the inverse isomophism is $\varphi_B \colon x \mapsto x \wedge 0_B$.

Proof. If the elements 1_A and 0_B are comparable, then $1_A < 0_B$ and the statement is true with the singletons S_A and S_B. So we can assume that 1_A and 0_B are incomparable. By symmetry, we can also assume that 0_B is to the right of 1_A.

Let $a_0 = 1_A \wedge 0_B$ and $b_0 = 0_B$. If $a_0 = 0_B$, then $a_0 \in A \cap B$, a contradiction, since $A \prec B$ in $L/\boldsymbol{\alpha}$, so A and B are disjoint. Hence, $a_0 < b_0$.

We claim that $a_0 \prec b_0$. Indeed, let there be an element z of L with $a_0 < z < b_0$. If $z \in A$, then $z \leq 1_A$, so $z = a_0$, a contradiction. If $z \in B$, then 0_B is not the smallest element of B, a contradiction. Therefore, $A < z/\boldsymbol{\alpha} < B$ in $L/\boldsymbol{\alpha}$, contradicting the assumption that $A \prec B$ in $L/\boldsymbol{\alpha}$. This verifies the claim.

By Kelly and Rival [26], this implies that a_0 is on the boundary of A, say, on the right boundary. This allows us to take a maximal chain

$$S_A = \{a_0 \prec a_1 \prec \cdots \prec a_t = 1_A\}$$

of $[a_0, 1_A]$ on the right boundary of A. Put $b_i = a_i \vee b_0$ for $i = 0, \ldots, t$. Since $A \vee B = B$ in $L/\boldsymbol{\alpha}$, we get that $b_i \in B$ for $i = 0, \ldots, t$. So $a_i < b_i$. By semimodularity, $a_i \prec b_i$ for $i = 0, \ldots, t$. Again, by semimodularity, we obtain that $b_i \preceq b_{i+1}$. Since $1_A = a_t \prec b_t$, we can see that

$$\{a_0 \prec a_1 \prec \cdots \prec a_t = 1_A \prec b_t\}$$

is a maximal chain in the interval $[a_0, b_t]$ of length is t. The chain

$$\{a_0 \prec b_0 \preceq b_1 \preceq \cdots \preceq b_t\}$$

is a maximal chain in the same interval, so by the Jordan-Hölder property of finite semimodular lattices, we obtain that it is also of length t. Now it follows that

$$S_B = \{b_0 \prec b_1 \prec \cdots \prec b_t\}$$

satisfies the requirements of the lemma, since all the squares depicted on the right of Figure 6 are covering squares. ∎

We call $S_A \times \mathsf{C}_2$ the *ladder* associated with $A \prec B$. Note that it has a single rung if $1_A < 0_B$ (equivalently, if $1_A \prec 0_B$).

6. Prime congruences

A congruence $\boldsymbol{\pi}$ of a lattice L is *prime* if it has exactly two blocks. Clearly, one of its blocks is a prime ideal P. Since P determines $\boldsymbol{\pi}$, we use the notation $\boldsymbol{\pi}(P)$ for $\boldsymbol{\pi}$. Every prime congruence of L is a dual atom in $\mathrm{Con}(L)$. Also, if a congruence has only two congruence classes, then it is prime.

Theorem 18. *Let L be a slim rectangular lattice and let the congruence $\boldsymbol{\pi}$ of L be a dual atom in* $\mathrm{Con}\, L$*. Then the congruence $\boldsymbol{\pi}$ is prime.*

Proof. Let $\boldsymbol{\pi}$ be a dual atom in $\mathrm{Con}\, L$. Let $\boldsymbol{\pi}$ partition the upper left boundary into b blocks.

Case 1. $b = 1$. Equivalently, $c_l \equiv 1 \pmod{\boldsymbol{\pi}}$. Meeting both sides with c_r, we obtain that $0 \equiv c_r \pmod{\boldsymbol{\pi}}$. By Corollary 6, for every $x \in L$, the element $x \vee c_r$ is in the upper right boundary of L, so $x \equiv x \vee c_r \pmod{\boldsymbol{\pi}}$. Thus we can choose a subchain C of $[c_r, 1]$ with the property that every congruence class of $\boldsymbol{\alpha}$ contains exactly one element of C. By the First Isomorphism Theorem * (see, for instance, [12, Exercise I.3.61]), we have the isomophism $L/\boldsymbol{\pi} \cong C$, so $L/\boldsymbol{\pi}$ is a chain. Since the congruence $\boldsymbol{\pi}$ of L is a dual atom in $\mathrm{Con}\, L$, by the Second Isomorphism Theorem ((see, for instance, [12, Theorem 220])) the lattice $L/\boldsymbol{\pi}$ is simple. A simple distributive lattice has two elements, so $\boldsymbol{\pi}$ is prime, as required.

Case 2. $b = 2$. Equivalently, there is a prime interval $\mathfrak{p}$ on the upper left boundary of L, such that $c_l \equiv 0_\mathfrak{p} \pmod{\boldsymbol{\pi}}$, $1_\mathfrak{p} \equiv 1 \pmod{\boldsymbol{\pi}}$, and $0_\mathfrak{p} \not\equiv 1_\mathfrak{p} \pmod{\boldsymbol{\pi}}$, or symmetrically.

For $c_l = 0_\mathfrak{p}$ or $1_\mathfrak{p} = 1$ or both, define $c_l = 0_\mathfrak{p}$. Then $L/\boldsymbol{\pi} \cong Q/\boldsymbol{\pi}$.

Otherwise, $c_l < 0_\mathfrak{p} \prec 1_\mathfrak{p} < 1$.

We use the ladder of Lemma 17, see Figure 6. Let q be the cover of $0_\mathfrak{p} \wedge c_r$ on the lower right boundary. Note the ideal $P = [0, 0_\mathfrak{p}]$ and the filter $Q = [q, 1]$. The two sides of the ladder are

$$S_A = \{0_\mathfrak{p} \wedge c_r = a_0 \prec \cdots \prec a_t = 0_\mathfrak{p}\},$$
$$S_B = \{0_B = q \prec \cdots \prec b_t = 1_\mathfrak{p}\},$$

using the notation of Figure 7. The chain S_A is shaded black and the chain S_B is shaded gray.

We argue as in Case 1, *mutatis mutandis*, that for every element $x \in P$, there is an element $y \in S_A$ such that $x \equiv y \pmod{\boldsymbol{\pi}}$. The same way, for every element $x \in [1_\mathfrak{p} \wedge c_r, 1]$, there is an element $y \in [c_r, 1]$ such that $x \equiv y \pmod{\boldsymbol{\pi}}$.

Since the corresponding prime intervals of S_A and S_B are perspective (as illustrated by Figure 6), it follows that $S_A/\boldsymbol{\pi}$ and $S_B/\boldsymbol{\pi}$ are isomorphic. Therefore, $L/\boldsymbol{\pi}$ can be obtained by gluing together $[0, 0_\mathfrak{p}]/\boldsymbol{\pi}$ and $[q, 1]/\boldsymbol{\pi}$ over a chain $S_A/\boldsymbol{\pi} \cong S_B/\boldsymbol{\pi}$. Both lattices $[0, 0_\mathfrak{p}]/\boldsymbol{\pi}$ and $[q, 1]/\boldsymbol{\pi}$ are slim rectangular lattices so their

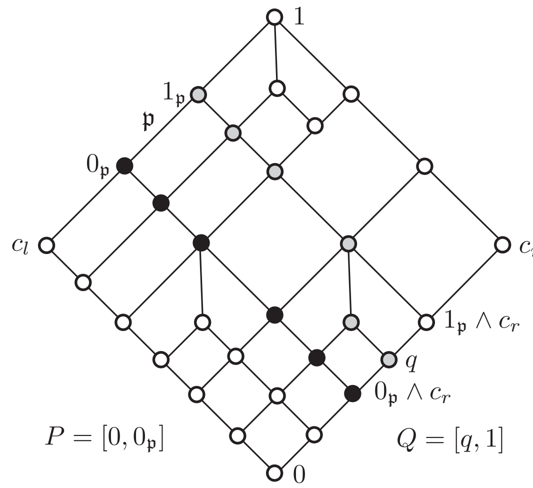

Figure 7. Notation for the proof of Theorem 18, Case 2; the chain S_A is shaded black and the chain S_B is shaded gray.

gluing over $S_A/\boldsymbol{\pi} \cong S_B/\boldsymbol{\pi}$ is also a slim rectangular lattice by Grätzer and Knapp [21, Lemma 5].

Since $[0, 0_p]/\boldsymbol{\pi}$ and $[q, 1]/\boldsymbol{\pi}$ are isomorphic to the chains $C_A/\boldsymbol{\pi}$ and $C_B/\boldsymbol{\pi}$, respectively, the lattices $[0, 0_p]/\boldsymbol{\pi}$ and $[q, 1]/\boldsymbol{\pi}$ are distributive. Gluing these two lattices over $S_A/\boldsymbol{\pi} \cong S_B/\boldsymbol{\pi}$, the Second Isomorphism Theorem gives again that $L/\boldsymbol{\pi}$ is a simple distributive lattice and so $\boldsymbol{\pi}$ is prime, and the statement follows.

Case 3. $b \geq 3$. Equivalently, there are prime intervals $\mathfrak{p}$ and $\mathfrak{q}$ on the upper left boundary of L (or symmetrically) such that $1_{\mathfrak{p}} < 0_{\mathfrak{q}}$, $0_{\mathfrak{p}} \not\equiv 1_{\mathfrak{p}} \pmod{\boldsymbol{\pi}}$, and $0_{\mathfrak{q}} \not\equiv 1_{\mathfrak{q}} \pmod{\boldsymbol{\pi}}$. Since $\boldsymbol{\pi}$ is a dual atom in $\operatorname{Con} L$, it follows that

$$\boldsymbol{\pi} \vee \operatorname{con}(\mathfrak{p}) = \boldsymbol{\pi} \vee \operatorname{con}(\mathfrak{q}) = \mathbf{1}.$$

Therefore, $\operatorname{con}(\mathfrak{p}) \leq \boldsymbol{\pi} \vee \operatorname{con}(\mathfrak{q})$. Since $\mathfrak{p}$ is a prime interval, we get that $\operatorname{con}(\mathfrak{p}) \leq \boldsymbol{\pi}$ or $\operatorname{con}(\mathfrak{p}) \leq \operatorname{con}(\mathfrak{q})$. The inequality $\operatorname{con}(\mathfrak{p}) \leq \boldsymbol{\pi}$ contradicts the assumption that $0_{\mathfrak{p}} \not\equiv 1_{\mathfrak{p}} \pmod{\boldsymbol{\pi}}$, so we conclude that $\operatorname{con}(\mathfrak{p}) \leq \operatorname{con}(\mathfrak{q})$ holds.

By the Swing Lemma (Lemma 7), there is a sequence of prime intervals (2) (also satisfying (3)). Since $\mathfrak{p}$ is on the upper boundary of L, we cannot "climb up" from $\mathfrak{p}$; it follows that $\mathfrak{p} = \mathfrak{r}$. Therefore, $1_{\mathfrak{p}} = 1_{\mathfrak{r}} \geq 1_{\mathfrak{q}}$ by (3), contradicting our assumption that $1_{\mathfrak{p}} < 0_{\mathfrak{q}} \prec 1_{\mathfrak{q}}$. ∎

Now we are ready to prove our main result. Let $t = \operatorname{length}[c_l, 1] + \operatorname{length}[c_r, 1]$. By Theorem 15, the lattice L has exactly t prime ideals, and each prime ideal has an associated prime congruence, a dual atom. So $\operatorname{Con} L$ has at least t dual atoms. By Theorem 18, all dual atoms of $\operatorname{Con} L$ are prime congruences, so $\operatorname{Con} L$ has exactly t dual atoms.

7. Meet semidistributive lattices

A lattice L is *meet-semidistributive*, if the following implication holds:

(SD$_\wedge$) $\quad x \wedge y = x \wedge z$ implies that $x \wedge y = x \wedge (y \vee z)$ for all $x, y, z \in L$.

This implication was introduced by Whitman [29] and [30] as a property of free lattices. It also holds for SPS lattices.

Lemma 19. *Let L be an SPS lattice. Then the implication (SD$_\wedge$) holds in L.*

Proof. Assume that it does not hold. Then there are elements $a, b, c \in L$ such that $a \wedge b = a \wedge c$ but $a \wedge b \neq a \wedge (b \vee c)$. Then $x \neq y \in \{a \wedge (b \vee c), b, c\}$ satisfy that $x \wedge y = a \wedge b$, so we cab choose elements $a', b', c' \in L$ so that $a \wedge b \prec a' \leq a \wedge (b \vee c)$, $a \wedge b \prec b' \leq b$, $a \wedge b \prec c' \leq c$, contradicting Lemma 3(i). ■

For some references about semidistributive lattices, see Adaricheva, Gorbunov, Tumanov [1], Czédli, Ozsvárt, and Udvari [9], Avann [2], and Dilworth [11].

In the rest of this section, we outline the proof of the following variant of Theorem 1.

Theorem 1'. *If L is a finite meet-semidistributive lattice, then the meet of the dual atoms is the least congruence $\boldsymbol{\delta}$ with $L/\boldsymbol{\delta}$ distributive.*

This result and its proof is due to Ralph Freese, who emailed me after this paper was completed. Prossor Freese kindly suggested to me to "feel free to use it in your paper".

The following sketch of the proof (slightly edited) is from his email.

Since the class $\boldsymbol{D}$ of distributive lattices is closed under subdirect products, we get the first statement.

Lemma 20. *Every lattice L has a unique minimal congruence $\boldsymbol{\delta}$ such that $L/\boldsymbol{\delta}$ is distributive.*

$L/\boldsymbol{\delta}$ is called the *reflection* of L into $\boldsymbol{D}$.

Lemma 21. *For the congruence $\boldsymbol{\delta}$ of Lemma 20, we have*

$$\boldsymbol{\delta} = \bigwedge \mathcal{C},$$

where $\mathcal{C}$ is the set of those dual atoms of $\operatorname{Con} L$*, whose corresponding quotient is* C_2*, the two-element chain.*

Lemma 22. *Every meet-semidistributive lattice with* 0 *has* C_2 *as a homomorphic image.*

We apply these lemmas to prove Theorem 1'. Since the lattice L is finite and meet-semidistributive, it follows that every homomorphic image of L is also meet-semidistributive, and so every dual atom of $\mathrm{Con}\, L$ is in $\mathcal{C}$.

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