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# LEFT ANNIHILATOR OF IDENTITIES WITH GENERALIZED DERIVATIONS IN PRIME AND SEMIPRIME RINGS

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#### Abstract

Let R be a noncommutative prime ring of char  $(R) \neq 2$ , F a generalized derivation of R associated to the derivation d of R and I a nonzero ideal of R. Let  $S \subseteq R$ . The left annihilator of S in R is denoted by  $l_R(S)$  and defined by  $l_R(S) = \{x \in R \mid xS = 0\}$ . In the present paper, we study the left annihilator of the sets  $\{F(x) \circ_n F(y) - x \circ_n y \mid x, y \in I\}$  and  $\{F(x) \circ_n F(y) - d(x \circ_n y) \mid x, y \in I\}$ .

**Keywords:** prime ring, derivation, generalized derivation, extended centroid, Utumi quotient ring.

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### 1. INTRODUCTION

Throughout this paper, R always denotes an associative ring with center Z(R). For any  $a, b \in R$ , a ring R is said to be prime, if aRb = (0) implies either a = 0or b = 0 and is semiprime if for any  $a \in R$ , aRa = (0) implies a = 0. A mapping f is said to be an additive mapping on R if f(x + y) = f(x) + f(y) holds for all  $x, y \in R$ . An additive mapping  $d : R \to R$  defined by d(xy) = d(x)y + xd(y) for all  $x, y \in R$  is called a derivation on R. The map d(x) = [a, x] for all  $x \in R$  and for some fixed  $a \in R$ , is called an inner derivation of R. For any  $x, y \in R$ , the symbol [x, y] stands for the commutator xy - yx and the symbol  $x \circ y$  stands for the anti-commutator xy + yx. For given  $x, y \in R$ , we set  $x \circ_0 y = x, x \circ_1 y = xy + yx$ , and inductively  $x \circ_n y = (x \circ_{n-1} y) \circ y$  for n > 1. Let  $S \subseteq R$ . Then  $r_R(S)$ denotes the right annihilator of S in R, that is,  $r_R(S) = \{x \in R | Sx = 0\}$  and  $l_R(S)$  denotes the left annihilator of S in R that is,  $l_R(S) = \{x \in R | xS = 0\}$ . If  $r_R(S) = l_R(S)$ , then  $r_R(S)$  is called an annihilator ideal of R and is written as  $ann_R(S)$ .

Ashraf and Rehman [3] proved that if R is a prime ring of char  $(R) \neq 2$ , I is a nonzero ideal of R and  $d \neq 0$  such that  $d(x) \circ d(y) = x \circ y$  for all  $x, y \in I$ , then R is commutative.

In [1], Ali and Huang studied the case for semiprime ring. They proved that if R is a 2-torsion free semiprime ring and I a nonzero ideal of R and  $d(I) \neq (0)$ such that  $d(x) \circ d(y) = x \circ y$  for all  $x, y \in I$ , then R has a nonzero central ideal.

There is ongoing interest to investigate the situations replacing derivations with generalized derivations. An additive mapping  $F : R \to R$  is said to be a generalized derivation of R, if there exists a derivation d of R such that F(xy) =F(x)y + xd(y) holds for all  $x, y \in R$ . In particular, when d = 0, then F becomes a left multiplier map on R. Thus a generalized derivation covers both the concept of derivation and left multiplier map.

In [12], Huang proved that if R is a prime ring with char  $(R) \neq 2$ , L is a square closed Lie ideal of R and F a generalized derivation associated with derivation d of R such that  $d(x) \circ F(y) = x \circ y$  for all  $x, y \in L$ , then either d = 0or  $L \subseteq Z(R)$ . In [4], Bell and Rehman studied the situation for prime ring Rthat  $F(x) \circ F(y) = x \circ y$  for all  $x, y \in R$ .

Ashraf *et al.* [2] proved that the prime ring R must be commutative if R admits a generalized derivation F associated with a nonzero derivation d satisfying  $d(x) \circ F(y) = x \circ y$  for all  $x, y \in I$ , where I is a nonzero ideal of R. Then Dhara *et al.* [9] studied the same situation in 2-torsion free semiprime ring and obtained that R has a nonzero central ideal. Recently, in [18], Raza and Rehman studied the cases  $F(x) \circ_m F(y) = (x \circ y)^n$  for all  $x, y \in I$  and  $F(x) \circ_m d(y) = d(x \circ y)^n$  for all  $x, y \in I$  in prime and semiprime rings, where I is a nonzero ideal of R, F is a generalized derivation of R with associated derivation d and m, n are fixed positive integers. In the present paper, we investigate the left annihilator condition of the identities, that is  $a\{F(x) \circ_n F(y) - (x \circ_n y)\} = 0$  for all  $x, y \in I$  and  $a\{F(x) \circ_n F(y) - d(x \circ_n y)\} = 0$  for all  $x, y \in I$ .

Let R be a prime ring with center Z(R) and U is the Utumi quotient ring of R. It is well known that any derivation of R can be uniquely extended to a derivation of U, and so any derivation of R can be defined on the whole of U. Moreover U is a prime ring as well as R and the extended centroid C of Rcoincides with the center of U. Note that C is a field. We refer to [16] for more details.

We mention a very important result which will be used quite frequently as follows.

**Theorem 1.1** (Kharchenko [14]). Let R be a prime ring, d a nonzero derivation on R and I a nonzero ideal of R. If I satisfies the differential identity

$$f(r_1, r_2, \dots, r_n, d(r_1), d(r_2), \dots, d(r_n)) = 0$$

for any  $r_1, r_2, \ldots, r_n \in I$ , then either

(i) I satisfies the generalized polynomial identity

$$f(r_1, r_2, \ldots, r_n, x_1, x_2, \ldots, x_n) = 0$$

or

(ii) d is inner i.e., for some  $q \in U$ , d(x) = [q, x] for all  $x \in R$  and I satisfies the generalized polynomial identity

$$f(r_1, r_2, \dots, r_n, [q, r_1], [q, r_2], \dots, [q, r_n]) = 0.$$

#### 2. Main results

We begin with the theorem.

**Theorem 2.1.** Let R be a noncommutative prime ring of char  $(R) \neq 2$  with its Utumi ring of quotients U, C = Z(U) the extended centroid of R, F a generalized derivation of R and I a nonzero ideal of R. Suppose that there exists  $0 \neq a \in R$  such that  $a(F(x)^n - x^n) = 0$  for all  $x \in I$ , where  $n \geq 1$  is a fixed integer. Then one of the following holds.

- (1) n = 1 and there exists  $b \in U$  such that F(x) = bx for all  $x \in R$  with a(b-1) = 0;
- (2)  $n \ge 2$  and there exists  $\lambda \in C$  such that  $F(x) = \lambda x$  for all  $x \in R$  with  $\lambda^n = 1$ .

**Proof.** In light of [15, Theorem 3], there exist  $b \in U$  and derivation d of U such that F(x) = bx + d(x). Since I, R and U satisfy the same generalized polynomial identities (see [8]) as well as the same differential identities (see [16]), without loss of generality, we have  $a(F(x)^n - x^n) = 0$  for all  $x \in U$ , where d is the derivation of U, that is,

$$a\Big((bx+d(x))^n - x^n\Big) = 0$$

for all  $x \in U$ .

If F = 0, then our hypothesis reduces to  $ax^n = 0$  for all  $x \in U$ . Replacing x with xa yields  $(xa)^{n+1} = 0$  for all  $x \in U$ . Since R is a prime ring, by Levitzki's Lemma [11, Lemma 1.1], a = 0, a contradiction.

Now we assume  $F \neq 0$ . By Kharchenko's theorem, we divide the proof in two cases.

Case 1. Let d be an outer derivation of U. Then by Kharchenko's theorem [14], we have by our assumption that

$$a\Big((bx+u)^n - x^n\Big) = 0$$

for all  $x, u \in U$ . In particular, for x = 0, we have  $au^n = 0$  for all  $u \in U$ . Again, this implies a = 0, a contradiction.

Case 2. Let d be inner derivation of U, that is, d(x) = [c, x] for all  $x \in R$ and for some  $c \in U$ . Since  $d \neq 0$ ,  $c \notin C$ . Thus  $a((bx + [c, x])^n - x^n) =$ 0 is a nontrivial generalized polynomial identity (GPI) for U. Denote by E either the algebraic closure of C or C according as C is either infinite or finite, respectively. Then, by a standard argument (see for instance, [16, Proposition]),  $a((bx + [c, x])^n - x^n) = 0$  is also a GPI for  $U \otimes_C E$ . Since  $U \otimes_C E$  is centrally closed prime E-algebra [10, Theorem 2.5 and Theorem 3.5], by replacing R, C with  $U \otimes_C E$  and E, respectively, we may assume R is centrally closed and C is either finite or algebraically closed. By Martindale's theorem [17], R is then a primitive ring having nonzero socle H with C as the associated division ring. Hence by Jacobson's theorem [13, p.75] R is isomorphic to a dense ring of linear transformations of some vector space V over C, and H consists of the linear transformations in R of finite rank.

If V is finite dimensional over C, then by the density of R on V,  $R \cong M_k(C)$ where  $k = \dim_C V$ .

Since R is noncommutative,  $\dim_C V \ge 2$ .

We show that v and cv are linearly C-dependent for any  $v \in V$ . Suppose that v and cv are linearly independent for some  $v \in V$ . By the density there exists  $x \in R$  such that

$$xv = 0, \quad xcv = v.$$

Then

$$0 = a \Big( (bx + [c, x])^n - x^n \Big) v = av.$$

If for some  $u \in V$ ,  $\{u, v\}$  is linearly *C*-dependent, then au = 0. Since  $a \neq 0$ , there exists  $w \in V$  such that  $aw \neq 0$  and so  $\{w, v\}$  is linearly *C*-independent. Moreover,  $a(w + v) = aw \neq 0$  and  $a(w - v) = aw \neq 0$ . By the above argument, it follows that w and cw are linearly *C*-dependent, as are  $\{w + v, c(w + v)\}$  and  $\{w - v, c(w - v)\}$ . Therefore there exist  $\alpha_w, \alpha_{w+v}, \alpha_{w-v} \in C$  such that

$$cw = \alpha_w w, \quad c(w+v) = \alpha_{w+v}(w+v), \quad c(w-v) = \alpha_{w-v}(w-v).$$

This gives

(1) 
$$\alpha_w w + cv = \alpha_{w+v} w + \alpha_{w+v} v$$

and

(2) 
$$\alpha_w w - cv = \alpha_{w-v} w - \alpha_{w-v} v.$$

Now (1) and (2) together yields

(3) 
$$(2\alpha_w - \alpha_{w+v} - \alpha_{w-v})w + (\alpha_{w-v} - \alpha_{w+v})v = 0$$

and

(4) 
$$2cv = (\alpha_{w+v} - \alpha_{w-v})w + (\alpha_{w+v} + \alpha_{w-v})v.$$

By (3), and since  $\{w, v\}$  are *C*-independent and char  $(R) \neq 2$ , we have  $\alpha_w = \alpha_{w+v} = \alpha_{w-v}$ . Thus by (4), we have  $cv = \alpha_w v$ . This leads to a contradiction with the fact that  $\{v, cv\}$  is linear *C*-independent.

In light of this, we may assume that for any  $v \in V$  there exists a suitable  $\alpha_v \in C$  such that  $cv = \alpha_v v$ . By standard argument, there is  $\alpha \in C$  such that  $cv = \alpha v$  for all  $v \in V$ . Now let  $r \in R$ ,  $v \in V$ . Since  $cv = \alpha v$ ,

(5) 
$$[c,r]v = (cr)v - (rc)v = c(rv) - r(cv) = \alpha(rv) - r(\alpha v) = 0.$$

Thus [c, r]v = 0 for all  $v \in V$ , i.e., [c, r]V = 0. Since [c, r] acts faithfully as a linear transformation on the vector space V, [c, r] = 0 for all  $r \in R$ . Therefore,  $c \in Z(R)$ . Then

$$a\Big((bx)^n - x^n\Big) = 0$$

for all  $x \in R$ . If n = 1, then a(b-1)R = (0) implying a(b-1) = 0.

So, let n > 1. Suppose that v and bv are linearly independent for some  $v \in V$ . By the density there exists  $x \in R$  such that

$$xv = v, \quad xbv = 0,$$

and hence

$$0 = a\Big((bx)^n - x^n\Big)v = -av.$$

Since  $a \neq 0$ , by the same argument as earlier, it yields  $b \in C$ . Then  $a(b^n x^n - x^n) = 0$ , i.e.,  $a(b^n - 1)x^n = 0$  for all  $x \in R$ . By the same argument as before, we have  $a(b^n - 1) = 0$ . Since  $a \neq 0$ , it yields  $b^n = 1$ . This completes the proof.

**Corollary 2.2.** Let R be a noncommutative prime ring of char  $R \neq 2$  with its Utumi ring of quotients U, C = Z(U) the extended centroid of R, F a generalized derivation of R and I a nonzero ideal of R. Suppose that there exists  $0 \neq a \in R$  such that  $a(F(x) \circ_n F(y) - x \circ_n y) = 0$  for all  $x, y \in I$ , where  $n \geq 0$  is a fixed integer. Then one of the following holds.

(1) n = 0 and there exists  $b \in U$  such that F(x) = bx for all  $x \in R$  with a(b-1) = 0;

(2)  $n \ge 1$  and there exists  $\lambda \in C$  such that  $F(x) = \lambda x$  for all  $x \in R$  with  $\lambda^{n+1} = 1$ .

**Proof.** In particular, for x = y, we have  $a(F(x)^{n+1} - x^{n+1}) = 0$  for all  $x \in I$ . Then by Theorem 2.1, we conclude one of the following.

(1) n = 0 and there exists  $b \in U$  such that F(x) = bx for all  $x \in R$  with a(b-1) = 0;

(2)  $n \ge 1$  and there exists  $\lambda \in C$  such that  $F(x) = \lambda x$  for all  $x \in R$  with  $\lambda^{n+1} = 1$ .

In particular, for F = d, we have the following corollary.

**Corollary 2.3.** Let R be a prime ring of char  $(R) \neq 2$  and  $0 \neq a \in R$ . Suppose that d is a nonzero derivation of R and  $n \geq 0$  a fixed integer such that  $a(d(x) \circ_n d(y) - x \circ_n y) = 0$  for all  $x, y \in I$ , then R is commutative.

**Theorem 2.4.** Let R be a noncommutative prime ring of char  $(R) \neq 2$  with its Utumi ring of quotients U, C = Z(U) the extended centroid of R, F a generalized derivation of R with associated derivation d of R and I a nonzero ideal of R. Suppose that there exists  $0 \neq a \in R$  such that  $a(F(x)^n - d(x^n)) = 0$  for all  $x \in I$ , where  $n \geq 1$  is a fixed integer. Then there exists  $b \in U$  such that F(x) = bx for all  $x \in R$  with ab = 0.

**Proof.** In light of [15, Theorem 3], we may assume that there exist  $b \in U$  and derivation d of U such that F(x) = bx + d(x). Since I, R and U satisfy the same generalized polynomial identities (see [8]) as well as the same differential identities (see [16]), without loss of generality, we have  $a(F(x)^n - d(x^n)) = 0$  for all  $x \in U$ , where d is derivation on U, that is

$$a\left\{(bx+d(x))^{n} - \sum_{i=0}^{n} x^{i} d(x) x^{n-i-1}\right\} = 0$$

for all  $x \in U$ .

In light of Kharchenko's theorem, we divide the proof in two cases.

Case 1. If d is not U-inner, then by Kharchenko's theorem [14] we have

$$a\left\{(bx+y)^{n} - \sum_{i=0}^{n-1} x^{i}yx^{n-i-1}\right\} = 0$$

for all  $x, y \in U$ . If n > 1, then in particular for x = 0, we have  $ay^n = 0$  for all  $y \in U$ . This yields a = 0, a contradiction.

On the other hand, if n = 1, then abx = 0 for all  $x \in U$ , implying ab = 0.

Case 2. We assume the case when d is U-inner derivation, that is for some  $c \in U$ , d(x) = [c, x] for all  $x \in U$ . Since  $d \neq 0$ ,  $c \notin C$ . Hence  $a((bx + [c, x])^n - [c, x^n]) = 0$  is a nontrivial generalized polynomial identity (GPI) for U. Denote by E either the algebraic closure of C or C according as C is either infinite or finite, respectively. Then, by a standard argument (see for instance, [16, Proposition]),  $a((bx + [c, x])^n - [c, x^n]) = 0$  is also a GPI for  $U \otimes_C E$ . Since  $U \otimes_C E$  is centrally closed prime E-algebra [10, Theorem 2.5 and Theorem 3.5], by replacing R, C with  $U \otimes_C E$  and E, respectively, we may assume that R is centrally closed and C is either finite or algebraically closed. By Martindale's theorem [17], R is then a primitive ring having nonzero socle H with C as the associated division ring. Hence by Jacobson's theorem [13, p.75] R is isomorphic to a dense ring of linear transformations of some vector space V over C, and H consists of the linear transformations in R of finite rank. If V is finite dimensional over C, then by the density of R on V gives  $R \cong M_k(C)$ , where  $k = \dim_C V$ .

Since R is noncommutative,  $\dim_C V \ge 2$ .

We prove now that for any  $v \in V$ , v and cv are linearly C-dependent. Suppose on the contrary that v and cv are linearly independent for some  $v \in V$ . By the density, there exists  $x \in R$  such that

$$xv = 0, xcv = v.$$

Then

$$0 = a \Big( (bx + [c, x])^n - [c, x^n] \Big) v = av.$$

If for any  $u \in V$ ,  $\{u, v\}$  is linearly *C*-dependent, then au = 0. Since  $a \neq 0$ , there exists  $w \in V$  such that  $aw \neq 0$  and so  $\{w, v\}$  are linearly *C*-independent. Also  $a(w + v) = aw \neq 0$  and  $a(w - v) = aw \neq 0$ . By the above argument, it follows that w and cw are linearly *C*-dependent, as are  $\{w + v, c(w + v)\}$  and  $\{w - v, c(w - v)\}$ . Therefore there exist  $\alpha_w, \alpha_{w+v}, \alpha_{w-v} \in C$  such that

$$cw = \alpha_w w, \quad c(w+v) = \alpha_{w+v}(w+v), \quad c(w-v) = \alpha_{w-v}(w-v).$$

Thus we have

(6) 
$$\alpha_w w + cv = \alpha_{w+v} w + \alpha_{w+v} v$$

and

(7) 
$$\alpha_w w - cv = \alpha_{w-v} w - \alpha_{w-v} v.$$

Now (6) and (7) together yields

(8) 
$$(2\alpha_w - \alpha_{w+v} - \alpha_{w-v})w + (\alpha_{w-v} - \alpha_{w+v})v = 0$$

(9) 
$$2cv = (\alpha_{w+v} - \alpha_{w-v})w + (\alpha_{w+v} + \alpha_{w-v})v.$$

By (8), and since  $\{w, v\}$  are C-independent,  $2\alpha_w - \alpha_{w+v} - \alpha_{w-v} = 0$  and  $\alpha_{w-v} - \alpha_{w-v} = 0$  $\alpha_{w+v} = 0$ . These relations imply by using char  $(R) \neq 2$ , that  $\alpha_w = \alpha_{w+v} = \alpha_{w-v}$ . By (9) it follows  $cv = \alpha_w v$ . This leads to a contradiction with the fact that  $\{v, cv\}$ is linear C-independent.

In light of this, we may assume that for any  $v \in V$  there exists a suitable  $\alpha_v \in C$  such that  $cv = \alpha_v v$ , and standard argument shows that there is  $\alpha \in C$ such that  $cv = \alpha v$  for all  $v \in V$ . Then by the same argument as in Theorem 2.1,  $c \in Z(R)$ . Then  $a(bx)^n = 0$  for all  $x \in R$ . Replacing x with xa, we have  $(xab)^{n+1} = 0$  for all  $x \in R$ . By Levitzki's Lemma [11, Lemma 1.1], ab = 0.

**Corollary 2.5.** Let R be a noncommutative prime ring of char  $(R) \neq 2$  with its Utumi ring of quotients U, C = Z(U) the extended centroid of R, F a generalized derivation of R with associated derivation d and I a nonzero ideal of R. Suppose that there exists  $0 \neq a \in R$  such that  $a(F(x) \circ_n F(y) - d(x \circ_n y)) = 0$  for all  $x, y \in I$ , where  $n \geq 0$  is a fixed integer. Then d = 0 and there exists  $b \in U$  such that F(x) = bx for all  $x \in R$  with ab = 0.

**Proof.** In particular, for x = y, we have  $a(F(x)^{n+1} - d(x^{n+1})) = 0$  for all  $x \in I$ . Then by Theorem 2.4, we conclude that there exists  $b \in U$  such that F(x) = bxfor all  $x \in R$  with ab = 0.

In particular, for F = d, we have the following corollary.

**Corollary 2.6.** Let R be a prime ring of char  $(R) \neq 2$  and  $0 \neq a \in R$ . Suppose that d is a nonzero derivation of R and  $n \ge 0$  a fixed integer such that  $a(d(x) \circ_n$  $d(y) - d(x \circ_n y) = 0$  for all  $x, y \in I$ . Then R is commutative.

*Example 2.7.* Let  $\mathbb{Z}$  be the set of all integers. Consider

$$R = \left\{ \left( \begin{array}{ccc} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{array} \right) \mid x, y, z \in \mathbb{Z} \right\}.$$

Notice that R is not prime ring. We define maps  $F, d: R \to R$  by  $F\begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix}$ 

 $= \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } d \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$  Then it is easy to verify

that F is a generalized derivation associated with a derivation d on R. We

and

choose  $0 \neq a = \begin{pmatrix} 0 & a_1 & 0 \\ 0 & 0 & b_1 \\ 0 & 0 & 0 \end{pmatrix} \in R$  such that  $a(F(x) \circ_n F(y) - x \circ_n y) = 0$ and  $a(F(x) \circ_n F(y) - d(x \circ_n y)) = 0$  for all  $x, y \in R$  and for any integer  $n \ge 1$ . But  $d \neq 0$  and so F can not be written as F(x) = bx for all  $x \in R$ , for some  $b \in R$ . Thus the primeness hypothesis in Corollary 2.2 and Corollary 2.5 is not superfluous.

#### 3. The results on semiprime rings

In this section we extend Corollary 2.3 and Corollary 2.6 to the semiprime ring. Let R be a semiprime ring and U be its left Utumi ring of quotients. Then C = Z(U) is the extended centroid of R [7, p. 38]. We know the fact.

**Fact 1.** Any derivation of a semiprime ring R can be uniquely extended to a derivation of its left Utumi quotient ring U and so any derivation of R can be defined on the whole of U [16, Lemma 2].

Let M(C) be the set of all maximal ideals of C.

By the standard theory of orthogonal completions for semiprime rings, we have the following lemma.

**Lemma 3.1** ([6], Lemma 1 and Theorem 1). Let R be a 2-torsion free semiprime ring and P a maximal ideal of C. Then PU is a prime ideal of U invariant under all derivations of U. Moreover,  $\bigcap \{PU : P \in M(C) \text{ with } U/PU \text{ 2-torsion free} \} = 0.$ 

**Theorem 3.2.** Let R be a noncommutative 2-torsion free semiprime ring, U the left Utumi quotient ring of R and  $0 \neq a \in R$ . Let d be a nonzero derivation of R such that  $a(d(x) \circ_n d(y) - x \circ_n y) = 0$  for all  $x, y \in R$ . Then R contains a nonzero central ideal.

**Proof.** By Fact 1 and since U and R satisfy the same differential identities (see [16]), we have  $a(d(x) \circ_n d(y) - x \circ_n y) = 0$  for all  $x, y \in U$ . Let  $P \in M(C)$  be such that U/PU is 2-torsion free. It is clear that U is 2-torsion free semiprime ring. Then PU is a prime ideal of U invariant under d by Lemma 3.1. Denote  $\overline{U} = U/PU$  and  $\overline{d}$  the derivation induced by d on  $\overline{U}$ , that is  $\overline{d}(\overline{x}) = \overline{d(x)}$  for all  $x \in U$ . For any  $\overline{x}, \overline{y} \in \overline{U}$ , we get  $\overline{a}(\overline{d}(\overline{x}) \circ_n \overline{d}(\overline{y}) - \overline{x} \circ_n \overline{y}) = 0$ . Moreover  $\overline{U}$  is a prime ring so by Corollary 2.3, we get either  $\overline{d} = 0$  or  $[\overline{U}, \overline{U}] = 0$ . In any case we have  $d(U)[U,U] \subseteq PU$  for all  $P \in M(C)$ . In view of Lemma 3.1,  $\bigcap \{PU : P \in M(C) \text{ with } U/PU \text{ 2-torsion free}\} = 0$ . Then d(U)[U,U] = 0. In particular we get d(R)[R, R] = 0. These imply that  $0 = d(R)[R^2, R] = d(R)R[R, R] + d(R)[R, R]R = d(R)R[R, R]$ . In particular d(R)R[R, d(R)] = 0.

Thus [d(R), R]R[d(R), R] = 0. Since R is semiprime, we obtain that [d(R), R] = 0. Then by [5, Theorem 3], R contains a nonzero central ideal.

Similarly, we have

**Theorem 3.3.** Let R be noncommutative 2-torsion free semiprime ring, U the left Utumi quotient ring of R and  $0 \neq a \in R$ . Let d be a nonzero derivation of R such that  $a(d(x) \circ_n d(y) - d(x \circ_n y)) = 0$  for all  $x, y \in R$ . Then R contains a nonzero central ideal.

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