# LEFT ANNIHILATOR OF IDENTITIES WITH GENERALIZED DERIVATIONS IN PRIME AND SEMIPRIME RINGS 

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#### Abstract

Let $R$ be a noncommutative prime ring of $\operatorname{char}(R) \neq 2, F$ a generalized derivation of $R$ associated to the derivation $d$ of $R$ and $I$ a nonzero ideal of $R$. Let $S \subseteq R$. The left annihilator of $S$ in $R$ is denoted by $l_{R}(S)$ and defined by $l_{R}(S)=\{x \in R \mid x S=0\}$. In the present paper, we study the left annihilator of the sets $\left\{F(x) \circ_{n} F(y)-x \circ_{n} y \mid x, y \in I\right\}$ and $\left\{F(x) \circ_{n} F(y)-d\left(x \circ_{n} y\right) \mid x, y\right.$ $\in I\}$.


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## 1. InTRODUCTION

Throughout this paper, $R$ always denotes an associative ring with center $Z(R)$. For any $a, b \in R$, a ring $R$ is said to be prime, if $a R b=(0)$ implies either $a=0$ or $b=0$ and is semiprime if for any $a \in R, a R a=(0)$ implies $a=0$. A mapping $f$ is said to be an additive mapping on $R$ if $f(x+y)=f(x)+f(y)$ holds for all $x, y \in R$. An additive mapping $d: R \rightarrow R$ defined by $d(x y)=d(x) y+x d(y)$ for all $x, y \in R$ is called a derivation on $R$. The map $d(x)=[a, x]$ for all $x \in R$ and for some fixed $a \in R$, is called an inner derivation of $R$. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $x y-y x$ and the symbol $x \circ y$ stands for the anti-commutator $x y+y x$. For given $x, y \in R$, we set $x \circ_{0} y=x, x \circ_{1} y=x y+y x$, and inductively $x \circ_{n} y=\left(x \circ_{n-1} y\right) \circ y$ for $n>1$. Let $S \subseteq R$. Then $r_{R}(S)$ denotes the right annihilator of $S$ in $R$, that is, $r_{R}(S)=\{x \in R \mid S x=0\}$ and $l_{R}(S)$ denotes the left annihilator of $S$ in $R$ that is, $l_{R}(S)=\{x \in R \mid x S=0\}$. If
$r_{R}(S)=l_{R}(S)$, then $r_{R}(S)$ is called an annihilator ideal of $R$ and is written as $a n n_{R}(S)$.

Ashraf and Rehman [3] proved that if $R$ is a prime ring of char $(R) \neq 2, I$ is a nonzero ideal of $R$ and $d \neq 0$ such that $d(x) \circ d(y)=x \circ y$ for all $x, y \in I$, then $R$ is commutative.

In [1], Ali and Huang studied the case for semiprime ring. They proved that if $R$ is a 2 -torsion free semiprime ring and $I$ a nonzero ideal of $R$ and $d(I) \neq(0)$ such that $d(x) \circ d(y)=x \circ y$ for all $x, y \in I$, then $R$ has a nonzero central ideal.

There is ongoing interest to investigate the situations replacing derivations with generalized derivations. An additive mapping $F: R \rightarrow R$ is said to be a generalized derivation of $R$, if there exists a derivation $d$ of $R$ such that $F(x y)=$ $F(x) y+x d(y)$ holds for all $x, y \in R$. In particular, when $d=0$, then $F$ becomes a left multiplier map on $R$. Thus a generalized derivation covers both the concept of derivation and left multiplier map.

In [12], Huang proved that if $R$ is a prime ring with char $(R) \neq 2, L$ is a square closed Lie ideal of $R$ and $F$ a generalized derivation associated with derivation $d$ of $R$ such that $d(x) \circ F(y)=x \circ y$ for all $x, y \in L$, then either $d=0$ or $L \subseteq Z(R)$. In [4], Bell and Rehman studied the situation for prime ring $R$ that $F(x) \circ F(y)=x \circ y$ for all $x, y \in R$.

Ashraf et al. [2] proved that the prime ring $R$ must be commutative if $R$ admits a generalized derivation $F$ associated with a nonzero derivation $d$ satisfying $d(x) \circ F(y)=x \circ y$ for all $x, y \in I$, where $I$ is a nonzero ideal of $R$. Then Dhara et al. [9] studied the same situation in 2-torsion free semiprime ring and obtained that $R$ has a nonzero central ideal. Recently, in [18], Raza and Rehman studied the cases $F(x) \circ_{m} F(y)=(x \circ y)^{n}$ for all $x, y \in I$ and $F(x) \circ_{m} d(y)=d(x \circ y)^{n}$ for all $x, y \in I$ in prime and semiprime rings, where $I$ is a nonzero ideal of $R, F$ is a generalized derivation of $R$ with associated derivation $d$ and $m, n$ are fixed positive integers. In the present paper, we investigate the left annihilator condition of the identities, that is $a\left\{F(x) \circ_{n} F(y)-\left(x \circ_{n} y\right)\right\}=0$ for all $x, y \in I$ and $a\left\{F(x) \circ_{n} F(y)-d\left(x \circ_{n} y\right)\right\}=0$ for all $x, y \in I$.

Let $R$ be a prime ring with center $Z(R)$ and $U$ is the Utumi quotient ring of $R$. It is well known that any derivation of $R$ can be uniquely extended to a derivation of $U$, and so any derivation of $R$ can be defined on the whole of $U$. Moreover $U$ is a prime ring as well as $R$ and the extended centroid $C$ of $R$ coincides with the center of $U$. Note that $C$ is a field. We refer to [16] for more details.

We mention a very important result which will be used quite frequently as follows.

Theorem 1.1 (Kharchenko [14]). Let $R$ be a prime ring, $d$ a nonzero derivation on $R$ and $I$ a nonzero ideal of $R$. If $I$ satisfies the differential identity

$$
f\left(r_{1}, r_{2}, \ldots, r_{n}, d\left(r_{1}\right), d\left(r_{2}\right), \ldots, d\left(r_{n}\right)\right)=0
$$

for any $r_{1}, r_{2}, \ldots, r_{n} \in I$, then either
(i) I satisfies the generalized polynomial identity

$$
f\left(r_{1}, r_{2}, \ldots, r_{n}, x_{1}, x_{2}, \ldots, x_{n}\right)=0
$$

or
(ii) $d$ is inner i.e., for some $q \in U, d(x)=[q, x]$ for all $x \in R$ and $I$ satisfies the generalized polynomial identity

$$
f\left(r_{1}, r_{2}, \ldots, r_{n},\left[q, r_{1}\right],\left[q, r_{2}\right], \ldots,\left[q, r_{n}\right]\right)=0
$$

## 2. Main results

We begin with the theorem.
Theorem 2.1. Let $R$ be a noncommutative prime ring of char $(R) \neq 2$ with its Utumi ring of quotients $U, C=Z(U)$ the extended centroid of $R, F$ a generalized derivation of $R$ and $I$ a nonzero ideal of $R$. Suppose that there exists $0 \neq a \in R$ such that $a\left(F(x)^{n}-x^{n}\right)=0$ for all $x \in I$, where $n \geq 1$ is a fixed integer. Then one of the following holds.
(1) $n=1$ and there exists $b \in U$ such that $F(x)=b x$ for all $x \in R$ with $a(b-1)=0$;
(2) $n \geq 2$ and there exists $\lambda \in C$ such that $F(x)=\lambda x$ for all $x \in R$ with $\lambda^{n}=1$.

Proof. In light of [15, Theorem 3], there exist $b \in U$ and derivation $d$ of $U$ such that $F(x)=b x+d(x)$. Since $I, R$ and $U$ satisfy the same generalized polynomial identities (see [8]) as well as the same differential identities (see [16]), without loss of generality, we have $a\left(F(x)^{n}-x^{n}\right)=0$ for all $x \in U$, where $d$ is the derivation of $U$, that is,

$$
a\left((b x+d(x))^{n}-x^{n}\right)=0
$$

for all $x \in U$.
If $F=0$, then our hypothesis reduces to $a x^{n}=0$ for all $x \in U$. Replacing $x$ with $x a$ yields $(x a)^{n+1}=0$ for all $x \in U$. Since $R$ is a prime ring, by Levitzki's Lemma [11, Lemma 1.1], $a=0$, a contradiction.

Now we assume $F \neq 0$. By Kharchenko's theorem, we divide the proof in two cases.

Case 1. Let $d$ be an outer derivation of $U$. Then by Kharchenko's theorem [14], we have by our assumption that

$$
a\left((b x+u)^{n}-x^{n}\right)=0
$$

for all $x, u \in U$. In particular, for $x=0$, we have $a u^{n}=0$ for all $u \in U$. Again, this implies $a=0$, a contradiction.

Case 2. Let $d$ be inner derivation of $U$, that is, $d(x)=[c, x]$ for all $x \in R$ and for some $c \in U$. Since $d \neq 0, c \notin C$. Thus $a\left((b x+[c, x])^{n}-x^{n}\right)=$ 0 is a nontrivial generalized polynomial identity (GPI) for $U$. Denote by $E$ either the algebraic closure of $C$ or $C$ according as $C$ is either infinite or finite, respectively. Then, by a standard argument (see for instance, [16, Proposition]), $a\left((b x+[c, x])^{n}-x^{n}\right)=0$ is also a GPI for $U \otimes_{C} E$. Since $U \otimes_{C} E$ is centrally closed prime $E$-algebra [10, Theorem 2.5 and Theorem 3.5], by replacing $R, C$ with $U \otimes_{C} E$ and $E$, respectively, we may assume $R$ is centrally closed and $C$ is either finite or algebraically closed. By Martindale's theorem [17], $R$ is then a primitive ring having nonzero socle $H$ with $C$ as the associated division ring. Hence by Jacobson's theorem [13, p.75] $R$ is isomorphic to a dense ring of linear transformations of some vector space $V$ over $C$, and $H$ consists of the linear transformations in $R$ of finite rank.

If $V$ is finite dimensional over $C$, then by the density of $R$ on $V, R \cong M_{k}(C)$ where $k=\operatorname{dim}_{C} V$.

Since $R$ is noncommutative, $\operatorname{dim}_{C} V \geq 2$.
We show that $v$ and $c v$ are linearly $C$-dependent for any $v \in V$. Suppose that $v$ and $c v$ are linearly independent for some $v \in V$. By the density there exists $x \in R$ such that

$$
x v=0, \quad x c v=v
$$

Then

$$
0=a\left((b x+[c, x])^{n}-x^{n}\right) v=a v .
$$

If for some $u \in V,\{u, v\}$ is linearly $C$-dependent, then $a u=0$. Since $a \neq 0$, there exists $w \in V$ such that $a w \neq 0$ and so $\{w, v\}$ is linearly $C$-independent. Moreover, $a(w+v)=a w \neq 0$ and $a(w-v)=a w \neq 0$. By the above argument, it follows that $w$ and $c w$ are linearly $C$-dependent, as are $\{w+v, c(w+v)\}$ and $\{w-v, c(w-v)\}$. Therefore there exist $\alpha_{w}, \alpha_{w+v}, \alpha_{w-v} \in C$ such that

$$
c w=\alpha_{w} w, \quad c(w+v)=\alpha_{w+v}(w+v), \quad c(w-v)=\alpha_{w-v}(w-v) .
$$

This gives

$$
\begin{equation*}
\alpha_{w} w+c v=\alpha_{w+v} w+\alpha_{w+v} v \tag{1}
\end{equation*}
$$

and

$$
\begin{equation*}
\alpha_{w} w-c v=\alpha_{w-v} w-\alpha_{w-v} v . \tag{2}
\end{equation*}
$$

Now (1) and (2) together yields

$$
\begin{equation*}
\left(2 \alpha_{w}-\alpha_{w+v}-\alpha_{w-v}\right) w+\left(\alpha_{w-v}-\alpha_{w+v}\right) v=0 \tag{3}
\end{equation*}
$$

and

$$
\begin{equation*}
2 c v=\left(\alpha_{w+v}-\alpha_{w-v}\right) w+\left(\alpha_{w+v}+\alpha_{w-v}\right) v \tag{4}
\end{equation*}
$$

By (3), and since $\{w, v\}$ are $C$-independent and char $(R) \neq 2$, we have $\alpha_{w}=$ $\alpha_{w+v}=\alpha_{w-v}$. Thus by (4), we have $c v=\alpha_{w} v$. This leads to a contradiction with the fact that $\{v, c v\}$ is linear $C$-independent.

In light of this, we may assume that for any $v \in V$ there exists a suitable $\alpha_{v} \in C$ such that $c v=\alpha_{v} v$. By standard argument, there is $\alpha \in C$ such that $c v=\alpha v$ for all $v \in V$. Now let $r \in R, v \in V$. Since $c v=\alpha v$,

$$
\begin{equation*}
[c, r] v=(c r) v-(r c) v=c(r v)-r(c v)=\alpha(r v)-r(\alpha v)=0 \tag{5}
\end{equation*}
$$

Thus $[c, r] v=0$ for all $v \in V$, i.e., $[c, r] V=0$. Since $[c, r]$ acts faithfully as a linear transformation on the vector space $V,[c, r]=0$ for all $r \in R$. Therefore, $c \in Z(R)$. Then

$$
a\left((b x)^{n}-x^{n}\right)=0
$$

for all $x \in R$. If $n=1$, then $a(b-1) R=(0)$ implying $a(b-1)=0$.
So, let $n>1$. Suppose that $v$ and $b v$ are linearly independent for some $v \in V$. By the density there exists $x \in R$ such that

$$
x v=v, \quad x b v=0
$$

and hence

$$
0=a\left((b x)^{n}-x^{n}\right) v=-a v
$$

Since $a \neq 0$, by the same argument as earlier, it yields $b \in C$. Then $a\left(b^{n} x^{n}-x^{n}\right)=0$, i.e., $a\left(b^{n}-1\right) x^{n}=0$ for all $x \in R$. By the same argument as before, we have $a\left(b^{n}-1\right)=0$. Since $a \neq 0$, it yields $b^{n}=1$. This completes the proof.

Corollary 2.2. Let $R$ be a noncommutative prime ring of char $R \neq 2$ with its Utumi ring of quotients $U, C=Z(U)$ the extended centroid of $R, F$ a generalized derivation of $R$ and $I$ a nonzero ideal of $R$. Suppose that there exists $0 \neq a \in R$ such that $a\left(F(x) \circ_{n} F(y)-x \circ_{n} y\right)=0$ for all $x, y \in I$, where $n \geq 0$ is a fixed integer. Then one of the following holds.
(1) $n=0$ and there exists $b \in U$ such that $F(x)=b x$ for all $x \in R$ with $a(b-1)=0$;
(2) $n \geq 1$ and there exists $\lambda \in C$ such that $F(x)=\lambda x$ for all $x \in R$ with $\lambda^{n+1}=1$.

Proof. In particular, for $x=y$, we have $a\left(F(x)^{n+1}-x^{n+1}\right)=0$ for all $x \in I$. Then by Theorem 2.1, we conclude one of the following.
(1) $n=0$ and there exists $b \in U$ such that $F(x)=b x$ for all $x \in R$ with $a(b-1)=0$;
(2) $n \geq 1$ and there exists $\lambda \in C$ such that $F(x)=\lambda x$ for all $x \in R$ with $\lambda^{n+1}=1$.

In particular, for $F=d$, we have the following corollary.
Corollary 2.3. Let $R$ be a prime ring of char $(R) \neq 2$ and $0 \neq a \in R$. Suppose that $d$ is a nonzero derivation of $R$ and $n \geq 0$ a fixed integer such that $a\left(d(x) \circ_{n}\right.$ $\left.d(y)-x \circ_{n} y\right)=0$ for all $x, y \in I$, then $R$ is commutative.

Theorem 2.4. Let $R$ be a noncommutative prime ring of char $(R) \neq 2$ with its Utumi ring of quotients $U, C=Z(U)$ the extended centroid of $R, F$ a generalized derivation of $R$ with associated derivation $d$ of $R$ and $I$ a nonzero ideal of $R$. Suppose that there exists $0 \neq a \in R$ such that $a\left(F(x)^{n}-d\left(x^{n}\right)\right)=0$ for all $x \in I$, where $n \geq 1$ is a fixed integer. Then there exists $b \in U$ such that $F(x)=b x$ for all $x \in R$ with $a b=0$.

Proof. In light of [15, Theorem 3], we may assume that there exist $b \in U$ and derivation $d$ of $U$ such that $F(x)=b x+d(x)$. Since $I, R$ and $U$ satisfy the same generalized polynomial identities (see [8]) as well as the same differential identities (see [16]), without loss of generality, we have $a\left(F(x)^{n}-d\left(x^{n}\right)\right)=0$ for all $x \in U$, where $d$ is derivation on $U$, that is

$$
a\left\{(b x+d(x))^{n}-\sum_{i=0}^{n} x^{i} d(x) x^{n-i-1}\right\}=0
$$

for all $x \in U$.
In light of Kharchenko's theorem, we divide the proof in two cases.
Case 1. If $d$ is not $U$-inner, then by Kharchenko's theorem [14] we have

$$
a\left\{(b x+y)^{n}-\sum_{i=0}^{n-1} x^{i} y x^{n-i-1}\right\}=0
$$

for all $x, y \in U$. If $n>1$, then in particular for $x=0$, we have $a y^{n}=0$ for all $y \in U$. This yields $a=0$, a contradiction.

On the other hand, if $n=1$, then $a b x=0$ for all $x \in U$, implying $a b=0$.

Case 2. We assume the case when $d$ is $U$-inner derivation, that is for some $c \in U, d(x)=[c, x]$ for all $x \in U$. Since $d \neq 0, c \notin C$. Hence $a\left((b x+[c, x])^{n}-\right.$ $\left.\left[c, x^{n}\right]\right)=0$ is a nontrivial generalized polynomial identity (GPI) for $U$. Denote by $E$ either the algebraic closure of $C$ or $C$ according as $C$ is either infinite or finite, respectively. Then, by a standard argument (see for instance, [16, Proposition]), $a\left((b x+[c, x])^{n}-\left[c, x^{n}\right]\right)=0$ is also a GPI for $U \otimes_{C} E$. Since $U \otimes_{C} E$ is centrally closed prime $E$-algebra [10, Theorem 2.5 and Theorem 3.5], by replacing $R, C$ with $U \otimes_{C} E$ and $E$, respectively, we may assume that $R$ is centrally closed and $C$ is either finite or algebraically closed. By Martindale's theorem [17], $R$ is then a primitive ring having nonzero socle $H$ with $C$ as the associated division ring. Hence by Jacobson's theorem [13, p.75] $R$ is isomorphic to a dense ring of linear transformations of some vector space $V$ over $C$, and $H$ consists of the linear transformations in $R$ of finite rank. If $V$ is finite dimensional over $C$, then by the density of $R$ on $V$ gives $R \cong M_{k}(C)$, where $k=\operatorname{dim}_{C} V$.

Since $R$ is noncommutative, $\operatorname{dim}_{C} V \geq 2$.
We prove now that for any $v \in V, v$ and $c v$ are linearly $C$-dependent. Suppose on the contrary that $v$ and $c v$ are linearly independent for some $v \in V$. By the density, there exists $x \in R$ such that

$$
x v=0, x c v=v
$$

Then

$$
0=a\left((b x+[c, x])^{n}-\left[c, x^{n}\right]\right) v=a v
$$

If for any $u \in V,\{u, v\}$ is linearly $C$-dependent, then $a u=0$. Since $a \neq 0$, there exists $w \in V$ such that $a w \neq 0$ and so $\{w, v\}$ are linearly $C$-independent. Also $a(w+v)=a w \neq 0$ and $a(w-v)=a w \neq 0$. By the above argument, it follows that $w$ and $c w$ are linearly $C$-dependent, as are $\{w+v, c(w+v)\}$ and $\{w-v, c(w-v)\}$. Therefore there exist $\alpha_{w}, \alpha_{w+v}, \alpha_{w-v} \in C$ such that

$$
c w=\alpha_{w} w, \quad c(w+v)=\alpha_{w+v}(w+v), \quad c(w-v)=\alpha_{w-v}(w-v)
$$

Thus we have

$$
\begin{equation*}
\alpha_{w} w+c v=\alpha_{w+v} w+\alpha_{w+v} v \tag{6}
\end{equation*}
$$

and

$$
\begin{equation*}
\alpha_{w} w-c v=\alpha_{w-v} w-\alpha_{w-v} v \tag{7}
\end{equation*}
$$

Now (6) and (7) together yields

$$
\begin{equation*}
\left(2 \alpha_{w}-\alpha_{w+v}-\alpha_{w-v}\right) w+\left(\alpha_{w-v}-\alpha_{w+v}\right) v=0 \tag{8}
\end{equation*}
$$

and

$$
\begin{equation*}
2 c v=\left(\alpha_{w+v}-\alpha_{w-v}\right) w+\left(\alpha_{w+v}+\alpha_{w-v}\right) v \tag{9}
\end{equation*}
$$

By (8), and since $\{w, v\}$ are $C$-independent, $2 \alpha_{w}-\alpha_{w+v}-\alpha_{w-v}=0$ and $\alpha_{w-v}-$ $\alpha_{w+v}=0$. These relations imply by using char $(R) \neq 2$, that $\alpha_{w}=\alpha_{w+v}=\alpha_{w-v}$. By (9) it follows $c v=\alpha_{w} v$. This leads to a contradiction with the fact that $\{v, c v\}$ is linear $C$-independent.

In light of this, we may assume that for any $v \in V$ there exists a suitable $\alpha_{v} \in C$ such that $c v=\alpha_{v} v$, and standard argument shows that there is $\alpha \in C$ such that $c v=\alpha v$ for all $v \in V$. Then by the same argument as in Theorem 2.1, $c \in Z(R)$. Then $a(b x)^{n}=0$ for all $x \in R$. Replacing $x$ with $x a$, we have $(x a b)^{n+1}=0$ for all $x \in R$. By Levitzki's Lemma [11, Lemma 1.1], $a b=0$.

Corollary 2.5. Let $R$ be a noncommutative prime ring of char $(R) \neq 2$ with its Utumi ring of quotients $U, C=Z(U)$ the extended centroid of $R, F$ a generalized derivation of $R$ with associated derivation $d$ and $I$ a nonzero ideal of $R$. Suppose that there exists $0 \neq a \in R$ such that $a\left(F(x) \circ_{n} F(y)-d\left(x \circ_{n} y\right)\right)=0$ for all $x, y \in I$, where $n \geq 0$ is a fixed integer. Then $d=0$ and there exists $b \in U$ such that $F(x)=b x$ for all $x \in R$ with $a b=0$.

Proof. In particular, for $x=y$, we have $a\left(F(x)^{n+1}-d\left(x^{n+1}\right)\right)=0$ for all $x \in I$. Then by Theorem 2.4, we conclude that there exists $b \in U$ such that $F(x)=b x$ for all $x \in R$ with $a b=0$.

In particular, for $F=d$, we have the following corollary.
Corollary 2.6. Let $R$ be a prime ring of $\operatorname{char}(R) \neq 2$ and $0 \neq a \in R$. Suppose that $d$ is a nonzero derivation of $R$ and $n \geq 0$ a fixed integer such that $a\left(d(x) \circ_{n}\right.$ $\left.d(y)-d\left(x \circ_{n} y\right)\right)=0$ for all $x, y \in I$. Then $R$ is commutative.

Example 2.7. Let $\mathbb{Z}$ be the set of all integers. Consider

$$
R=\left\{\left.\left(\begin{array}{ccc}
0 & x & y \\
0 & 0 & z \\
0 & 0 & 0
\end{array}\right) \right\rvert\, x, y, z \in \mathbb{Z}\right\}
$$

Notice that $R$ is not prime ring. We define maps $F, d: R \rightarrow R$ by $F\left(\begin{array}{lll}0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0\end{array}\right)$ $=\left(\begin{array}{lll}0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{array}\right)$ and $d\left(\begin{array}{ccc}0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0\end{array}\right)=\left(\begin{array}{ccc}0 & 0 & z \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{array}\right)$. Then it is easy to verify that $F$ is a generalized derivation associated with a derivation $d$ on $R$. We
choose $0 \neq a=\left(\begin{array}{ccc}0 & a_{1} & 0 \\ 0 & 0 & b_{1} \\ 0 & 0 & 0\end{array}\right) \in R$ such that $a\left(F(x) \circ_{n} F(y)-x \circ_{n} y\right)=0$ and $a\left(F(x) \circ_{n} F(y)-d\left(x \circ_{n} y\right)\right)=0$ for all $x, y \in R$ and for any integer $n \geq 1$. But $d \neq 0$ and so $F$ can not be written as $F(x)=b x$ for all $x \in R$, for some $b \in R$. Thus the primeness hypothesis in Corollary 2.2 and Corollary 2.5 is not superfluous.

## 3. The Results on semiprime Rings

In this section we extend Corollary 2.3 and Corollary 2.6 to the semiprime ring. Let $R$ be a semiprime ring and $U$ be its left Utumi ring of quotients. Then $C=Z(U)$ is the extended centroid of $R[7, \mathrm{p} .38]$. We know the fact.

Fact 1. Any derivation of a semiprime ring $R$ can be uniquely extended to a derivation of its left Utumi quotient ring $U$ and so any derivation of $R$ can be defined on the whole of $U$ [16, Lemma 2].

Let $M(C)$ be the set of all maximal ideals of $C$.
By the standard theory of orthogonal completions for semiprime rings, we have the following lemma.

Lemma 3.1 ([6], Lemma 1 and Theorem 1). Let $R$ be a 2-torsion free semiprime ring and $P$ a maximal ideal of $C$. Then $P U$ is a prime ideal of $U$ invariant under all derivations of $U$. Moreover,$\bigcap\{P U: P \in M(C)$ with $U / P U$ 2-torsion free $\}$ $=0$.

Theorem 3.2. Let $R$ be a noncommutative 2-torsion free semiprime ring, $U$ the left Utumi quotient ring of $R$ and $0 \neq a \in R$. Let $d$ be a nonzero derivation of $R$ such that $a\left(d(x) \circ_{n} d(y)-x \circ_{n} y\right)=0$ for all $x, y \in R$. Then $R$ contains a nonzero central ideal.

Proof. By Fact 1 and since $U$ and $R$ satisfy the same differential identities (see [16]), we have $a\left(d(x) \circ_{n} d(y)-x \circ_{n} y\right)=0$ for all $x, y \in U$. Let $P \in M(C)$ be such that $U / P U$ is 2 -torsion free. It is clear that $U$ is 2 -torsion free semiprime ring. Then $P U$ is a prime ideal of $U$ invariant under $d$ by Lemma 3.1. Denote $\bar{U}=U / P U$ and $\bar{d}$ the derivation induced by $d$ on $\bar{U}$, that is $\bar{d}(\bar{x})=\overline{d(x)}$ for all $x \in U$. For any $\bar{x}, \bar{y} \in \bar{U}$, we get $\bar{a}\left(\bar{d}(\bar{x}) \circ_{n} \bar{d}(\bar{y})-\bar{x} \circ_{n} \bar{y}\right)=0$. Moreover $\bar{U}$ is a prime ring so by Corollary 2.3, we get either $\bar{d}=0$ or $[\bar{U}, \bar{U}]=0$. In any case we have $d(U)[U, U] \subseteq P U$ for all $P \in M(C)$. In view of Lemma 3.1, $\bigcap\{P U: P \in M(C)$ with $U / P U$ 2-torsion free $\}=0$. Then $d(U)[U, U]=0$. In particular we get $d(R)[R, R]=0$. These imply that $0=d(R)\left[R^{2}, R\right]=$ $d(R) R[R, R]+d(R)[R, R] R=d(R) R[R, R]$. In particular $d(R) R[R, d(R)]=0$.

Thus $[d(R), R] R[d(R), R]=0$. Since $R$ is semiprime, we obtain that $[d(R), R]=$ 0 . Then by [5, Theorem 3], $R$ contains a nonzero central ideal.

Similarly, we have
Theorem 3.3. Let $R$ be noncommutative 2-torsion free semiprime ring, $U$ the left Utumi quotient ring of $R$ and $0 \neq a \in R$. Let $d$ be a nonzero derivation of $R$ such that $a\left(d(x) \circ_{n} d(y)-d\left(x \circ_{n} y\right)\right)=0$ for all $x, y \in R$. Then $R$ contains $a$ nonzero central ideal.

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